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## On $\tau$ -supplemented subgroups of finite groups

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## ON $\tau$ -SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

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*Abstract.* Let  $H$  be a subgroup of a finite group  $G$ . We say that: (1)  $H$  is  $\tau$ -quasinormal in  $G$  if  $H$  permutes with every Sylow subgroup  $Q$  of  $G$  such that  $(|H|, |Q|) = 1$  and  $(|H|, |Q^G|) \neq 1$ ; (2)  $H$  is  $\tau$ -supplemented in  $G$  if  $G$  has a subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T \leq H_{\tau}G$ , where  $H_{\tau}G$  is the subgroup generated by all those subgroups of  $H$  which are  $\tau$ -quasinormal in  $G$ . We investigate the influence of  $\tau$ -supplemented subgroups on the structure of finite groups. Some recent known results are generalized and unified.

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### 1. INTRODUCTION

This paper deals with finite groups. We use the standard terminology as in [10].  $G$  denotes always a group,  $|G|$  is the order of  $G$  and the set of distinct primes dividing  $|G|$  will be denoted by  $\pi(G)$ . A group  $G$  is called  $p$ -supersolvable if it is  $p$ -solvable and all its  $G$ -chief  $p$ -factors are cyclic. A group  $G$  is called  $p$ -nilpotent if it is  $p$ -solvable and all its  $G$ -chief  $p$ -factors are central in  $G$ . Obviously, a  $p$ -nilpotent group is also a  $p$ -supersolvable group and  $G$  is supersolvable (or nilpotent) if and only if  $G$  is  $p$ -supersolvable (or  $p$ -nilpotent) for any  $p \in \pi(G)$ . If  $G = HK$  and  $K$  is  $p$ -supersolvable (or supersolvable,  $p$ -nilpotent), then we call that  $H$  has a  $p$ -supersolvable (or supersolvable,  $p$ -nilpotent) supplement  $K$  in  $G$ .

A subgroup  $H$  of a group  $G$  is said to be  $S$ -quasinormal (or  $S$ -permutable) in  $G$  if  $H$  permutes with all Sylow subgroups of  $G$ , i.e.,  $HS = SH$  for any Sylow subgroup  $S$  of  $G$ . This concept was first introduced by Kegel in [12]. Later, many authors generalized  $S$ -quasinormal concept; see, for example, [5, 13, 14, 34]. A subgroup  $H$  is said to be  $s$ -semipermutable in  $G$  if  $H$  permutes with every Sylow  $p$ -subgroup of  $G$  such that  $(p, |H|) = 1$ . More recently, Lukanenko and Skiba [19] introduced the concept of  $\tau$ -quasinormal subgroup as follows: A subgroup  $H$  of  $G$  is said to be  $\tau$ -quasinormal in  $G$  if  $H$  permutes with every Sylow subgroup  $Q$  of  $G$  such that  $(|H|, |Q|) = 1$  and  $(|H|, |Q^G|) \neq 1$ . It is clear that  $s$ -semipermutability implies  $\tau$ -quasinormality by definition; however, the converse is not true, as seen in [20, Example 1].

On the other hand, some subgroups with supplemented properties were introduced recently. In particular, Ballester-Bolinches, Wang and Guo [7] called that a subgroup  $H$  of a group  $G$  is  $c$ -supplemented in  $G$  if there is a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G$ , where  $H_G$  is the normal core of  $H$  in  $G$ . In 2007, Skiba [25] again gave the concept of  $S$ -supplemented subgroup as follows: A subgroup  $H$  of  $G$  is called  $S$ -supplemented in  $G$  if there exists a subgroup  $K$  such that  $G = HK$  and  $H \cap K \leq H_{sG}$ , where  $H_{sG}$  is the subgroup of  $H$  generated by all those subgroups of  $H$  which are  $S$ -quasinormal in  $G$ . If we take  $G = \langle a, b \mid a^{16} = b^4 = 1, ba = a^3b \rangle$ , then  $\langle b^2 \rangle$  is an  $S$ -supplemented subgroup of  $G$ . However  $\langle b^2 \rangle$  is not  $c$ -supplemented in  $G$ . Hence,  $S$ -supplemented subgroups generalize  $c$ -supplemented subgroups.

There is no obvious general relationship between  $\tau$ -quasinormal subgroups and  $S$ -supplemented subgroups. Hence it is meaningful to unify and generalize above series subgroups. On the basis of these definitions, we now introduce the following new concept:

**Definition 1.** A subgroup  $H$  of a group  $G$  is said to be  $\tau$ -supplemented in  $G$  if  $G$  has a subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T \leq H_{\tau G}$ , where  $H_{\tau G}$  is the subgroup generated by all those subgroups of  $H$  which are  $\tau$ -quasinormal in  $G$ .

The next two examples show that the class of all  $\tau$ -supplemented subgroups is wider than the class of all  $\tau$ -quasinormal subgroups and the class of all  $S$ -supplemented subgroups.

*Example 1.* Let  $G = S_4$  be the symmetric group of degree 4 and  $H = \langle (14) \rangle$ . Obviously,  $H$  is  $\tau$ -supplemented in  $G$ . However,  $H$  is not  $\tau$ -quasinormal in  $G$ .

*Example 2.* Let  $G = \langle a, b, c \mid a^5 = b^4 = c^5 = 1, b^{-1}ab = a^2, [a, c] = [b, c] = 1 \rangle$  and  $H = \langle b^2 \rangle$ . It is easy to see that  $H$  is  $\tau$ -supplemented in  $G$ , but not  $S$ -supplemented in  $G$ .

Let  $\mathcal{F}$  be a class of groups. We call  $\mathcal{F}$  a formation provided that (i) if  $G \in \mathcal{F}$  and  $H \trianglelefteq G$ , then  $G/H \in \mathcal{F}$ , and (ii) if  $G/M$  and  $G/N$  are in  $\mathcal{F}$ , then  $G/(M \cap N)$  is in  $\mathcal{F}$  for any normal subgroups  $M, N$  of  $G$ . A formation  $\mathcal{F}$  is said to be saturated if  $G/\Phi(G) \in \mathcal{F}$  implies that  $G \in \mathcal{F}$ . In this paper,  $\mathcal{U}$  and  $\mathcal{N}$  will denote the class of all supersolvable groups and the class of all nilpotent groups, respectively. It is well known that both  $\mathcal{U}$  and  $\mathcal{N}$  are saturated formations. A chief factor  $H/K$  of a group  $G$  is called  $\mathcal{F}$ -central provided  $[H/K](G/C_G(H/K)) \in \mathcal{F}$ . The product of all normal subgroups of  $G$  whose  $G$ -chief factors are  $\mathcal{F}$ -central in  $G$  is called the  $\mathcal{F}$ -hypercentre of  $G$  and denoted by  $Z_{\mathcal{F}}(G)$ . Noticing that for any  $\mathcal{N}$ -central chief factor  $H/K$  of  $G$  we have  $C_G(H/K) = G$ . Hence the  $\mathcal{N}$ -hypercentre of  $G$  coincides with the hypercentre  $Z_{\infty}(G)$  of  $G$ . Another fact is that any chief factor of  $G$  under  $Z_{\mathcal{U}}(G)$  is of prime order. The major aim of the present paper is to find cyclicity conditions for  $G$ -chief factors of normal subgroups of a group  $G$  by some

$\tau$ -supplemented subgroups. As their applications, we not only extend some known results in [1, 2, 4, 7, 9, 15, 16, 22, 28, 29, 32], but also give more simple proofs.

## 2. PRELIMINARIES

**Lemma 1** ([19, Lemma 2.3]). *Let  $G$  be a group and  $E \leq K \leq G$ .*

- (1) *If  $E$  is  $\tau$ -quasinormal in  $G$ , then  $E$  is  $\tau$ -quasinormal in  $K$ .*
- (2) *Suppose that  $E$  is normal in  $G$  and  $\pi(K/E) = \pi(K)$ . If  $K$  is  $\tau$ -quasinormal in  $G$ , then  $K/E$  is  $\tau$ -quasinormal in  $G/E$ .*
- (3) *Suppose that  $E$  is normal in  $G$ . Then  $HE/E$  is  $\tau$ -quasinormal in  $G/E$  for every  $\tau$ -quasinormal subgroup  $E$  in  $G$  satisfying  $(|H|, |E|) = 1$ .*
- (4) *If  $E$  is  $\tau$ -quasinormal in  $G$  and  $E \leq O_p(G)$  for some prime  $p$ , then  $E$  is  $S$ -quasinormal in  $G$ .*

**Lemma 2.** *Let  $H$  be a  $\tau$ -supplemented subgroup of a group  $G$ .*

- (1) *If  $H \leq L \leq G$ , then  $H$  is  $\tau$ -supplemented in  $L$ .*
- (2) *If  $E \trianglelefteq G$ ,  $E \leq H \leq G$  and  $H$  is a  $p$ -group for some prime  $p$ , then  $H/E$  is  $\tau$ -supplemented in  $G/E$ .*
- (3) *If  $H$  is a  $\pi$ -subgroup and  $E$  is a normal  $\pi'$ -subgroup of  $G$ , then  $HE/E$  is  $\tau$ -supplemented in  $G/E$ .*

*Proof.* By the hypothesis, there is a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_{\tau G}$ .

(1)  $L = L \cap HK = H(L \cap K)$  and  $H \cap (L \cap K) = H \cap K \leq H_{\tau G} \leq H_{\tau K}$ . Hence  $H$  is  $\tau$ -supplemented in  $L$ .

(2) We have  $G/E = HK/E = H/E \cdot EK/E$  and  $(H/E) \cap (KE/E) = (H \cap KE)/E = (H \cap K)E/E \leq H_{\tau G}E/E = H_{\tau G}/E \leq (H/E)_{\tau(G/E)}$  by Lemma 1. Hence  $H/E$  is  $\tau$ -supplemented in  $G/E$ .

(3) Since  $(|G : K|, |E|) = 1$  and  $E \trianglelefteq G$ , we have  $E \leq K$ . It is easy to see that  $G/E = HE/E \cdot KE/E = HE/E \cdot K/E$  and  $(HE/E) \cap (K/E) = (HE \cap K)/E = (H \cap K)E/E \leq H_{\tau G}E/E \leq (HE/E)_{\tau(G/E)}$  by Lemma 1. Hence  $HE/E$  is  $\tau$ -supplemented in  $G/E$ .  $\square$

**Lemma 3** ([33, p.38, Theorem 7.19]). *Let  $H$  be a normal subgroup of  $G$ . Then  $H \leq Z_{\mathcal{U}}(G)$  if and only if  $H/\Phi(H) \leq Z_{\mathcal{U}}(G/\Phi(H))$ .*

**Lemma 4** ([26, Lemma A]). *Let  $E$  be a normal subgroup of a group  $G$ . Suppose that for every non-cyclic Sylow subgroup  $P$  of  $E$ , either all maximal subgroups of  $P$  or all cyclic subgroups of  $P$  of prime order and order 4 are  $S$ -supplemented in  $G$ . Then each  $G$ -chief factor below  $E$  is cyclic.*

**Lemma 5** ([8, p.362, Proposition 3.11]). *If  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are two saturated formations such that  $\mathcal{F}_1 \subseteq \mathcal{F}_2$ , then  $Z_{\mathcal{F}_1}(G) \leq Z_{\mathcal{F}_2}(G)$ .*

The generalized Fitting subgroup  $F^*(G)$  of  $G$  is the unique maximal normal quasinilpotent subgroup of  $G$ .  $F^*(G)$  is an important subgroup of  $G$  and it is a natural

generalization of  $F(G)$ . The definition and important properties can be found in [11, Chapter X].

**Lemma 6** ([11, X, 13]). *Let  $G$  be a group. Then:*

(1) *If  $F^*(G)$  is solvable, then  $F^*(G) = F(G)$ .*

(2)  $C_G(F^*(G)) \leq F(G)$ .

**Lemma 7** ([27, Theorem C]). *Let  $E$  be a normal subgroup of a group  $G$ . If  $F^*(E) \leq Z_{\mathcal{U}}(G)$ , then  $E \leq Z_{\mathcal{U}}(G)$ .*

**Lemma 8** ([10, p.434, Satz 5.4 and p.281, Satz 5.2]). *If  $G$  is a group which is not  $p$ -nilpotent but all of its proper subgroups are  $p$ -nilpotent, then it is a minimal non-nilpotent group (that is,  $G$  is not nilpotent but all of its proper subgroups are nilpotent). Then*

(1)  *$G$  has a normal Sylow  $p$ -subgroup  $P$  for some prime  $p$  and  $G = PQ$ , where  $Q$  is a non-normal cyclic  $q$ -subgroup for some prime  $q \neq p$ .*

(2)  $P/\Phi(P)$  is a minimal normal subgroup of  $G/\Phi(P)$ .

(3) *If  $P$  is non-abelian and  $p > 2$ , then the exponent of  $P$  is  $p$ ; If  $P$  is non-abelian and  $p = 2$ , then the exponent of  $P$  is 4.*

(4) *If  $P$  is abelian, then the exponent of  $P$  is  $p$ .*

**Lemma 9** ([28, Lemma 2.8]). *Let  $M$  be a maximal subgroup of  $G$ ,  $P$  a normal  $p$ -subgroup of  $G$  such that  $G = PM$ , where  $p$  is a prime. Then  $P \cap M$  is a normal subgroup of  $G$ .*

**Lemma 10** ([33, p.220, Theorem 6.3]). *Let  $P$  be a normal  $p$ -subgroup of  $G$  such that  $|G/C_G(P)|$  is a power of  $p$ . Then  $P \leq Z_{\infty}(G)$ .*

### 3. MAIN RESULTS

**Theorem 1.** *Suppose that  $p$  is the smallest prime dividing the order of a group  $G$  and  $G$  has a normal subgroup  $E$  such that  $G/E$  is  $p$ -nilpotent. If every cyclic subgroup  $H$  of  $E$  with prime order  $p$  or order 4 (if  $p = 2$ ) having no  $p$ -supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ , then  $G$  is  $p$ -nilpotent.*

*Proof.* Suppose that the theorem is false, and let  $G$  be a counterexample of minimal order.

(1) The hypotheses are inherited by all proper subgroups of  $G$  and  $G$  is a group which is not nilpotent but whose proper subgroups are all nilpotent.

In fact,  $\forall K < G$ , since  $G/E$  is  $p$ -nilpotent,  $K/(K \cap E) \cong KE/E$  is also  $p$ -nilpotent. Let  $H$  be any cyclic subgroup of  $K \cap E$  with prime order  $p$  or order 4 (if  $p = 2$ ). Obviously,  $H$  is a cyclic subgroup of  $E$  with prime order  $p$  or order 4. If  $H$  has a  $p$ -supersoluble supplement  $T$  in  $G$ , then  $H$  has a  $p$ -supersoluble supplement  $T \cap K$  in  $K$ . If  $H$  is  $\tau$ -supplemented in  $G$ , then  $H$  is  $\tau$ -supplemented in  $K$  by Lemma

2(1). Thus  $K$  satisfies the hypotheses of the theorem. By the choice of  $G$ ,  $K$  is  $p$ -nilpotent. Then  $G$  is a group which is not  $p$ -nilpotent but whose proper subgroups are all  $p$ -nilpotent. By Lemma 8,  $G = PQ$  and  $P \trianglelefteq G$ .

(2)  $G/(P \cap E)$  is  $p$ -nilpotent.

Since  $G/P \cong Q$  is  $p$ -nilpotent,  $G/E$  is  $p$ -nilpotent and  $G/(P \cap E) \lesssim G/P \times G/E$ , we have  $G/(P \cap E)$  is  $p$ -nilpotent.

(3)  $P \leq E$ .

If  $P \not\leq E$ , then  $P \cap E < P$ . So  $Q(P \cap E) < QP = G$ . Thus  $Q(P \cap E)$  is nilpotent by Step (1) and so  $Q(P \cap E) = Q \times (P \cap E)$ . Since  $G/(P \cap E) = P/(P \cap E) \cdot Q(P \cap E)/(P \cap E)$ , it follows that  $Q(P \cap E)/(P \cap E) \trianglelefteq G/(P \cap E)$  by Step (2). Now  $Q \text{ char } Q(P \cap E) \trianglelefteq G$  implies that  $G = P \times Q$ , a contradiction.

(4) For every cyclic subgroup  $L$  of  $P$  with order  $p$  or 4, if there is a subgroup  $T$  of  $G$  such that  $G = LT$ , then  $T = G$ .

Let  $L$  be a cyclic subgroup of prime order  $p$  or of order 4 in  $P$  and assume that there exists a subgroup  $T$  of  $G$  such that  $G = LT$ . Obviously,  $P = P \cap G = P \cap LT = L(P \cap T)$ . Since  $P/\Phi(P)$  is abelian, we have  $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ . By Step (1),  $P \cap T \leq \Phi(P)$  or  $P = (P \cap T)\Phi(P) = P \cap T$ . If the former holds, then  $L = P$  and so  $G$  is  $p$ -nilpotent by [24, P.280, Theorem 10.1.9], a contradiction. Hence  $P = P \cap T$  and  $T = G$ .

(5) Every cyclic subgroup  $L$  of  $P$  with prime order  $p$  or of order 4 is  $S$ -quasinormal in  $G$ .

If  $L$  has a  $p$ -supersoluble supplement  $T$  in  $G$ , then  $G = T$  is  $p$ -supersoluble by Step (4) and so  $G$  is  $p$ -nilpotent since  $p$  is the smallest prime dividing  $|G|$ , a contradiction. Thus we may assume all cyclic subgroups of  $P$  with order  $p$  or 4 are  $\tau$ -supplemented in  $G$ . In view of Step (4), all cyclic subgroups of  $P$  with order  $p$  or 4 are  $\tau$ -quasinormal in  $G$ . By Lemma 1(4), all cyclic subgroups of  $P$  with order  $p$  or 4 are  $S$ -quasinormal in  $G$ .

(6) Final contradiction.

For every  $x \in P$ , we have  $| \langle x \rangle | = p$  or 4 by Step (1), and so  $\langle x \rangle$  is  $S$ -quasinormal in  $G$  by Step (5). By [24, P.280, Theorem 10.1.9], we have  $\langle x \rangle Q$  is a proper subgroup of  $G$ , and so  $\langle x \rangle Q = \langle x \rangle \times Q$  by Step (1). Then we conclude that  $G = P \times Q$ , a contradiction.  $\square$

**Theorem 2.** *Suppose that  $P$  is a normal  $p$ -subgroup of a group  $G$ . If every cyclic subgroup of  $P$  with order  $p$  or 4 (if  $p = 2$ ) having no  $p$ -supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ , then  $P \leq Z_{\mathcal{U}}(G)$ .*

*Proof.* We distinguish two cases:

Case I.  $p = 2$ .

Pick an arbitrary Sylow  $q$ -subgroup  $G_q$  of  $G$ , where  $q \neq 2$ . Consider the subgroup  $W = G_q P$ . Let  $L$  be a cyclic subgroup of  $P$  with order 2 or 4. If  $L$  has a 2-supersoluble supplement  $T$  in  $G$ , then  $L$  has a 2-supersoluble supplement  $T \cap W$  in  $W$ . If  $L$  is  $\tau$ -supplemented in  $G$ , then  $L$  is also  $\tau$ -supplemented in  $W$  by Lemma

2(1). Applying Theorem 1, we have  $W$  is 2-nilpotent. Hence  $W = P \times G_q$ . This implies that  $O^p(G) \leq C_G(P)$ . In view of Lemma 10,  $P \leq Z_\infty(G)$ . Consequently,  $P \leq Z_{\mathcal{U}}(G)$  from Lemma 5.

Case II.  $p > 2$ .

First suppose that some cyclic subgroup  $L$  of  $P$  with order  $p$  has a  $p$ -supersoluble supplement  $T$  in  $G$ . So  $G = LT = PT$ . If  $T = G$ , then  $P \leq Z_{\mathcal{U}}(G)$  and we are done. Hence we may assume that  $T < G$ , which shows that  $L \cap T = 1$  and  $T$  is a maximal subgroup of  $G$ . Clearly,  $P = P \cap LT = L(P \cap T)$  and  $P \cap T$  is a maximal subgroup of  $P$ . By Lemma 9,  $P \cap T$  is normal in  $G$ . Since every cyclic subgroup of  $P \cap T$  with order  $p$  having no  $p$ -supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ ,  $P \cap T \leq Z_{\mathcal{U}}(G)$  by induction. Noticing that  $P/(P \cap T)$  is a normal subgroup of  $G/(P \cap T)$  with order  $p$ , we have  $P/(P \cap T) \leq Z_{\mathcal{U}}(G/(P \cap T))$ . It follows that  $P \leq Z_{\mathcal{U}}(G)$ . Now we may assume that all cyclic subgroups of  $P$  with order  $p$  are  $\tau$ -supplemented in  $G$ . From Lemma 1(4) we have that all cyclic subgroups of  $P$  with order  $p$  are  $S$ -supplemented in  $G$  since  $P \leq O_p(G)$ . In view of Lemma 4, we have also  $P \leq Z_{\mathcal{U}}(G)$ .  $\square$

**Theorem 3.** *Let  $E$  be a normal subgroup of a group  $G$ . Suppose that for each  $p \in \pi(E)$  and non-cyclic Sylow  $p$ -subgroup  $P$  of  $E$ , all cyclic subgroups of  $P$  with order  $p$  or 4 (if  $p = 2$ ) having no  $p$ -supersoluble supplement in  $G$  are  $S$ -supplemented in  $G$ . Then  $E \leq Z_{\mathcal{U}}(G)$ .*

*Proof.* Let  $q$  be the smallest prime dividing  $|E|$  and  $L$  a cyclic subgroup of the Sylow  $q$ -subgroup  $Q$  of  $E$  with order  $q$  or 4 (if  $q = 2$ ). If  $L$  has a  $q$ -supersoluble supplement  $T$  in  $G$ , then  $L$  has a  $q$ -supersoluble supplement  $T \cap E$  in  $E$ . If  $L$  is  $\tau$ -supplemented in  $G$ , then  $L$  is also  $\tau$ -supplemented in  $E$  by Lemma 2(1). In view of Theorem 1,  $E$  is  $q$ -nilpotent. Let  $E_{q'}$  be the normal  $q'$ -complement of  $E$ . If  $E = Q$ , then  $E \leq Z_{\mathcal{U}}(G)$  by Theorem 2. Hence we may assume that  $E_{q'} \neq 1$ . Since  $E_{q'} \text{ char } E \trianglelefteq G$ ,  $E_{q'} \trianglelefteq G$ . By the hypothesis of the theorem, for each  $p \in \pi(E_{q'})$  and non-cyclic Sylow  $p$ -subgroup  $P$  of  $E_{q'}$ , all cyclic subgroups of  $P$  with order  $p$  having no  $p$ -supersoluble supplement in  $G$  are  $S$ -supplemented in  $G$ . By induction,  $E_{q'} \leq Z_{\mathcal{U}}(G)$ . By Lemma 2(3), it is easy to see that all cyclic subgroups of  $QE_{q'}/E_{q'}$  with order  $q$  or 4 (if  $q = 2$ ) having no  $q$ -supersoluble supplement in  $G/E_{q'}$  are  $S$ -supplemented in  $G/E_{q'}$ . By induction, we have also  $E/E_{q'} \leq Z_{\mathcal{U}}(G/E_{q'})$ . It follows that  $E \leq Z_{\mathcal{U}}(G)$ .  $\square$

**Corollary 1.** *Let  $E$  be a normal subgroup of a group  $G$ . Suppose that every cyclic subgroup of each non-cyclic Sylow subgroup of  $E$  with prime order or order 4 having no supersoluble supplement in  $G$  are  $S$ -supplemented in  $G$ . Then  $E \leq Z_{\mathcal{U}}(G)$ .*

**Theorem 4.** *Let  $P$  be a normal  $p$ -subgroup of a group  $G$ , where  $p$  is a prime dividing the order of  $G$ . Suppose that every maximal subgroup of  $P$  having no  $p$ -supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ . Then  $P \leq Z_{\mathcal{U}}(G)$ .*

*Proof.* We distinguish two cases:

Case I.  $\Phi(P) \neq 1$ .

Obviously,  $P/\Phi(P)$  is a normal  $p$ -subgroup of  $G/\Phi(P)$ . Let  $P_1/\Phi(P)$  be a maximal subgroup of  $P/\Phi(P)$ . Then  $P_1$  is a maximal subgroup of  $P$ . If  $P_1$  has a  $p$ -supersoluble supplement  $T$  in  $G$ , then  $P_1/\Phi(P)$  has a  $p$ -supersoluble supplement  $T\Phi(P)/\Phi(P)$  in  $G/\Phi(P)$ . If  $P_1$  is  $\tau$ -supplemented in  $G$ , then  $P_1/\Phi(P)$  is  $\tau$ -supplemented in  $G/\Phi(P)$  by Lemma 2. Therefore,  $G/\Phi(P)$  satisfies the hypothesis of the theorem. By induction,  $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ . In view of Lemma 3, we have  $P \leq Z_{\mathcal{U}}(G)$ .

Case II.  $\Phi(P) = 1$ .

This shows that  $P$  is abelian. First suppose that some maximal subgroup  $V$  of  $P$  has a  $p$ -supersoluble supplement  $T$  in  $G$ . Then  $G = VT = PT$  and  $P \cap T \neq 1$ . Since  $P \cap T \trianglelefteq T$ , we may assume that  $T$  has a minimal normal subgroup  $N$  contained in  $P \cap T$ . It is clear that  $|N| = p$ . From  $G = PT$ , we have  $N$  is also normal in  $G$ . With the similar argument in Case I, the hypothesis of the theorem holds for  $(G/N, P/N)$ . By induction, we have  $P/N \leq Z_{\mathcal{U}}(G/N)$ . It follows that  $P \leq Z_{\mathcal{U}}(G)$ . Now we may assume that every maximal subgroup of  $P$  is  $\tau$ -supplemented in  $G$ . In view of Lemma 1, every maximal subgroup of  $P$  is  $S$ -supplemented in  $G$  since  $P \leq O_p(G)$ . Then we have also  $P \leq Z_{\mathcal{U}}(G)$  by Lemma 4.  $\square$

**Corollary 2.** *Let  $P$  be a normal  $p$ -subgroup of  $G$ . Suppose that every maximal subgroup of  $P$  having no supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ . Then  $P \leq Z_{\mathcal{U}}(G)$ .*

In connection with Theorem 3 and Lemma 4 the following natural question arises:

**Question.** *Let  $E$  be a normal subgroup of a group  $G$ . Suppose that for every non-cyclic Sylow subgroup  $P$  of  $E$  all maximal subgroups of  $P$  having no supersoluble supplement in  $G$  are  $\tau$ -supplemented in  $G$ . Is then  $E \leq Z_{\mathcal{U}}(G)$ ?*

#### 4. SOME APPLICATIONS

**Theorem 5.** *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every maximal subgroup of each non-cyclic Sylow subgroup of  $F(E)$  having no supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

*Proof.* By Theorem 4,  $F(E) \leq Z_{\mathcal{U}}(G)$ . Since  $\mathcal{U} \subseteq \mathcal{F}$ , we have that  $F(E) \leq Z_{\mathcal{F}}(G)$  by Lemma 5. In view of the solvability of  $E$  and Lemma 6,  $F^*(E) = F(E) \leq Z_{\mathcal{F}}(G)$ . By Lemma 7,  $E \leq Z_{\mathcal{F}}(G)$ . Since  $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$ , we have  $G \in \mathcal{F}$ .  $\square$

**Theorem 6.** *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . If there is a normal subgroup  $E$  of a group  $G$  such that  $G/E \in \mathcal{F}$  and every cyclic subgroup of  $E$  with*

prime order or order 4 having no supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .

*Proof.* Since  $E \leq Z_{\mathcal{U}}(G)$  by Theorem 3 and  $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$  by Lemma 5, we have  $E \leq Z_{\mathcal{F}}(G)$ . Hence  $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$ . It follows that  $G \in \mathcal{F}$ .  $\square$

**Theorem 7.** *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$  and let  $G$  be a group. If there is a normal subgroup  $E$  such that  $G/E \in \mathcal{F}$  and every cyclic subgroup of  $F^*(E)$  with prime order or order 4 having no supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

*Proof.* By Theorem 3,  $F^*(E) \leq Z_{\mathcal{U}}(G)$  and since  $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$  by Lemma 5, we have  $F^*(E) \leq Z_{\mathcal{F}}(G)$ . In view of Lemma 7,  $E \leq Z_{\mathcal{F}}(G)$ . Hence  $G \in \mathcal{F}$  since  $G/E \in \mathcal{F}$ .  $\square$

**Theorem 8.** *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every cyclic subgroup of  $F(E)$  with prime order or order 4 having no supersoluble supplement in  $G$  is  $\tau$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

*Proof.* Since  $E$  is solvable,  $F^*(E) = F(E)$  by Lemma 6. Applying Theorem 7, we have  $G \in \mathcal{F}$ .  $\square$

**Corollary 3** ([22, Theorem 3.1]). *Assume that  $G$  is solvable and every maximal subgroup of the Sylow subgroups of  $F(G)$  is normal in  $G$ . Then  $G$  is supersolvable.*

**Corollary 4** ([4, Corollary 4.4]). *Suppose that  $G$  is a solvable group with a normal subgroup  $E$  such that  $G/E$  is supersolvable. If all maximal subgroups of any Sylow subgroup of  $F(E)$  are  $S$ -quasinormal in  $G$ , then  $G$  is supersolvable.*

A subgroup  $H$  of a group  $G$  is  $c$ -normal in  $G$  if there is a normal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G$ , where  $H_G$  is the normal core of  $H$  in  $G$ .

**Corollary 5** ([15, Theorem 2]). *Let  $G$  be a group and  $E$  a solvable normal subgroup of  $G$  such that  $G/E$  is supersolvable. If every maximal subgroup of each Sylow subgroup of  $F(E)$  is  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 6** ([34, Theorem 2]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ , the class of all supersolvable groups. Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If all maximal subgroups of all Sylow subgroups of  $F(E)$  are  $s$ -semipermutable in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 7** ([16, Theorem 1.2]). *Suppose that  $G$  is a solvable group with a normal subgroup  $E$  such that  $G/E$  is supersolvable. If every maximal subgroup of each Sylow subgroup of  $F(E)$  is complement in  $G$ , then  $G$  is supersolvable.*

**Corollary 8** ([30, Theorem 1]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every maximal subgroup of each Sylow subgroup of  $F(E)$  is  $c$ -normal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 9** ([1, Theorem 1.4]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a solvable group with a normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every maximal subgroup of each Sylow subgroup of  $F(E)$  is  $S$ -quasinormal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 10** ([28, Theorem 4.5]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every maximal subgroup of each Sylow subgroup of  $F(E)$  is  $c$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 11** ([9, Theorem 1.6]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every maximal subgroup of each Sylow subgroup of  $F(E)$  is complemented in  $G$ , then  $G \in \mathcal{F}$ .*

A subgroup  $H$  is called  $Q$ -supplemented in a group  $G$ , if there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K$  is contained in  $H_Q G$ , where  $H_Q G$  is the maximal quasinormal subgroup of  $G$  contained in  $H$ .

**Corollary 12** ([21, Theorem 3.6]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If every maximal subgroup of each Sylow subgroup of  $F(E)$  is  $Q$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 13** ([34, Theorem 3]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . If there is a normal subgroup  $E$  of  $G$  such that  $G/E \in \mathcal{F}$  and every cyclic subgroup of  $E$  with prime order or order 4 is  $s$ -semipermutable in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 14** ([29, Theorem 4.2]). *If every cyclic subgroup of a group  $G$  with prime order or order 4 is  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 15** ([35, Theorem 3.1]). *Let  $G$  be a group and  $E$  a normal subgroup of a group  $G$  such that  $G/E$  is supersolvable. If every minimal subgroup of  $E$  is  $c$ -supplemented in  $G$  and if every cyclic subgroup of  $E$  with order 4 is  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 16** ([7, Theorem 4.1]). *If every cyclic subgroup of  $G^{\mathcal{U}}$  with prime order or order 4 is  $c$ -supplemented in  $G$ , then  $G$  is supersolvable.*

**Corollary 17** ([23, Theorem 3.9]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Then  $G \in \mathcal{F}$  if and only if there is a normal subgroup  $E$  of  $G$  such that  $G/E \in \mathcal{F}$  and the subgroups prime order or order 4 of  $E$  with are  $c$ -normal in  $G$ .*

**Corollary 18** ([2, Theorem 1]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . If there is a normal subgroup  $E$  of  $G$  such that  $G/E \in \mathcal{F}$  and every cyclic subgroup of  $E$  with prime order or order 4 is  $S$ -quasinormal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 19** ([6, Theorem 3.4]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . If every cyclic subgroup of  $G^{\mathcal{F}}$  with prime order or order 4 is  $c$ -normal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 20** ([31, Theorem 3.2]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$  and let  $G$  be a group. If there is a normal subgroup  $E$  such that  $G/E \in \mathcal{F}$  and the subgroups of  $F^*(E)$  with prime order or order 4 are  $c$ -normal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 21** ([18, Theorem 3.3]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$  and let  $G$  be a group. If there is a normal subgroup  $E$  such that  $G/E \in \mathcal{F}$  and the subgroups of  $F^*(E)$  with prime order or order 4 are  $S$ -quasinormal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 22** ([32, Theorem 1.2]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$  and let  $G$  be a group. If there is a normal subgroup  $E$  such that  $G/E \in \mathcal{F}$  and the subgroups of  $F^*(E)$  with prime order or order 4 are  $c$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 23** ([3, Corollary 1]). *Suppose that  $G$  is a group with a normal solvable subgroup  $E$  such that  $G/E$  is supersolvable. If every subgroup of  $F(E)$  of prime order or order 4 is  $S$ -quasinormal in  $G$ , then  $G$  is supersolvable.*

**Corollary 24** ([15, Theorem 3]). *Let  $G$  be a group and  $E$  a solvable normal subgroup of  $G$  such that  $G/E$  is supersolvable. If all minimal subgroups and all cyclic subgroups of  $F(E)$  with order 4 are  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 25** ([28, Theorem 4.1]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If all minimal subgroups and all cyclic subgroups of  $F(E)$  with order 4 is  $c$ -supplemented in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 26** ([30, Theorem 2]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If all minimal subgroups and all cyclic subgroups of  $F(E)$  with order 4 is  $c$ -normal in  $G$ , then  $G \in \mathcal{F}$ .*

**Corollary 27** ([17, Theorem 3]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . A group  $G \in \mathcal{F}$  if and only if there is a solvable normal subgroup  $E$  of  $G$  such that  $G/E \in \mathcal{F}$  and the subgroups of  $F(E)$  with prime order or order 4 is  $c$ -normal in  $G$ .*

**Corollary 28** ([3, Theorem ]). *A group  $G \in \mathcal{F}$  if and only if there is a solvable normal subgroup  $E$  of  $G$  such that  $G/E \in \mathcal{F}$  and the subgroups of  $F(E)$  with prime order or order 4 is  $S$ -quasinormal in  $G$ .*

**Corollary 29** ([34, Theorem 4]). *Let  $\mathcal{F}$  be a saturated formation containing  $\mathcal{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $E$  such that  $G/E \in \mathcal{F}$ . If all minimal subgroups and all cyclic subgroups of  $F(E)$  with order 4 is  $s$ -semipermutable in  $G$ , then  $G \in \mathcal{F}$ .*

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