Existence and uniqueness results for best proximity points

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EXISTENCE AND UNIQUENESS RESULTS FOR BEST PROXIMITY POINTS

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Abstract. Let us consider a non-self mapping \( T : A \to B \), where \( A \) and \( B \) are two nonempty subsets of a metric space \((X, d)\). The aim of this paper is to solve the nonlinear programming problem that consists in minimizing the real valued function \( x \mapsto d(x, Tx) \), where \( T \) belongs to a new class of non-self mappings. In especial case, existence and uniqueness of fixed point for Kannan type self mappings are also obtained.

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1. Introduction

Let \( A \) and \( B \) be two nonempty subsets of a metric space \( X \). A non-self mapping \( T : A \to B \) is said to be a contraction if there exists a constant \( r \in [0, 1) \), such that \( d(Tx, Ty) \leq rd(x, y) \), for all \( x, y \in A \). The well-known Banach contraction principle states that if \( A \) is a complete subset of \( X \) and \( T \) is a contraction self-mapping, then the fixed point equation \( Tx = x \) has exactly one solution.

The Banach contraction principle is a very important tools in nonlinear analysis and there are many extensions of this principle; see, e.g., [13] and the references therein.

Let \( (X, d) \) be a metric space. A self-mapping \( T : X \to X \) is called Kannan mapping if there exists \( \alpha \in [0, \frac{1}{2}) \) such that

\[
d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],
\]

for all \( x, y \in X \). We know that if \( X \) is complete metric space, every Kannan self-mapping defined on \( X \) has a unique fixed point ([12]). Note that, the notion of contraction mappings and Kannan mappings are independent. That is, there exists a contraction mapping, which is not Kannan and a Kannan mapping, which is not a contraction. Therefore, we cannot compare these two class of mappings directly.

Recently, Kikkawa and Suzuki in [14], established the following fixed point theorem, which is an extension of Kannan’s fixed point theorem.
Theorem 1 ([14]). Define a non-increasing function $\varphi$ from $[0, 1)$ into $(\frac{1}{2}, 1]$ by

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$. Let $\alpha \in [0, \frac{1}{2})$ and put $r := \frac{\alpha}{1-\alpha} \in [0, 1)$. Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. Then $T$ has a unique fixed point $z$ and $\lim_{n \to \infty} T^nx = z$ holds for every $x \in X$.

It is interesting to note that the function $\varphi(r)$ defined in Theorem 1 is the best constant for every $r$ (see Theorem 2.4 of [14]).

2. Preliminaries

Consider the non-self mapping $T : A \to X$, in which $A$ is a nonempty subset of a metric space $(X, d)$. Clearly, the fixed point equation $Tx = x$ may not have solution. Hence, it is contemplated to find an approximate $x \in A$ such that the error $d(x, Tx)$ is minimum. Indeed, best approximation theory has been derived from this idea. Here, we state the following well-known best approximation theorem due to Kay Fan.

Theorem 2 ([8]). Let $A$ be a nonempty compact convex subset of a normed linear space $X$ and $T : A \to X$ be a continuous mapping. Then there exists $x \in A$ such that

$$\|x - Tx\| = \text{dist}(Tx, A) := \inf\{\|Tx - a\| : a \in A\}.$$

A point $x \in A$ in the above theorem is called a best approximant point of $T$ in $A$.

The notion of best proximity point for non-self mappings is introduced in a similar fashion:

Definition 1. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \to B$ be a non-self mapping. A point $p \in A$ is called a best proximity point of $T$ if

$$d(p, Tp) = \text{dist}(A, B) := \{d(x, y) : (x, y) \in A \times B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x \in A} d(x, Tx),$$

has at least one solution.

Best proximity point theory is an interesting topic in optimization theory which recently attracted the attention of many authors (see for instance [1–9, 16]).

Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. Let us fix the following notations which will be needed throughout this article:

$$A_0 := \{x \in A : d(x, y) = \text{dist}(A, B) \quad \text{for some} \quad y \in B\},$$
$B_0 := \{ y \in B : d(x,y) = dist(A,B) \text{ for some } x \in A \}.$

It is easy to see that if $(A,B)$ is a nonempty and weakly compact pair of subsets of a Banach space $X$, then $(A_0 : B_0)$ is also nonempty pair $X$.

The notion of proximal contractions was defined by Sadiq Basha, as follows.

**Definition 2** ([15]). Let $(A, B)$ be a pair of nonempty subsets of a metric space $(X,d)$. A mapping $T : A \rightarrow B$ is said to be a proximal contraction if there exists a non-negative real number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2 \in A$,

$$
\begin{align*}
&d(u_1, Tx_1) = dist(A,B) \\
&d(u_2, Tx_2) = dist(A,B) \\
&d(u_1, u_2) \leq \alpha d(x_1, x_2).
\end{align*}
$$

**Definition 3** ([15]). Let $A, B$ be two nonempty subsets of a metric space $(X,d)$. $A$ is said to be approximatively compact with respect to $B$ if every sequence $\{x_n\}$ of $A$ satisfying the condition that $d(y, x_n) \rightarrow D(y, A)$ for some $y \in B$ has a convergent subsequence.

Next theorem is the main result of [15].

**Theorem 3.** Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty and $B$ is approximatively compact with respect to $A$. Assume that $T : A \rightarrow B$ is a proximal contraction such that $T(A_0) \subseteq B_0$.

Then $T$ has a unique best proximity point.

We mention that in [10], the current author extended Theorem 3 and established a best proximity point theorem under weaker conditions with respect to Theorem 3, due to Sadiq Basha (see Theorem 2.1 and Corollary 2.1 of [10]).

In this article, we introduce a new class of mappings called weak proximal Kannan non-self mappings and obtain a similar result of Theorem 1 for this new class of non-self mappings.

### 3. Main Results

To establish our main results, we introduce the following new class of non-self mappings.

**Definition 4.** Define a strictly decreasing function $\theta$ from $[0, \frac{1}{2})$ onto $(\frac{1}{2}, 1]$ by

$$
\theta(r) = 1 - r.
$$

Let $(A,B)$ be a nonempty pair of subsets of a metric space $(X,d)$. Let $\alpha \in [0, \frac{1}{2})$ and put $r := \frac{\alpha}{1 - \alpha}$. Then $T : A \rightarrow B$ is said to be a weak proximal Kannan non-self mapping if for all $u, v, x, y \in A$ with

$$
d(u, Tx) = dist(A,B) \quad & d(v, Ty) = dist(A,B),
$$

we have

$$
\theta(r)d^*(x, Tx) \leq d(x, y) \text{ implies } d(u, v) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)]. \quad (3.1)
$$
The notion of proximal Kannan non-self mapping can be defined as below.

**Definition 5.** Let \((A, B)\) be a nonempty pair of subsets of a metric space \((X, d)\). Then \(T : A \rightarrow B\) is said to be a proximal Kannan non-self mapping if there exists \(\alpha \in [0, \frac{1}{2})\) such that for all \(u, v, x, y \in A\) with

\[
d(u, Tx) = \text{dist}(A, B) \quad \& \quad d(v, Ty) = \text{dist}(A, B),
\]

we have

\[
d(u, v) \leq \alpha [d^*(x, Tx) + d^*(y, Ty)].
\]

It is clear that the class of weak proximal Kannan non-self mappings contains the class of proximal Kannan non-self mappings as a subclass. Also, the class of proximal Kannan non-self mappings contains the class of Kannan non-self mappings.

We now state our main result of this article.

**Theorem 4.** Let \((A, B)\) be a nonempty pair of subsets of a complete metric space \((X, d)\) such that \(A_0\) is nonempty and closed. Assume that \(T : A \rightarrow B\) is a weak proximal Kannan non-self mapping such that \(T(A_0) \subseteq B_0\). Then there exists a unique point \(x^* \in A\) such that \(d(x^*, Tx^*) = \text{dist}(A, B)\). Moreover, if \(\{x_n\}\) is a sequence in \(A\) such that \(d(x_{n+1}, Tx_n) = \text{dist}(A, B)\), then \(x_n \rightarrow x^*\).

**Proof.** Assume \(x_0 \in A_0\). Since \(T(A_0) \subseteq B_0\), there exists \(x_1 \in A_0\) such that \(d(x_1, Tx_0) = \text{dist}(A, B)\). Again, since \(Tx_1 \in B_0\), there exists \(x_2 \in A_0\) such that \(d(x_2, Tx_1) = \text{dist}(A, B)\). Continuing this process, we can find a sequence \(\{x_n\}\) in \(A_0\) such that

\[
d(x_{n+1}, Tx_n) = \text{dist}(A, B), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}
\]

Thus,

\[
d(x_0, Tx_0) \leq d(x_0, x_1) + d(x_1, Tx_0) = d(x_0, x_1) + \text{dist}(A, B),
\]

and so,

\[
\theta(r)d^*(x_0, Tx_0) \leq d^*(x_0, Tx_0) \leq d(x_0, x_1) \quad \& \quad \begin{cases} d(x_1, Tx_0) = \text{dist}(A, B), \\ d(x_2, Tx_1) = \text{dist}(A, B). \end{cases}
\]

Since \(T\) is a weak proximal Kannan non-self mapping, we conclude that

\[
d(x_1, x_2) \leq \alpha [d^*(x_0, Tx_0) + d^*(x_1, Tx_1)]
\]

\[
\leq \alpha [d(x_0, x_1) + d^*(x_1, Tx_0) + d(x_1, x_2) + d^*(x_2, Tx_1)]
\]

\[
= \alpha [d(x_0, x_1) + d(x_1, x_2)].
\]

Therefore,

\[
d(x_1, x_2) \leq \frac{\alpha}{1 - \alpha} d(x_0, x_1) = rd(x_0, x_1).
\]
Similarly, we can see that
\[
\theta(r) d^*(x_1, T x_1) \leq d(x_1, x_2) \quad \text{and} \quad \begin{cases} 
  d(x_2, T x_1) = \text{dist}(A, B), \\
  d(x_3, T x_2) = \text{dist}(A, B).
\end{cases}
\]

This implies that
\[
d(x_2, x_3) \leq \alpha [d^*(x_1, T x_1) + d^*(x_2, T x_2)] \\
\leq \alpha [d(x_1, x_2) + d^*(x_2, T x_1) + d(x_2, x_3) + d^*(x_3, T x_2)] \\
= \alpha [d(x_1, x_2) + d(x_2, x_3)].
\]

So,
\[
d(x_2, x_3) \leq \frac{\alpha}{1 - \alpha} d(x_1, x_2) = r d(x_1, x_2) \leq r^2 d(x_0, x_1).
\]

Hence, by induction, we conclude that
\[
d(x_n, x_{n+1}) \leq r^n d(x_0, x_1),
\]

which implies that
\[
\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} r^n d(x_0, x_1) < \infty.
\]

That is, \( \{x_n\} \) is a Cauchy sequence in \( A_0 \). Since \( A_0 \) is closed and \( X \) is complete metric space, we deduce that \( \{x_n\} \) is a convergent sequence. Let \( x^* \in A_0 \) be such that \( x_n \to x^* \). We claim that \( x^* \) is a unique best proximity point of \( T \). At first, we prove that
\[
d^*(x^*, T x) \leq \alpha d(x^*, x), \quad \forall x \in A_0 \quad \text{with} \quad x \neq x^*. \tag{3.3}
\]

Let \( x \in A_0 \) and \( x \neq x^* \). Since \( T(A_0) \subseteq B_0 \), there exists \( y \in A_0 \) such that \( d(y, T x) = \text{dist}(A, B) \). By the fact that \( x_n \to x^* \), there exists \( N_1 \in \mathbb{N} \) such that
\[
d(x_n, x^*) \leq \frac{1}{3} d(x, x^*), \quad \forall n \geq N_1.
\]

We now have
\[
\theta(r) d^*(x_n, T x_n) \leq d^*(x_n, T x_n) = d(x_n, T x_n) - \text{dist}(A, B) \\
\leq d(x_n, x^*) + d(x^*, x_{n+1}) + d(x_{n+1}, T x_n) - \text{dist}(A, B) \\
= d(x_n, x^*) + d(x^*, x_{n+1}) \leq \frac{2}{3} d(x, x^*) \\
= d(x, x^*) - \frac{1}{3} d(x, x^*) \leq d(x, x^*) - d(x_n, x^*) \\
\leq d(x_n, x).
\]

Thus,
\[
\theta(r) d^*(x_n, T x_n) \leq d(x_n, x) \quad \text{and} \quad \begin{cases} 
  d(x_{n+1}, T x_n) = \text{dist}(A, B), \\
  d(y, T x) = \text{dist}(A, B).
\end{cases}
\]
Again, since $T$ is weak proximal Kannan non-self mapping we conclude that
\[ d(x_{n+1}, y) \leq \alpha [d^*(x_n, Tx_n) + d^*(x, Tx)] \leq \alpha [d(x_n, x_{n+1}) + d^*(x, Tx)]. \]

Thereby,
\[
\begin{align*}
d(x^*, Tx) &= \lim_{n \to \infty} d(x_n, Tx) \\
&\leq \lim_{n \to \infty} [d(x_n, x_{n+1}) + d(x_{n+1}, y) + d(y, Tx)] \\
&\leq \lim_{n \to \infty} [1 + \alpha]d(x_n, x_{n+1}) + \alpha d^*(x, Tx) + d(y, Tx)] \\
&\leq \lim_{n \to \infty} [1 + \alpha r^n d(x_0, x_1) + \alpha d^*(x, Tx) + dist(A, B)] \\
&= \alpha d^*(x, Tx) + dist(A, B).
\end{align*}
\]

Thus,
\[
\begin{align*}
d^*(x^*, Tx) \leq \alpha d^*(x, Tx), \quad \forall x \in A_0, \quad \text{with} \quad x \neq x^*.
\end{align*}
\]

That is, (3.3) holds. It now follows from (3.3) that
\[
\begin{align*}
d^*(x_n, Tx_n) &\leq d(x_n, x^*) + d^*(x^*, Tx_n) \\
&\leq d(x_n, x^*) + \alpha d^*(x_n, Tx_n).
\end{align*}
\]

Thus,
\[
\begin{align*}
\theta(r) d^*(x_n, Tx_n) &= (1 - r) d^*(x_n, Tx_n) \leq (1 - \alpha)d^*(x_n, Tx_n) \leq d(x_n, x^*). \quad (3.4)
\end{align*}
\]

On the other hand, since $x^* \in A_0$ and $T(A_0) \subseteq B_0$, there exists $y^* \in B_0$ such that $d(y^*, Tx^*) = dist(A, B)$. Therefore,
\[
\begin{align*}
\theta(r) d^*(x_n, Tx_n) &\leq d(x_n, x^*) \quad & \text{with} \quad d(x_{n+1}, Tx_{n+1}) = dist(A, B), \\
d(y^*, Tx^*) &\leq dist(A, B),
\end{align*}
\]

which implies that
\[
\begin{align*}
d(x_{n+1}, y^*) &\leq \alpha [d^*(x_n, Tx_n) + d^*(x^*, Tx^*)] \\
&\leq \alpha [d(x_n, x_{n+1}) + d^*(x_{n+1}, Tx_n) + d^*(x^*, Tx^*)] \\
&= \alpha d^*(x^*, Tx^*) + d(y^*, Tx^*) = \alpha d(x^*, y^*).
\end{align*}
\]

This deduces that $d(x^*, y^*) = 0$ or $x^* = y^*$. Hence $x^*$ is a best proximity point of the mapping $T$. The uniqueness of best proximity point follows from the condition that $T$ is weak proximal Kannan non-self mapping. That is, suppose that $x^*_1, x^*_2$ are two distinct points in $A$ such that $d(x^*_i, Tx^*_i) = dist(A, B)$, for $i = 1, 2$. So,
\[
\begin{align*}
\theta(r) d^*(x^*_1, Tx^*_1) &\leq d(x^*_1, x^*_2) \quad & \text{with} \quad d(x^*_1, Tx^*_1) = dist(A, B), \\
d(x^*_2, Tx^*_2) &\leq dist(A, B),
\end{align*}
\]
Then,

\[ 0 < d(x_1^*, x_2^*) \leq \alpha [d^*(x_1^*, Tx_1^*) + d^*(x_2^*, Tx_2^*)] = 0, \]

which is a contradiction. Hence, the best proximity point is unique.

\[ \square \]

The following corollaries are obtained from Theorem 4.

**Corollary 1.** Let \((A, B)\) be a nonempty pair of subsets of a complete metric space \((X, d)\) such that \(A_0\) is nonempty and closed. Assume that \(T : A \to B\) is a proximal Kannan non-self mapping such that \(T(A_0) \subseteq B_0\). Then there exists a unique point \(x^* \in A\) such that \(d(x^*, Tx^*) = dist(A, B)\). Moreover, if \(\{x_n\}\) is a sequence in \(A\) such that \(d(x_{n+1}, Tx_n) = dist(A, B)\) then, \(x_n \to x^*\).

**Corollary 2.** Let \((A, B)\) be a nonempty pair of a complete metric space \((X, d)\) such that \(A_0\) is nonempty and closed. Assume that \(T : A \to B\) is a Kannan non-self mapping such that \(T(A_0) \subseteq B_0\). Then there exists a unique point \(x^* \in A\) such that \(d(x^*, Tx^*) = dist(A, B)\). Moreover, if \(\{x_n\}\) is a sequence in \(A\) such that \(d(x_{n+1}, Tx_n) = dist(A, B)\) then, \(x_n \to x^*\).

**Corollary 3.** Let \(A\) be a nonempty and closed subset of a complete metric space \((X, d)\). Assume that \(T : A \to A\) is a self mapping such that

\[ \theta(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq \alpha [d(x, Tx) + d(y, Ty)], \]

for all \(x, y \in A\), where \(\theta(r)\) is defined as in the Definition 4. Then \(T\) has a unique fixed point \(x^* \in A\). Moreover, if \(x_0 \in A\) and we define \(x_{n+1} = Tx_n\), then \(x_n \to x^*\).

**Corollary 4** (Kannan fixed point theorem). Let \(A\) be a nonempty and closed subset of a complete metric space \((X, d)\). Assume that \(T : A \to A\) is a Kannan mapping. Then \(T\) has a unique fixed point. Moreover, for each \(x_0 \in A\) if we define \(x_{n+1} = Tx_n\) then the sequence \(\{x_n\}\) converges to the fixed point of \(T\).

**Example 1.** Suppose that \(X = \mathbb{R}\) with the usual metric. Suppose that

\[ A := [0, 2] \cup \{5\} \quad \& \quad B := [3, 4]. \]

Then \(A\) and \(B\) are nonempty closed subsets of \(X\) and \(A_0 = \{2.5\}\) and \(B_0 = \{3, 4\}\). Note that \(dist(A, B) = 1\). Let \(T : A \to B\) be a mapping defined as

\[ T(x) = \begin{cases} 
    7 & \text{if } x = 0, \\
    2 & \text{if } x \neq 0.
\end{cases} \]

It is easy to see that \(T\) is weak proximal Kannan non-self mapping for each \(\alpha \in [0, \frac{1}{2}]\). Indeed, it is sufficient to note that \(d(u, Tx) = dist(A, B)\), holds for \(u = 5\) and \(x \in A - \{0\}\). Therefore, Theorem 4 guaranties the existence and uniqueness of a best proximity point for \(T\) and this point is \(x^* = 5\).
Example 2. Suppose that $X = \mathbb{R}$ with the usual metric. Suppose that
\[ A := [0, \frac{1}{100}] \cup \{1\} \quad \& \quad B := [2, 3]. \]
Then $A$ and $B$ are nonempty closed subsets of $X$ and $dist(A, B) = 1$. Define a non-self mapping $T : A \to B$ as follows
\[ T(x) = \begin{cases} 2 & \text{if } x \in \mathbb{Q} \cap A, \\ \frac{101}{50} & \text{if } x \in \mathbb{Q}^c \cap A. \end{cases} \]
Note that $T$ is not continuous. We claim that $T$ is Kannan non-self mapping with $\alpha = \frac{1}{3}$. For this purpose, it is sufficient to consider two following cases.

Case 1. If $x \in \mathbb{Q} \cap A - \{1\}$ and $y \in \mathbb{Q}^c \cap A$, then
\[ \alpha[d^*(x, Tx) + d^*(y, Ty)] = \frac{1}{3}[\frac{101}{50} - (x + y)] \geq \frac{2}{3} > \frac{1}{50} = d(Tx, Ty). \]

Case 2. If $x = 1$ and $y \in \mathbb{Q}^c \cap A$, then
\[ \alpha[d^*(x, Tx) + d^*(y, Ty)] = \frac{1}{3}[\frac{51}{50} - y] \geq \frac{1}{3} \times \frac{101}{100} > \frac{1}{50} = d(Tx, Ty). \]
It now follows from Corollary 2 that $T$ has a unique best proximity point and this point is $x^* = 1$.

Remark 1. We mention that in [11] the author studied the existence of best proximity points in metric spaces with a partial order, where weak proximal Kannan non-self mappings are satisfied only for comparable elements.

References


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