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ON MANNHEIM PARTNER CURVES IN THREE DIMENSIONAL LIE GROUPS

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Abstract. In this paper, we define Mannheim partner curves in a three dimensional Lie group G with a bi-invariant metric. The main result of the paper is given as (Theorem 4): A curve $\alpha : I \subset \mathbb{R} \to G$ with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ is a Mannheim partner curve if and only if

$$\lambda\kappa\left(1 + H^2\right) = 1$$

where λ, μ are constants and H is the harmonic curvature function of the curve α.

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1. INTRODUCTION

In the classical diferential geometry of curves, J. Bertrand studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve. In [1] he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other word, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves* (see [10]).

Another kind of associated curve whose principal normal vector field is the binormal vector field of another curve is called Mannheim curve. Mannheim partner curves was studied by Liu and Wang (see [12]) in Euclidean 3− space and in the Minkowski 3−space. After these papers lots of papers have been published about Mannheim curves in Euclidean 3−space, Minkowski 3−space, dual 3−space and Galilean spaces (see [7, 9, 15, 17, 18]). Matsuda and Yorozu [13] gave a definition of generalized Mannheim curve in Euclidean 4−space. They show some characterizations and examples of generalized Mannheim curves. Ersoy *et.al.* gave a definition of generalized Mannheim curve in Minkowski 4−space.

The degenarete semi-Riemannian geometry of Lie group is studied by Çöken and Çiftçi [4]. Moreover, they obtanied a naturally reductive homogeneous semi-Riemannian space using the Lie group. Then Çiftçi [3] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem and gave a relation between the geodesics of the so-called cylinders and general helices.

Recently, *Izumiya and Takeuchi* [8] have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ was defined by the property that its principal normal vector field makes a constant angle with a fixed direction. Also, Izumiya and Takeuchi showed that α is a slant helix if and only if the geodesic curvature of spherical image of principal normal indicatrix (N) of a space curve α

$$\sigma_N(s) = \left(\frac{\kappa^2}{\left(\kappa^2+\tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

is a constant function.

Harmonic curvature functions have been defined by Özdamar and Hacısalihoğlu [16]. Recently, many studies have been reported on generalized helices and slant helices using the harmonic curvatures in Euclidean spaces and Minkowski spaces [2, 6, 11]. Then, Okuyucu et al. [14] have defined slant helices in three dimensional Lie groups with a bi-invariant metric and obtained some characterizations using their harmonic curvature functions.

In this paper, first of all, we define Mannheim partner curves in a three dimensional Lie group G with a bi-invariant metric and we obtain the necessary and sufficient conditions for the Mannheim partner curves in a three dimensional Lie group G.

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle\,,\rangle$ and D be the Levi-Civita connection of Lie group G. If $\mathfrak{g}$ denotes the Lie algebra of G then we know that $\mathfrak{g}$ is issomorphic to T_eG where e is neutral element of G. If $\langle\,,\rangle$ is a bi-invariant metric on G then we have

$$\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X,Y] \tag{2.2}$$

for all X, Y and $Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted regular curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, we write that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^{n} w_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields W and Z

is given by

$$[W,Z]=\sum_{i=1}^{n} w_i z_i \left[X_i, X_j\right]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha'}W$ is given as follows

$$D_{\alpha'}W=\dot{W}+\frac{1}{2}[T,W] \tag{2.3}$$

where $T=\alpha'$ and $\dot{W}=\sum_{i=1}^{n}\dot{w}_i X_i$ or $\dot{W}=\sum_{i=1}^{n}\frac{dw}{dt}X_i$. Note that if W is the left-invariant vector field along to the curve α then $\dot{W}=0$ (see for details [5]).

Let G be a three dimensional Lie group and (T,N,B,κ,τ) denote the Frenet apparatus of the curve α. Then the Serret-Frenet formulas of the curve α satisfies:

$$D_T T=\kappa N,\quad D_T N=-\kappa T+\tau B,\quad D_T B=-\tau N$$

where D is Levi-Civita connection of Lie group G and $\kappa=\|\dot{T}\|$.

Definition 1. Let $\alpha: I\subset\mathbb{R}\to G$ be a parametrized curve. Then α is called a *general helix* if it makes a constant angle with a left-invariant vector field X. That is,

$$\langle T(s),X\rangle=\cos\theta \text{ for all } s\in I,$$

for the left-invariant vector field $X\in g$ is unit length and θ is a constant angle between X and T, which is the tangent vector field of the curve α (see [3]).

Definition 2. Let $\alpha: I\subset\mathbb{R}\to G$ be a parametrized curve with the Frenet apparatus (T,N,B,κ,τ) then

$$\tau_G=\frac{1}{2}\langle[T,N],B\rangle \tag{2.4}$$

or

$$\tau_G=\frac{1}{2\kappa^2\tau}\langle\ddot{T},[T,\dot{T}]\rangle+\frac{1}{4\kappa^2\tau}\|[T,\dot{T}]\|^2$$

(see [3]).

Proposition 1. *Let $\alpha: I\subset\mathbb{R}\to G$ be an arc length parametrized curve with the Frenet apparatus $\{T,N,B\}$. Then the following equalities*

$$[T,N]=\langle[T,N],B\rangle B=2\tau_G B$$
$$[T,B]=\langle[T,B],N\rangle N=-2\tau_G N$$

hold [14].

Definition 3. Let $\alpha: I\subset\mathbb{R}\to G$ be an arc length parametrized curve. Then α is called a *slant helix* if its principal normal vector field makes a constant angle with a left-invariant vector field X which is unit length. That is,

$$\langle N(s),X\rangle=\cos\theta \text{ for all } s\in I,$$

where $\theta \neq \frac{\pi}{2}$ is a constant angle between X and N which is the principal normal vector field of the curve α (see [14]).

Definition 4. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$. Then the *harmonic curvature function* of the curve α is defined by

$$H = \frac{\tau - \tau_G}{\kappa}$$

where $\tau_G = \frac{1}{2} \langle [T, N], B \rangle$ (see [14]).

Theorem 1. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T, N, B, κ, τ). The curve α is a general helix, if and only if,*

$$\tau = c\kappa + \tau_G$$

where c is a constant (see [3]) or using the definition of the harmonic curvature function of the curve α (see [14]) H is a constant function.

Theorem 2. *Let $\alpha : I \subset \mathbb{R} \to G$ be a unit speed curve with the Frenet apparatus (T, N, B, κ, τ). Then α is a slant helix if and only if*

$$\sigma_N = \frac{\kappa(1 + H^2)^{\frac{3}{2}}}{H'} = \tan\theta$$

is a constant where H is a harmonic curvature function of the curve α and $\theta \neq \frac{\pi}{2}$ is a constant (see [14]).

3. Mannheim partner curves in a three dimensional Lie groups

In this section, we define Mannheim partner curves and their characterizations are given in a three dimensional Lie group G with a bi-invariant metric $\langle , \rangle$. Also we give some characterizations of Mannheim partner curves using the special cases of G.

Definition 5. A curve α in 3-dimensional Lie group G is a *Mannheim curve* if there exists a special curve β in 3-dimensional Lie group G such that principal normal vector field of α is linearly dependent binormal vector field of β at corresponding point under ψ which is bijection from α to β. In this case β is called the *Mannheim partner curve* of α and (α, β) is called *Mannheim curve couple.*

The curve $\alpha : I \subset \mathbb{R} \to G$ in 3-dimensional Lie group G is parametrized by the arc-length parameter s and from the Definition 5 Mannheim mate curve of α is given $\beta : \overline{I} \subset \mathbb{R} \to G$ in 3-dimensional Lie group G with the help of Figure 1 such that

$$\beta(s) = \alpha(s) + \lambda(s) N(s), \; s \in I$$

where λ is a smooth function on I and N is the principal normal vector field of α.

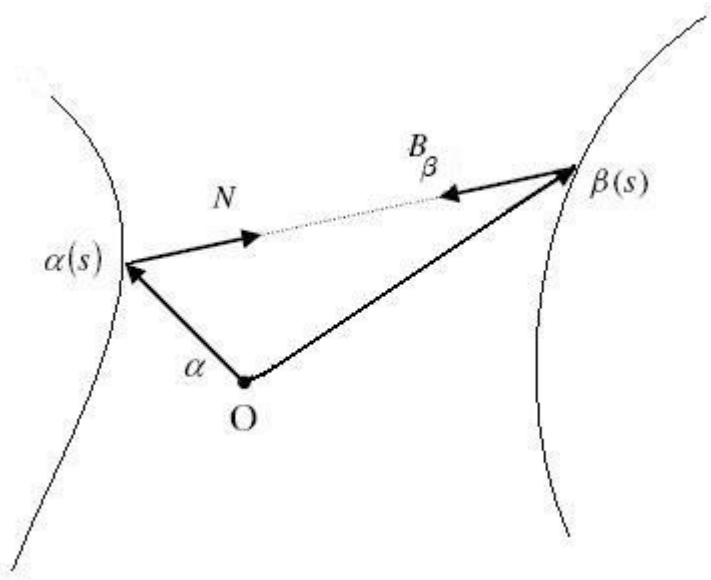

FIGURE 1. Mannheim Partner Curves

Theorem 3. *Let $\alpha : I \subset \mathbb{R} \to G$ and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim curve couple with arc-length parameter s and $\overline{s}$, respectively. Then corresponding points are a fixed distance apart for all $s \in I$, that is,*

$$d(\alpha(s), \beta(s)) = \text{constant}, \quad \text{for all } s \in I. \tag{3.1}$$

Proof. From Definition 5, we can simply write

$$\beta(s) = \alpha(s) + \lambda(s) N(s) \tag{3.2}$$

Differentiating Eq. (3.2) with respect to s and using Eq. (2.3), we get

$$\begin{aligned} \frac{d\beta(\overline{s})}{d\overline{s}} \frac{d\overline{s}}{ds} &= \frac{d\alpha(s)}{ds} + \lambda'(s) N(s) + \lambda(s) \dot{N}(s) \\ &= T(s) + \lambda'(s) N(s) + \lambda(s) \left[D_T N - \frac{1}{2}[T, N] \right] \end{aligned}$$

and with the help of Proposition 1 and Frenet equations, we obtain

$$\frac{d\beta(\overline{s})}{d\overline{s}} \frac{d\overline{s}}{ds} = (1 - \lambda(s)\kappa(s)) T(s) + \lambda'(s) N(s) + \lambda(s) ((\tau - \tau_G)(s)) B(s)$$

hence

$$T_\beta(\overline{s}) = \frac{ds}{d\overline{s}} \left[(1 - \lambda(s)\kappa(s)) T(s) + \lambda'(s) N(s) + \lambda(s) (\tau - \tau_G)(s) B(s) \right]. \tag{3.3}$$

And then, we know that $\{N(s), B_\beta(\overline{s})\}$ is a linearly dependent set, so we have

$$\langle T_\beta(\overline{s}), B_\beta(\overline{s})\rangle = \frac{ds}{d\overline{s}}\left[\begin{array}{c}(1-\lambda(s)\kappa(s))\langle T(s), B_\beta(\overline{s})\rangle + \lambda'(s)\langle N(s), B_\beta(\overline{s})\rangle \\ +\lambda(s)(\tau-\tau_G)(s)\langle B(s), B_\beta((\overline{s}))\rangle\end{array}\right]$$

hence

$$\lambda'(s) = 0$$

that is, $\lambda(s)$ is constant function on I. This completes the proof. □

Theorem 4. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s and the Frenet apparatus (T, N, B, κ, τ). Then, α is Mannheim curve if and only if*

$$\lambda\kappa\left(1+H^2\right) = 1, \text{ for all } s \in I \tag{3.4}$$

where λ is constant and H is the harmonic curvature function of the curve α.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Mannheim curve with arc length parameter s then we can write

$$\beta(s) = \alpha(s) + \lambda N(s)$$

Differentiating the above equality with respect to s and by using the Frenet equations, we get

$$\frac{d\beta(s)}{ds} = (1-\lambda\kappa(s))T(s) + \lambda(\tau-\tau_G)(s)B(s).$$

On the other hand, we have

$$T_\beta(\overline{s}) = \frac{d\beta}{ds}\frac{ds}{d\overline{s}} = [(1-\lambda\kappa(s))T(s) + \lambda(\tau-\tau_G)(s)B(s)]\frac{ds}{d\overline{s}}.$$

By taking the derivative of this equation with respect to $\overline{s}$ and using the Frenet equations we obtain

$$\begin{aligned}\frac{dT_\beta}{d\overline{s}} &= \left[-\lambda\frac{d\kappa}{ds}T(s) + \left(\kappa-\lambda\kappa^2-\lambda(\tau-\tau_G)^2\right)N(s) + \lambda(\tau-\tau_G)'B(s)\right]\left(\frac{ds}{d\overline{s}}\right)^2 \\ &\quad + [(1-\lambda\kappa(s))T(s) + \lambda(\tau-\tau_G)(s)B(s)]\frac{d^2s}{d\overline{s}^2}\end{aligned}$$

From this equation we get

$$\left(\kappa-\lambda\kappa^2-\lambda(\tau-\tau_G)^2\right) = 0,$$

$$\lambda\kappa\left(1+H^2\right) = 1.$$

Conversely, if $\lambda\kappa\left(1+H^2\right) = 1$ then we can easily see that α is a Mannheim curve.

This completes the proof. □

Corollary 1. *If G is Abelian Lie group then $\tau_G = 0$. And so, if $\alpha : I \subset \mathbb{R} \to G$ is a parametrized Mannheim curve with arc length parameter s and the Frenet apparatus (T, N, B, κ, τ) in Abelian Lie group G. Then, α satisfy the following equality*

$$\lambda\left(\kappa^2 + \tau^2\right) = \kappa$$

Proof. If G is Abelian Lie group then using the $\tau_G = 0$ and Theorem 4 we have the result. □

So, the above Corollary shows that the study is a generalization of Mannheim curves defined by Liu and Wang [12] in Euclidean 3-space.

Theorem 5. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s. Then β is the Mannheim partner curve of α if and only if the curvature κ_β and the torsion τ_β of β satisfy the following equation*

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}} = \frac{\kappa_\beta}{\mu}(1 + \mu^2 \kappa_\beta^2 H_\beta^2)$$

where μ is constant and H_β is the harmonic curvature function of the curve β.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Mannheim curve with arc length parameter s then we can write

$$\alpha(\bar{s}) = \beta(\bar{s}) + \mu(\bar{s}) B_\beta(\bar{s})$$

for some function $\mu(\bar{s})$. By taking the derivative of this equation with respect to $\bar{s}$ and using the Frenet equations we obtain

$$T\frac{ds}{d\bar{s}} = T_\beta(\bar{s}) + \mu'(\bar{s}) B_\beta(\bar{s}) - \mu(\bar{s})\left(\tau_\beta - \tau_{G_\beta}\right)(\bar{s}) N_\beta(\bar{s})$$

hence

$$T\frac{ds}{d\bar{s}} = T_\beta(\bar{s}) + \frac{d\mu(\bar{s})}{d\bar{s}} B_\beta(\bar{s}) - \mu(\bar{s}) \kappa_\beta H_\beta N_\beta(\bar{s})$$

where H_β is the harmonic curvature function of the curve β. And then, we know that $\{N(s), B_\beta(\bar{s})\}$ is a linearly dependent set, so we have

$$\frac{d\mu(\bar{s})}{d\bar{s}} = 0.$$

This means that $\mu(\bar{s})$ is a constant function. Thus we have

$$T\frac{ds}{d\bar{s}} = T_\beta - \mu(\bar{s}) \kappa_\beta H_\beta N_\beta(\bar{s}). \tag{3.5}$$

On the other hand, we have

$$T = T_\beta \cos\theta + N_\beta \sin\theta \tag{3.6}$$

where θ is the angle between T and T_β at the corresponding points of the curves α and β. By taking the derivative of this equation with respect to $\bar{s}$ and using the Frenet equations we obtain

$$\kappa N\frac{ds}{d\bar{s}}=-\left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\sin\theta T_\beta+\left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\cos\theta N_\beta+\kappa_\beta H_\beta\sin\theta B_\beta.$$

From this equation and since $\{N(s), B_\beta(\bar{s})\}$ is a linearly dependent set, we get

$$\begin{cases}\left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\sin\theta=0\\ \left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\cos\theta=0.\end{cases}$$

For this reason we have

$$\frac{d\theta}{d\bar{s}}=-\kappa_\beta. \tag{3.7}$$

From Eq. (3.5), Eq. (3.6) and notice that T_β is orthogonal to B_β, we find that

$$\frac{ds}{d\bar{s}}=\frac{1}{\cos\theta}=-\frac{\mu\kappa_\beta H_\beta}{\sin\theta}.$$

Then we have

$$\mu\kappa_\beta H_\beta=-\tan\theta.$$

By taking the derivative of this equation and applying Eq. (3.7), we get

$$\mu\frac{d\kappa_\beta H_\beta}{d\bar{s}}=\kappa_\beta\left(1+\mu^2\kappa_\beta^2H_\beta^2\right)$$

that is

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}}=\frac{\kappa_\beta}{\mu}\left(1+\mu^2\kappa_\beta^2H_\beta^2\right).$$

Conversely, if the curvature κ_β and torsion τ_β of the curve β in three dimensional Lie group G satisfy

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}}=\frac{\kappa_\beta}{\mu}\left(1+\mu^2\kappa_\beta^2H_\beta^2\right)$$

for constant $\mu(\bar{s})$, then we define a curve by

$$\alpha(\bar{s})=\beta(\bar{s})+\mu B_\beta(\bar{s}) \tag{3.8}$$

and we will show that $\{\alpha,\beta\}$ is Mannheim curve couple in three dimensional Lie group G. By taking the derivative of Eq. (3.8) with respect to $\bar{s}$ twice, we get

$$T\frac{ds}{d\bar{s}}=T_\beta-\mu\kappa_\beta H_\beta N_\beta, \tag{3.9}$$

$$\kappa N\left(\frac{ds}{d\bar{s}}\right)^2+T\frac{d^2s}{d\bar{s}^2}=\mu\kappa_\beta^2H_\beta T_\beta+\left(\kappa_\beta-\mu\frac{d\kappa_\beta H_\beta}{d\bar{s}}\right)N_\beta-\mu\kappa_\beta^2H_\beta^2B_\beta, \tag{3.10}$$

respectively. Taking the cross product of Eq. (3.9) with Eq. (3.10) and noticing that

$$\kappa_\beta - \mu \frac{d\kappa_\beta H_\beta}{d\bar{s}} + \mu^2 \kappa_\beta^3 H_\beta^2$$

we have

$$\kappa B \left(\frac{ds}{d\bar{s}}\right)^3 = \mu^2 \kappa_\beta^3 H_\beta^3 T_\beta + \mu \kappa_\beta^2 H_\beta^2 N_\beta. \tag{3.11}$$

By taking the cross product of Eq. (3.9) with Eq. (3.11), we get

$$\kappa N \left(\frac{ds}{d\bar{s}}\right)^4 = -\mu \kappa_\beta^2 H_\beta^2 \left(1 + \mu^2 \kappa_\beta^2 H_\beta^2\right) B_\beta.$$

This means that the principal normal vector field of the curve α and binormal vector field of the curve β are linearly dependent set. And so α is a Mannheim curve and β is Mannheim partner curve of the curve α in three dimensional Lie group G. □

Proposition 2. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve with the Frenet vector fields $\{T, N, B\}$ and $\beta : \bar{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α with the Frenet vector fields $\{T_\beta, N_\beta, B_\beta\}$. Then $\tau_G = \tau_{G_\beta}$ for the curves α and β where $\tau_G = \frac{1}{2}\langle [T, N], B\rangle$ and $\tau_{G_\beta} = \frac{1}{2}\langle [T_\beta, N_\beta], B_\beta\rangle$.*

Proof. Let (α, β) be a Mannheim curve couple. From Eq. (3.3) we have

$$T_\beta(\bar{s}) = [(1 - \lambda\kappa(s))T(s) + \lambda(\tau - \tau_G)(s)B(s)]\frac{ds}{d\bar{s}}.$$

We take the norm of this equation and by using Eq. (3.4), we obtain

$$\frac{d\bar{s}}{ds} = \lambda\kappa H\sqrt{1 + H^2}.$$

If we consider with together the last two equations, we get

$$T_\beta(\bar{s}) = \frac{H}{\sqrt{1+H^2}}T(s) + \frac{1}{\sqrt{1+H^2}}B(s). \tag{3.12}$$

Since (α, β) is a Mannheim curve couple we know $B_\beta(\bar{s}) = N(s)$. Then,

$$N_\beta(\bar{s}) = B_\beta(\bar{s}) \times T_\beta(\bar{s})$$

$$N_\beta(\bar{s}) = \frac{1}{\sqrt{1+H^2}}T(s) - \frac{H}{\sqrt{1+H^2}}B(s).$$

We know from Definition 2 $\langle [T_\beta, N_\beta], B_\beta\rangle = 2\tau_{G_\beta}$ for the curve β. Then with the help of above equations for $T_\beta(\bar{s})$, $N_\beta(\bar{s})$ and $B_\beta(\bar{s})$, we obtain

$$\left\langle \left[\frac{H}{\sqrt{1+H^2}}T + \frac{1}{\sqrt{1+H^2}}B, \frac{1}{\sqrt{1+H^2}}T(s) - \frac{H}{\sqrt{1+H^2}}B(s)\right], N\right\rangle = 2\tau_{G_\beta},$$

$$\frac{H^2}{1+H^2}\langle [T, N], B\rangle + \frac{1}{1+H^2}\langle [T, N], B\rangle = 2\tau_{G_\beta}.$$

Using the last equation of Proposition 1, we get

$$\tau_G = \tau_{G_\beta}.$$

This completes the proof. □

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Mannheim curve with curvature functions κ, τ and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α with curvatures functions κ_β, τ_β. Then the relations between these curvature functions are*

$$\kappa_\beta(\overline{s}) = \frac{H'(s)}{\lambda\kappa(s)H(s)\left(1+H^2(s)\right)^{3/2}},$$
$$\tau_\beta(\overline{s}) = \frac{1}{\lambda H(s)} + \tau_{G_\beta}.$$

Proof. If we differentiate Eq. (3.12) and using the Frenet formulas, we have

$$\kappa_\beta N_\beta \lambda\kappa H\sqrt{1+H^2} = \frac{H'}{\left(1+H^2\right)^{3/2}}(T - HB).$$

If we take the norm of the last equation, we get

$$\kappa_\beta = \frac{H'}{\lambda\kappa H\left(1+H^2\right)^{3/2}}.$$

Since $\{\alpha, \beta\}$ is a Mannheim curve couple, we know $B_\beta = N$. If we differentiate this equation and using the Frenet formulas, we have

$$-\left(\tau_\beta - \tau_{G_\beta}\right) N_\beta \lambda H\sqrt{1+H^2} = -T + HB.$$

If we take the norm of the last equation, we get

$$\tau_\beta = \frac{1}{\lambda H} + \tau_{G_\beta}.$$

This completes the proof. □

Theorem 7. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α. The Mannheim curve α is a slant helix if and only if its Mannheim mate β is a general helix.*

Proof. If Mannheim curve α is a slant helix, then using Theorem 2 we have, σ_N is a constant function. From Theorem 6 for the curve β, we have

$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \frac{\frac{1}{\lambda H}}{\frac{H'}{\lambda\kappa H\left(1+H^2\right)^{3/2}}}$$
$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \frac{\kappa\left(1+H^2\right)^{3/2}}{H'}$$

$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \sigma_N = \text{constant}.$$

So, the curve β which is Mannheim mate of α is a general helix.

Conversely, we assume that β, which is Mannheim mate of α, is a general helix. So we have

$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \text{constant}.$$

From the last equation and Theorem 6 we can easily see that σ_N is a constant function. This completes the proof. □

Theorem 8. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α. If α is a slant helix then the harmonic curvature function of the curve α: $H(s)$ is*

$$H(s) = \frac{1}{2}\left(ae^{bs} - \frac{1}{a}e^{-bs}\right) \tag{3.13}$$

for some nonzero constant a and b and s is the arc length parameter of the curve α. If we consider $a = b = 1$, we have the harmonic curvature funtion of the curve α is hyperbolic sine function in arc length s, that is, $H(s) = \sinh s$.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ in three dimensional Lie group G. Since the curve α is a slant helix, we have

$$\langle N, X\rangle = \cos\theta,\ \theta \neq \frac{\pi}{2} \tag{3.14}$$

for left invariant vector field X. Differentiating Eq. (3.14) twice, we have

$$-\kappa\langle T, X\rangle + (\tau - \tau_G)\langle B, X\rangle = 0 \tag{3.15}$$

and

$$-\kappa'\langle T, X\rangle + (\tau - \tau_G)'\langle B, X\rangle = \left\{\kappa^2 + (\tau - \tau_G)^2\right\}\langle N, X\rangle.$$

Since α is a Mannheim curve using Theorem 4, we rewrite the last equation

$$-\kappa'\langle T, X\rangle + (\tau - \tau_G)'\langle B, X\rangle = \frac{\kappa}{\lambda}\cos\theta \tag{3.16}$$

where λ is a non-zero constant. By a direct calculation using Eq. (3.15) and Eq. (3.16), we obtain

$$\langle T, X\rangle = \frac{H}{\lambda H'}\cos\theta \tag{3.17}$$

and

$$\langle B, X\rangle = \frac{1}{\lambda H'}\cos\theta. \tag{3.18}$$

Differentiating Eq. (3.17) and Eq. (3.18), we have

$$\kappa = \frac{1}{\lambda}\left(1 - \frac{HH''}{(H')^2}\right),$$

$$\tau - \tau_G = \frac{H''}{\lambda\left(H'\right)^2},$$

respectively. These equations give that

$$H = \frac{\tau - \tau_G}{\kappa} = \frac{H''}{\left(H'\right)^2 - HH''}.$$

Then we have the following differential equation

$$\left(1 + H^2\right) H'' - \left(H'\right)^2 = 0.$$

Solving the last equation, we obtain Eq. (3.13). This completes the proof. □

Theorem 9. *Let (α, β) be a Mannheim curve couple in three dimensional Lie group G with bi-invariant metric. Then α is general helix if and only if β is a geodesic.*

Proof. If Mannheim curve α is a general helix, then its harmonic curvature H is constant function. And so from Theorem 6,

$$\kappa_\beta = 0.$$

So, the curve β is a geodesic.

Conversely we assume that β is a geodesic curve. From Theorem 6 we can easily see that

$$H'(s) = 0$$

and so

$$H(s) = \text{constant}.$$

This completes the proof. □

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