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CHARACTERIZATIONS OF ATOMISTIC COMPLETE FINITE LATTICES RELATIVE TO GEOMETRIC ONES

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Abstract. S. Radeleczki in 2002 raised the open problem of the characterization of those atomistic complete lattices whose classification lattices are geometric. This paper solves the problem for the finite cases.

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1. Introduction

S.Radeleczki et al. pointed out [1,2,6,7] that the notion of the classification system can be applied in concept lattices. The dual of this notion is introduced by R.Wille [3, 4, 8]. In addition, S.Radeleczki et al. [1, 2, 6, 7] apply and study the properties of concept lattices in the process of construction of classification systems. Since any partition lattice is a particular geometric lattice, an open problem was arisen naturally (S.Radeleczki [7]): Characterize those atomistic complete lattices whose classification lattices are geometric. The solution of this open problem might be useful in the study of classification systems and concept lattices. Hence, this paper will characterize atomistic complete finite lattices whose classification lattices are geometric and answer the open problem for the finite case.

This paper is organized as follows. Section 2 presents some basic information relative to geometric lattices and classification systems. In Section 3, we describe certain properties on atomistic complete finite lattices related to geometric lattices. Afterwards, it answers the open problem suggested by S.Radeleczki [7] for the finite case.

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2. PRELIMINARIES

Some basic notions and results related to posets, lattices and classification systems are presented in this section.

2.1. *Posets and lattices*

We review some basic properties and notations of posets and lattices in this subsection. For more detail about posets and lattices, please refer to [5].

Lemma 1. *(1)*(in [5] p.232) *A lattice L is semimodular if and only if $x \wedge y \prec x$ implies that $y \prec x \vee y$.*
(2)(in [5] p.234) *A lattice L is called geometric if and only if L is complete, atomistic (that is, every element of L is a join of atoms), all atoms are compact, and L is semimodular.*
(3)(in [5] p.234) *Any interval $[y,x] = \{a \in L \mid y \leq a \leq x\}$ of a geometric lattice L is again a geometric lattice.*

We also need the following statements from [5].
(1) A finite lattice L is semimodular if for all $x, y \in L$: the property which x and y cover $x \wedge y$ implies that $x \vee y$ covers x and y.
(2) A finite lattice is geometric if it is semimodular and every element is a join of atoms.
(3) A finite semimodular lattice is characterized by the following property L is semimodular if and only if for all $x, y \in L$, all maximal chains between elements x, y have the same length, and the height function h of L satisfies $h(x) + h(y) \geq h(x \wedge y) + h(x \vee y)$.

Some notations 1. Let $(P, \leq)$ be a poset. In this paper, if there is no confusion in the text, then we use the notation P in stead of $(P, \leq)$. If P has the greatest element 1, then the height $h(1)$ is sometimes denoted by $h(P)$. In P, if y covers x, it is in notation $x \prec y$; if y does not cover x, it is in notation $x \nprec y$; if $y \leq x$ and $y \neq x$, it is in notation $y < x$; if y is not less than x, it is denoted by $y \nleq x$; if x and y are incomparable, it is denoted by $x || y$.

2.2. *Classification systems*

Let us recall some information of classification systems from [7]. For more detail for classification systems, please see [7].

Definition 1. (1) A nonzero element p of a lattice L is called *completely join-irreducible* if for any system of elements $x_i \in L\,(i \in I)$, the equality $p = \vee\{x_i : i \in I\}$ implies $p = x_{i_0}$ for some $i_0 \in I$. If any nonzero element of L is a join of completely join-irreducible elements, then L is called a *CJ-generated lattice*.

(2) Let L be a complete lattice. A set $S = \{a_j \mid j \in J\}$ ($J \neq \varnothing$) of nonzero elements of L is called a *classification system* of L if the following conditions are satisfied:

(2.1) $a_i \wedge a_j = 0$, for all $i \neq j$, where 0 is the least element in L.

(2.2) $x = \vee_{j \in J}(x \wedge a_j)$, for all $x \in L$.

Some notations 2. (1) Let L be a CJ-generated complete lattice. The set of all completely join-irreducible elements of L is denoted by $J(L)$. For $a \in L$, let $J(a) = \{p \in J(L) \mid p \leq a\}$ and set $\vee\varnothing = 0$.

(2) Let $A(L)$ denote the set of atoms of a lattice L.

Lemma 2. *(1) If $S = \{a_i \mid i \in I\}$ is a classification system of L, then $\pi_S = \{J(a_i), i \in I\}$ is a partition induced by S on $J(L)$.*

(2) If L is a CJ-generated lattice, then any classification system $S = \{a_i \mid i \in I\}$ of L is determined by the partition π_S, induced by S on $J(L)$.

(3) Any atomistic lattice L is a CJ-generated lattice with $J(L) = A(L)$.

Let L be an atomistic complete lattice. For any $x \in L \setminus 0$, the set $S_x = \{x\} \cup \{a \in A(L) \mid a \wedge x = 0\}$ is a classification system of L.

(4) Let L be a complete lattice and 1 be the greatest element in L. Then $S = \{1\}$ is a classification system.

Definition 2. Let L be a CJ-generated complete lattice and let S_p and S_q be two classification systems of L. We say that the system S_p is *finer* than S_q and we write $S_p \leq S_q$, if the partition π_{S_p} induced by S_p refines the partition π_{S_q} induced by S_q, that is, if $\pi_{S_p} \leq \pi_{S_q}$.

Lemma 3. *(1) Let $Cl_s(L)$ denote the set of all classification systems of a CJ-generated complete lattice L. Then, $(Cl_s(L), \leq)$ is a complete lattice.*

$(Cl_s(L), \leq)$ or $Cl_s(L)$ for short, is called the classification lattice of the lattice L.

(2) The least element of the lattice $Cl_s(L)$, that is, the finest classification system of L, is the same as $S_0 = \wedge\{S \mid S \in Cl_s(L)\}$.

Remark 1. Let L be an atomistic complete lattice.

(1) In this paper, S_0 stands for $\{a \mid a \in A(L)\}$, S_1 is $\{1\}$, and S_x represents $\{x\} \cup \{a \in A(L) \mid a \nleq x\}$ for any $x \in L \setminus 0$.

(2) For any $x \in L \setminus 0$, using the (3) in Lemma 1, we may easily know that $S_x = \{x\} \cup \{a \in A(L) \mid a \wedge x = 0\}$ is a classification system of L. In addition, in light of the (2) in Lemma 3, $S_0 = \{a \mid a \in A(L)\} \in Cl_s(L)$ holds, and $S_a = S_0$ is true for any $a \in A(L)$.

(3) Let $S \in Cl_s(L)$. Then according to the (2) of Definition 1, there is $S = S_x$ for some $x \in L \setminus 0$, or $S = \{x_i \in L \setminus (A(L) \cup 0) \mid i \in I\} \cup \{a \in A(L) \mid a \nleq x_i, i \in I\}$ where $x_p \wedge x_q = 0, (p \neq q; p, q \in I)$ and $|I| \geq 2$.

For convenient, if $S=\{x_i\in L\setminus(A(L)\cup 0)\mid i\in I\}\cup\{a\in A(L)\mid a\not\leq x_i, i\in I\}\in Cl_s(L)$, then it is denoted by $S_{x_i,i\in I}$. And further, when $I=\{1,2,\dots,n\}$, $S_{x_i,i\in I}$ is sometimes denoted as $S_{x_1x_2\dots x_n}$.

(4) Let $S=S_{x_1x_2\dots x_n}$. By Definition 1 and the (3) above, it is easily to find $S=S_{x_{i_1}x_{i_2}\dots x_{i_n}}$ where $i_1,i_2,\dots,i_n$ is an arbitrary permutation of $1,2,\dots,n$.

(5) Let $x\in L\setminus(A(L)\cup 0)$. By the discussion beyond, we easily obtain $Cl_s([0,x])=[S_0,S_x]$ in which $[S_0,S_x]$ is an interval in $Cl_s(L)$ and $[0,x]$ is an interval in L.

Some notations 3. Let L be an atomistic complete finite lattice.

(1) Let $\mathcal{F}^k$ denote $\{x\in L\mid x$ has height k in $L\}$, $(k=2,\dots,h(L))$.

(2) For any $x\in L$ and $l\leq h(x)$, let $\mathcal{F}^l(x)=\{y\in L\mid y\in\mathcal{F}^l$ and $y\leq x\}$ and $\mathcal{C}(x)=\{y\in L\mid x$ covers y in $L\}$. $\mathcal{F}^1$ is also in notation $A(x)=\{a\in A(L)\mid a\leq x\}$ for any $x\in L$.

3. Answer to the open problem

In this section, we will discover the characterizations of atomistic complete finite lattices whose classification lattices are geometric lattices. Therefore, the open problem, which is repeated in Section 1, will be solved for finite cases.

Theorem 1. *Let L be an atomistic complete finite lattice. If $|\mathcal{F}^2|=1$, i.e. L has only one element of height 2. Then $Cl_s(L)$ is geometric.*

Proof of Theorem 1. It is easily to find $Cl_s(L)=\{S_0,S_1\}$. Thus, $Cl_s(L)$ is geometric. □

In what follows, L always stands for an atomistic complete finite lattice with at least two elements of height 2. That is to say, $\mathcal{F}^2=\{x\in L\mid x$ has height 2 in $L\}$ satisfies $|\mathcal{F}^2|\geq 2$.

First, we may deal with some properties related to $Cl_s(L)$.

Lemma 4. *$Cl_s(L)$ possesses the following properties.*

(3.1) Let $x,y\in L\setminus 0$. Then $x\leq y\Leftrightarrow S_x\leq S_y$. Further, $x<y\Leftrightarrow S_x<S_y$.

(3.2) Let $x,y\in L\setminus(A(L)\cup 0)$. If $x\wedge y=0$, then $S_x\vee S_y=S_{xy}$.

Furthermore, let $x_j\in L\setminus(A(L)\cup 0),(j\in\mathcal{J};|\mathcal{J}|\geq 2)$. If $x_i\wedge x_j=0,(i,j\in\mathcal{J};i\neq j)$, then $\vee_{j\in\mathcal{J}}S_{x_j}=S_{x_j,j\in\mathcal{J}}$.

(3.3) Let $x,y\in L\setminus(A(L)\cup 0)$. If $x\wedge y\neq 0$, then $S_x\vee S_y=S_{x\vee y}$.

(3.4) $A(Cl_s(L))=\{S_d\mid d\in\mathcal{F}^2\}$.

(3.5) Let $S=S_{x_i,i\in I}$ with $2\leq|I|$. Let $S'\in Cl_s(L)$. Then the following items (3.5.1), (3.5.2) and (3.5.3) are true.

(3.5.1) If $S'=S_y\prec S$ holds for some $y\in L\setminus(A(L)\cup 0)$, then there are $I=\{1,2\}$, $y=x_1$ and $h(x_2)=2$.

(3.5.2) If $S'\neq S_y$ for any $y\in L\setminus(A(L)\cup 0)$, $S'\prec S$, and $h(x_n)=2$ with $x_n\notin S'$

in which $n \in I \setminus I_1$ for a $I_1 \subseteq I$ with $|I| = |I_1| + 1$. Then $S' = S_{x_i, i \in I_1}$.
(3.5.3) Let $S' = S_{x_i, i \in I_1}$ where $I_1 \subseteq I$ with $|I| = |I_1| + 1$. If S' satisfies $h(x_n) = 2$ where $n \in I \setminus I_1$. Then $S' \prec S$ holds.

Proof of Lemma 4. We will demonstrate all the results step by step.

Step 1. From Definition 2 with the atomistic property of L, we may easily obtain item (3.1).

Step 2. We verify item (3.2) using steps 2.1 and 2.2 as follows.

Step 2.1. To prove: for $x, y \in L \setminus (A(L) \cup 0)$,

$$x \wedge y = 0 \Rightarrow S_x \vee S_y = S_{xy}.$$

If $x \wedge y = 0$, then $S_{xy} \in Cl_s(L)$ and $S_x, S_y < S_{xy}$ according to Definition 2.

Let $S \in Cl_s(L)$ satisfy $S_x, S_y < S$. Using $x \wedge y = 0$ and Definition 2, we may find out $A(x) \subseteq A(x_s)$ and $A(y) \subseteq A(y_s)$ for some $x_s, y_s \in S$. This causes $S_{xy} \leq S$. Considering $S_x, S_y < S$ with $S_{xy} \leq S$, we may be assured $S_x \vee S_y = S_{xy}$.

Step 2.2. To prove: for $x_j \in L \setminus (A(L) \cup 0), (j \in \mathcal{J}; |\mathcal{J}| \geq 2)$,

$$x_i \wedge x_j = 0, (i, j \in \mathcal{J}; i \neq j) \Rightarrow \vee_{j \in \mathcal{J}} S_{x_j} = S_{x_j, j \in \mathcal{J}}.$$

Using induction on $|\mathcal{J}|$ and item (3.1), we may obtain $\vee_{j \in \mathcal{J}} S_{x_j} \leq S_{x_j, j \in \mathcal{J}}$ if $x_i \wedge x_j = 0, (i, j \in \mathcal{J}; i \neq j)$.

By the (3) in Remark 1, $\vee_{j \in \mathcal{J}} S_{x_j} = S_y$ holds for some $y \in L \setminus (A(L) \cup 0)$, or $\vee_{j \in \mathcal{J}} S_{x_j} = S_{z_1 \ldots z_m}$ holds for some $z_1, \ldots, z_m \in L \setminus (A(L) \cup 0)$ with $2 \leq m$.

Suppose that $\vee_{j \in \mathcal{J}} S_{x_j} = S_y$ holds. By item (3.1) and $S_{x_j} \leq \vee_{j \in \mathcal{J}} S_{x_j}, (j \in \mathcal{J})$, we may determine $x_j \leq y$ $(j \in \mathcal{J})$. Furthermore, $S_{x_j, j \in \mathcal{J}} \leq S_y$ is true by Definition 2. So, it follows $S_y = S_{x_j, j \in \mathcal{J}}$ with $|\mathcal{J}| \geq 2$, which is a contradiction to the (2) of Definition 1.

Suppose that $\vee_{j \in \mathcal{J}} S_{x_j} = S_{z_1 \ldots z_m}$ holds for $2 \leq m$. Using item (3.1) and $S_{x_j} \leq S_{z_1 \ldots z_m}$ $(j \in \mathcal{J})$, we point out $x_j \leq z_{j_t}, (j \in \mathcal{J}; j_t \in \{1, \ldots, m\})$. Utilizing $z_{j_t} \wedge z_{i_s} = 0\,(j_t \neq i_s; j_t, i_s = 1, \ldots, m)$ and the result in the above case, we follow that if $z_{j_t} \neq z_{i_s}$, then $x_j \neq x_i, (i, j \in \mathcal{J})$, where $x_j \leq z_{j_t}$ and $x_i \leq z_{i_s}$. This implies $A(x_j) \subseteq A(z_{j_t}), (j \in \mathcal{J}; j_t \in \{1, \ldots, m\})$ since L is atomistic. Combining items (1) and (2) in Lemma 2 with Definition 2, $\pi_{S_{x_j, j \in \mathcal{J}}} \leq \pi_{S_{z_1 \ldots z_m}}$ holds. Hence, $S_{x_j, j \in \mathcal{J}} \leq S_{z_1 \ldots z_m}$ is true. Additionally, we decide $S_{x_j} \leq S_{x_j, j \in \mathcal{J}}$ according to $A(x_j) \subseteq A(x_j), (j \in \mathcal{J})$, the definitions of $S_{x_j, j \in \mathcal{J}}$ and S_{x_j}, and Definition 2. Thus, it follows $\vee_{j \in \mathcal{J}} S_{x_j} \leq S_{x_j, j \in \mathcal{J}} \leq S_{z_1 \ldots z_m}$. Therefore, we may be assured $S_{x_j, j \in \mathcal{J}} = \vee_{j \in \mathcal{J}} S_{x_j}$.

Step 3. To prove item (3.3).

Let $x \wedge y \neq 0$. Then $\{x, y\} \nsubseteq S$ is true for any $S \in Cl_s(L)$. If $S_x \vee S_y = S_{z_1 z_2 \ldots z_n}$ for any $z_j \in L \setminus (A(L) \cup 0), (j = 1, 2, \ldots, n; 2 \leq n)$. Then, the properties of $x \leq z_1, y \leq z_2$ and $z_1 \neq z_2$ taken together follows $x \wedge y = 0$. This is a contradiction to $x \wedge y \neq 0$. Hence, we confirm $x, y \leq z_1$. However, $S_x, S_y \leq S_{z_1} < S_{z_1 \ldots z_n} = S_x \vee S_y$ follows $S_x \vee S_y \leq S_{z_1} < S_{z_1 \ldots z_n} = S_x \vee S_y$, a contradiction. Thus, $S_x \vee S_y = S_z$ is true for some $z \in L \setminus (A(L) \cup 0)$. Then, we attain $S_x, S_y \leq S_x \vee S_y$. Furthermore, we obtain $x, y \leq z$ using item (3.1). Moreover, $x \vee y \leq z$ holds. So, we determine $S_{x \vee y} \leq S_z$.

On the other hand, in view of $x, y \leq x \vee y$ and item (3.1), we obtain $S_x, S_y \leq S_{x \vee y}$. Furthermore, we attain $S_x \vee S_y \leq S_{x \vee y}$.

Therefore, we have demonstrated $S_x \vee S_y = S_{x \vee y}$.

Step 4. To prove item (3.4).

Let $d \in \mathcal{F}^2$. According to Remark 1, we obtain $S_d \in Cl_s(L)$. By item (3.1), it has $S_0 < S_d$. If $S_0 \prec S \leq S_d$ holds for some $S \in Cl_s(L)$. Then we obtain $x \in S$ and $x \in L \setminus (A(L) \cup 0)$. So, $2 \leq h(x)$ holds. Furthermore, we find the existence of $d_x \in \mathcal{F}^2(x)$. Thus, we may obtain $S_0 < S_{d_x} \leq S \leq S_d$. Combining $S_0 \prec S$ and $S_0 < S_{d_x} \leq S$, we find $S = S_{d_x}$. On the other hand, taking $S \leq S_d$, $S_{d_x} \leq S$ with $h(d_x) = h(d) = 2$ together, we follow $d_x = d$. Therefore, we attain $S_0 \prec S_d$. Hence, $\{S_d \mid d \in \mathcal{F}^2\} \subseteq A(Cl_s(L))$ is true.

If $A(Cl_s(L)) \setminus \{S_d \mid d \in \mathcal{F}^2\} \neq \varnothing$. Then, there is $S \in A(Cl_s(L)) \setminus \{S_d \mid d \in \mathcal{F}^2\}$. This causes $y \in S \setminus (A(L) \cup 0)$. So, $2 \leq h(y)$ is real. Considered the definition of height function of a lattice in [5], we find $d_y \in \mathcal{F}^2$ with $d_y \leq y$. Moreover, we obtain $S_{d_y} \leq S$. However, $S_0 \prec S_{d_y}$ is known according to the above proof. Hence, it follows $S = S_{d_y}$. This is a contradiction to $S \notin \{S_d \mid d \in \mathcal{F}^2\}$. In other words, $A(Cl_s(L)) = \{S_d \mid d \in \mathcal{F}^2\}$ is true.

Step 5. To prove item (3.5).

Combining $S \in Cl_s(L)$ with Definition 1, we confirm $\{x_i \mid i \in T\} \cup \{a \in A(L) \mid a \nleq x_i, i \in T\} \in Cl_s(L)$ for any $T \subseteq I$.

Step 5.1. This step verifies items (3.5.1) and (3.5.2).

If $S' \prec S$. Then, by the (3) in Remark 1, we may find $S' = S_y$ for some $y \in L \setminus (A(L) \cup 0)$ or $S' = S_{z_1 z_2 \ldots z_m}$ with $2 \leq m$ and $z_i \in L \setminus (A(L) \cup 0), (i = 1, \ldots, m)$.

Suppose $S' = S_y$ for some $y \in L \setminus (A(L) \cup 0)$. Then, $A(y) \subseteq A(x_{i_0})$ holds for some $i_0 \in I$. According to $x_i \wedge x_j = 0, (i \neq j; i, j \in I)$, we confirm that there is one and only one $i_0 \in I$ satisfying $A(y) \subseteq A(x_{i_0})$. In addition, owing to the atomistic property of $Cl_s(L)$, we may find $y \leq x_{i_0}$. And further, we obtain $S_y \leq S_{x_{i_0}}$ by item (3.1).

Under the assumption $S' \prec S$ and the closest result above, we may infer to $|I| = 2$. Otherwise, $3 \leq |I|$ will follow $S' = S_y \leq S_{x_{i_0}} < S_{x_{i_0} x_{i_1}} < S_{x_i, i \in I} = S$, a contradiction to $S' \prec S$.

We may assume $I = \{1, 2\}$ and $x_{i_0} = x_1$. In view of $S' = S_y \leq S_{x_1} < S$, $S_{x_1} \in Cl_s(L)$ and $S' \prec S$, we find $S_y = S_{x_1}$. Thus, $y = x_1$ holds. Since $x_2 \in L \setminus (A(L) \cup 0)$ causes $h(x_2) \geq 2$. Assume $h(x_2) > 2$. By the definition of height function in a lattice, we obtain $t < x_2$ and $h(t) = 2$. Additionally, $0 \leq t \wedge x_1 \leq x_2 \wedge x_1 = 0$ yields $t \wedge x_1 = 0$. So, $S_{x_1 t} \in Cl_s(L)$ holds. We may easily determine $S_{x_1} < S_{x_1 t} < S$, a contradiction to $S' = S_{x_1} \prec S$. Therefore, we confirm $h(x_2) = 2$.

Suppose $S' = S_{z_1 z_2 \ldots z_m}$ with $2 \leq m$. No harm to assume $I = \{1, 2, \ldots, n\}$. Taking $S' = S_{z_1 \ldots z_m} \prec S = S_{x_1 \ldots x_n}$ and $x_i \wedge x_j = 0, (i \neq j; i, j = 1, \ldots, n)$ together, we find $z_{i_{\alpha_1}} \leq x_{j_1}, (\alpha_1 \in I_1), \ldots, z_{i_{\alpha_p}} \leq x_{j_p}, (\alpha_p \in I_p)$ in which $\{i_{\alpha_t} \mid \alpha_t \in I_t\} \subseteq \{1, \ldots, m\}, (t = 1, \ldots, p); \{i_{\alpha_1} \mid \alpha_1 \in I_1\} \cup \ldots \cup \{i_{\alpha_p} \mid \alpha_p \in I_p\} = \{1, \ldots, m\}; \{i_{\alpha_k} \mid$

$\alpha_k \in I_k\} \cap \{i_{\alpha_l} \mid \alpha_l \in I_l\} = \varnothing, (k \neq l; k, l = 1, \ldots, p); \{j_1, j_2, \ldots, j_p\} \subseteq \{1, \ldots, n\}$ and $j_\alpha \neq j_\beta, (\alpha \neq \beta; \alpha, \beta = 1, \ldots, p)$. Thus, by the (4) of Remark 1, we may be assured $S_{z_1 \ldots z_m} = S_{z_{i_1}, i_1 \in I_1, \ldots, z_{i_p}, i_p \in I_p} \leq S_{x_{j_1} \ldots x_{j_p}} < S$. Since $S_{z_1 \ldots z_m} \prec S$, we obtain $S' = S_{x_{j_1} \ldots x_{j_p}} \prec S$. In addition, we attain $\{j_1, \ldots, j_p\} \subset \{1, \ldots, n\}$.

If $p < n-1$. Then, according to the (4) of Remark 1 and $\{j_1, \ldots, j_p\} \subset \{1, \ldots, n\}$, we may confirm $S_{x_{j_1} \ldots x_{j_p}} < S_{x_{j_1} \ldots x_{j_p} x_{j_{p+1}} \ldots x_{j_{n-1}}} < S_{x_{j_1} \ldots x_{j_n}} = S_{x_1 \ldots x_n} = S$ for a sub-arrangement $j_1, \ldots, j_p, j_{p+1}, \ldots, j_{n-1}$ of $1, 2, \ldots, n$. This causes a contradiction to $S' \prec S$. Thus, $p = n-1$ is true.

Certainly, $S' = S_{x_{j_1} \ldots x_{j_{n-1}}} \prec S$ shows that for any $t < x_{j_n}$, there is $t \wedge x_j = 0$ for any $j \in \{j_1, \ldots, j_{n-1}\}$ since $0 \leq t \wedge x_j \leq x_{j_n} \wedge x_j = 0$.

Hence, we may obtain $h(x_{j_n}) = 2$. Otherwise, it follows $S_{x_{j_1} \ldots x_{j_{n-1}}} < S_{x_{j_1} \ldots x_{j_{n-1}} b} < S$ for $b < x_{j_n}$ and $h(b) = 2$ according to Definition 2, a contradiction to $S' \prec S$.

Step 5.2. To prove item (3.5.3).

Let $S' = S_{x_{i_1} x_{i_2} \ldots x_{i_{n-1}}} = S_{x_i, i \in I_1} \in Cl_s(L)$. Then, it is easily to find $S' < S$.

Suppose $S' < S'' < S$ and $S'' = S_y$ for some $y \in L \setminus (A(L) \cup 0)$. Then by item (3.1), we obtain $x_j < y \leq x_{i_0}, (j \in I_1)$ and $i_0 \in I$. This follows that

if $j \neq i_0$, then $x_j \wedge x_{i_0} = x_j \in L \setminus (A(L) \cup 0)$, a contradiction to $x_j \wedge x_{i_0} = 0$;

if $j = i_0$, then $x_j < x_{i_0}$, a contradiction to $x_j = x_{i_0}$ when $j = i_0$.

Suppose $S' < S'' \prec S = S_{x_{i_1} \ldots x_{i_{n-1}} x_{i_n}}$ and $S'' = S_{z_1 z_2 \ldots z_m}$ with $2 \leq m$. $S' < S''$ follows $x_{i_{t_1}} \leq z_1, x_{i_{t_2}} \leq z_2, \ldots, x_{i_{t_k}} \leq z_k$, where $t_1 = 1, \ldots, m_1; t_2 = m_1 + 1, \ldots, m_2; \ldots; t_k = m_{k-1} + 1, \ldots, m_k; m_1 + \ldots + m_k = |I_1| = n-1$. $k \leq m$ holds since $x_i \wedge x_j = 0, (i \neq j; i, j \in I)$, $z_t \in L \setminus (A(L) \cup 0), (t = 1, \ldots, m)$ and $z_p \wedge z_q = 0, (p \neq q; p, q \in \{1, \ldots, m\})$. Meanwhile, combining $S_{x_j, j \in T} \in Cl_s(L), (T \subseteq \{1, \ldots, n\})$ with $S'' \prec S = S_{x_{i_1} \ldots x_{i_{n-1}} x_{i_n}}$, we attain $z_j \leq x_{i_j}, (j = 1, 2, \ldots, n-1)$. Hence, in view of this result with $x_i \wedge x_j = 0, (i \neq j; i, j \in I)$, $z_t \in L \setminus (A(L) \cup 0), (t = 1, \ldots, m)$ and $x_i \in L \setminus (A(L) \cup 0), (i \in I)$, we obtain $z_j = x_{i_j}, (j = 1, 2, \ldots, n-1)$ and $z_n < x_{i_n}$. However, by the known condition, it follows $h(x_{i_n}) = 2$. So, $z_n \in A(L) \cup 0$ holds. Hence, it causes $S'' = S'$, a contradiction to $S' < S''$.

Therefore, we confirm $S' \prec S$. □

Second, we will deal with some properties related to $Cl_s(L)$ if $Cl_s(L)$ is geometric for an atomistic complete finite lattice L.

Lemma 5. *Let L be an atomistic complete finite lattice with height 3. Then $Cl_s(L)$ is geometric if and only if*

(3.6) $|\mathcal{F}^2| \geq 2$, that is, L has at least two elements of height 2.

(3.7) If $d_1, d_2 \in \mathcal{F}^2$, then $d_1 \wedge d_2 \neq 0$.

Proof of Lemma 5. We will prove with two parts.

Part I. When $Cl_s(L)$ is geometric.

$h(L) = 3$ compels that there is $d \in \mathcal{F}^2$ according to the definition of height function in a lattice. If $\mathcal{F}^2 = \{d\}$. Then, we may obtain $Cl_s(L) = \{S_0, S_1, S_d\}$ with

$S_0 \prec S_d \prec S_1$. It is easily found that $Cl_s(L)$ is not geometric. This is a contradiction to the known supposition. Thus, it should have $|\mathcal{F}^2| \geq 2$. That is, item (3.6) is true.

Suppose that $d_1 \wedge d_2 = 0$ holds for any $d_1, d_2 \in \mathcal{F}^2 = \{d_j \mid j \in \mathcal{J}\}$. Then, this causes $S_{d_t, t\in T} \in Cl_s(L) < S_1$ for any $T \subseteq \mathcal{J}$ in virtue of the (3) of Remark 1 and the (2) in Definition 1. In addition, we may easily find $Cl_s(L) = \{S_0, S_1, S_{d_t, t\in T}, T \subseteq \mathcal{J}\}$. This causes $S_0 < S_{d_t, t\in T} \leq S_{d_j, j\in\mathcal{J}} < S_1$. So, $Cl_s(L)$ is not geometric since S_1 is not the join of atoms in $Cl_s(L)$. This is a contradiction to the geometry of $Cl_s(L)$.

Therefore, there are $d_6, d_7 \in \mathcal{F}^2$ satisfying $d_6 \neq d_7$ and $d_6 \wedge d_7 \neq 0$.

If there is $d_3 \in \mathcal{F}^2$ satisfying $d_3 \wedge d_7 = 0$. Then, we obtain $S_0 \prec S_{d_7} \prec S_{d_7 d_3} < S_1$ and $S_0 \prec S_{d_6} < S_1$ with $S_{d_6} || S_{d_7 d_3}$. In addition, $S_{d_7} \vee S_{d_6} = S_{d_1 \vee d_6} = S_1$ holds according to $d_7 \wedge d_6 \neq 0$ and item (3.3). This follows $S_{d_7} \not\prec S_1 = S_{d_7} \vee S_{d_6}$. So, $Cl_s(L)$ is not semimodular. This follows a contradiction to the geometry of $Cl_s(L)$. In other words, $d_7 \wedge d_j \neq 0$ holds for any $j \in \mathcal{J}$. Analogously, $d_6 \wedge d_j \neq 0$ holds for any $j \in \mathcal{J}$.

Moreover, $S_0 \prec S_{d_7} \prec S_1$ holds. Hence, this maximal chain $\{S_0, S_{d_7}, S_1\}$ in $Cl_s(L)$ has length 3. If there are $d_4, d_5 \in \mathcal{F}^2$ satisfying $d_4 \wedge d_5 = 0$. Then it causes $S_0 \prec S_{d_4} \prec S_{d_4 d_5} < S_1$. So, there is a maximal chain in $Cl_s(L)$ with length at least 4. Hence, there are two maximal chains with different lengths in $Cl_s(L)$. This follows a contradiction to the geometry of $Cl_s(L)$.

Summing up, we obtain $x \wedge y \neq 0$ for any $x, y \in \mathcal{F}^2$. That is, item (3.7) is true.

Part II. When L satisfies items (3.6) and (3.7).

Let d_j has height 2 in L, $(j \in \mathcal{J})$, that is, $\mathcal{F}^2 = \{d_j \mid j \in \mathcal{J}\}$.

Under the suppositions of items (3.6) and (3.7) with $h(L) = 3$, we may easily decide that $Cl_s(L)$ is $\{S_0, S_1, S_{d_j}, j \in \mathcal{J}\}$ in which $S_0 \prec S_{d_j} \prec S_1$, $S_{d_i} || S_{d_j}, (i, j \in \mathcal{J}; i \neq j)$ and $S_1 = \vee_{j\in\mathcal{J}} S_{d_j}$. Therefore, $Cl_s(L)$ is geometric. □

Considering the (3) in Lemma 1 with Lemma 5, we may easily express the following corollary.

Corollary 1. *Let L be an atomistic complete finite lattice and $x \in L \setminus (A(L) \cup 0 \cup \mathcal{F}^2)$ with $h(x) = 3$. If $Cl_s(L)$ is geometric, then there are $|\mathcal{F}^2(x)| \geq 2$ and $d_1 \wedge d_2 \neq 0$ for any $d_1, d_2 \in \mathcal{F}^2(x)$ and $x = \vee_{d\in\mathcal{F}^2(x)} d$.*

Lemma 6. *Let L be an atomistic complete finite lattice. If $Cl_s(L)$ is geometric, then L satisfies the following properties.*

(3.8) $|\mathcal{C}(x) \setminus A(L)| \geq 2$ for any $x \in L \setminus (A(L) \cup 0)$.

(3.9) $x = \vee_{d\in\mathcal{F}^2(x)} d$ for any $x \in L \setminus (A(L) \cup 0)$.

(3.10) Let $x, y \in L \setminus (A(L) \cup 0)$. If $x \wedge y \prec x, y$, then $x, y \prec x \vee y$.

(3.11) Let $x \in L$ with $h(x) = 3$. Then, there is $d_1 \wedge d_2 \neq 0$ for any $d_1, d_2 \in \mathcal{F}^2(x)$ with $d_1 \neq d_2$.

(3.12) Let $x \in L \setminus (A(L) \cup 0)$. If $3 \leq h(x)$, then there are $y, z \in \mathcal{C}(x) \setminus A(L)$ satisfying $y \neq z$ and $y \wedge z \neq 0$.

(3.13) Let $x, y \in L \setminus (A(L) \cup 0)$. *If* $2 \leq h(x \wedge y), x \wedge y \neq 0, x \wedge y \prec x, x \wedge y \prec y$ *and* $z_x \wedge (x \wedge y) \neq 0, z_y \wedge (x \wedge y) \neq 0$ *for any* $z_x, z_y \in L \setminus (A(L) \cup 0)$ *with* $z_x < x, z_y < y$. *Then,* $p \wedge x \neq 0$ *and* $q \wedge y \neq 0$ *for any* $p, q < x \vee y$ *and* $p, q \in L \setminus (A(L) \cup 0)$.

(3.14) Let $y_j \in L \setminus (A(L) \cup 0), (j = 1, 2, \ldots, n; 2 \leq n), y_i \wedge y_j = 0, (i \neq j; i, j = 1, 2, \ldots, n)$.

If $y_j \in \mathcal{F}^2, (j = 1, \ldots, n)$, *then* $S \prec S_{y_1 \ldots y_n} \Rightarrow S = S_{y_{i_1} \ldots y_{i_{n-1}}}$, *for some* $\{i_1, \ldots, i_{n-1}\} \subseteq \{1, \ldots, n\}$ *and* $|\{i_1, \ldots, i_{n-1}\}| = n - 1$.

If $S \prec S_{y_1 \ldots y_n}$. *We obtain that if there is* $y_{i_n} \notin \mathcal{F}^2$ *for some* $i_n \in \{1, \ldots, n\}$, *then* $S = S_{y_{i_1} \ldots y_{i_{n-1}} z_n}$ *holds where* $z_n \prec y_{i_n}$ *and* $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$*; or if* $y_{j_n} \in \mathcal{F}^2$ *for some* $j_n \in \{1, \ldots, n\}$, *then* $S = S_{y_{j_1} \ldots y_{j_{n-1}}}$ *where* $\{j_1, \ldots, j_{n-1}\} = \{1, \ldots, n\} \setminus j_n$.

(3.15) Let $y_i \in L, (i = 1, 2, \ldots, n)$ *satisfy* $y_i \wedge y_j = 0, (i \neq j; i, j = 1, 2, \ldots, n; 2 \leq n)$.

(α) Setting $x = \vee_{j=1}^{n-1} y_{i_j} \vee w_n$ *and* $w_n \prec y_{i_n}$ *with* $h(w_n) \geq 2$ *and* $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. *Suppose that for any* $m \leq n - 1$ *and* $p \in L$ *with* $\vee_{j=1}^{m} y_{i_{t_j}} \leq p < x$ *or* $\vee_{j=1}^{m} y_{i_{t_j}} \vee w_n \leq q < x$, *where* $\{i_{t_j} \mid j = 1, \ldots, m\} \subseteq \{i_1, \ldots, i_{n-1}\}$, *there exists* $s \in \{i_1, \ldots, i_{n-1}\} \setminus \{i_{t_j} \mid j = 1, \ldots, m\}$ *satisfying* $p \wedge y_s \neq 0$, *or* $p \wedge w_n \neq 0$, *or* $q \wedge y_s \neq 0$. *Then the following property (3.15.1) is true.*

(β) Setting $z = \vee_{t=1}^{n-1} y_{j_t}$, $h(y_{j_n}) = 2$ *and* $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. *Suppose that for any* $m \leq n - 1$ *and* $p \in L$ *with* $\vee_{j=1}^{m} y_{i_{t_j}} \leq p < z$ *where* $\{i_{t_j} \mid j = 1, \ldots, m\} \subseteq \{j_1, \ldots, j_{n-1}\}$, *there exists* $s \in \{j_1, \ldots, j_{n-1}\} \setminus \{i_{t_j} \mid j = 1, \ldots, m\}$ *satisfying* $p \wedge y_s \neq 0$. *Then the following property (3.15.2) is real.*

(3.15.1) If $x \wedge y_{i_n} \neq 0$. *Then* $x \prec x \vee y_{i_n} = \vee_{i=1}^{n} y_i$ *holds, and in addition, for any* $m \leq n$ *and* $\vee_{j=1}^{m} y_{i_{t_j}} \leq q < \vee_{i=1}^{n} y_i$ *where* $\{i_{t_j} \mid j = 1, \ldots, m\} \subset \{1, \ldots, n\}$, *there exists* $s \in \{1, \ldots, n\} \setminus \{i_{t_j} \mid j = 1, \ldots, m\}$ *satisfying* $q \wedge y_s \neq 0$.

(3.15.2) If $z \wedge y_{j_n} \neq 0$. *Then* $z \prec z \vee y_{j_n} = \vee_{j=1}^{n} y_j$ *holds, and in addition, for any* $m \leq n$ *and* $\vee_{j=1}^{m} y_{i_{t_j}} \leq q < \vee_{j=1}^{n} y_j$ *where* $\{i_{t_j} \mid j = 1, \ldots, m\} \subset \{1, \ldots, n\}$, *there exists* $s \in \{1, \ldots, n\} \setminus \{i_{t_j} \mid j = 1, \ldots, m\}$ *satisfying* $q \wedge y_s \neq 0$.

(3.16) Let $y_j \in L \setminus (A(L) \cup 0), (j = 1, 2, \ldots, n; 2 \leq n)$, $x_n \in L \setminus (A(L) \cup 0)$ *and* $x_n \neq y_n$ *satisfy* $y_i \wedge y_j = 0, (i \neq j; i, j = 1, 2, \ldots, n)$ *and* $y_j \wedge x_n = 0$, $(j = 1, 2, \ldots, n - 1)$. *Then the following statements hold.*

(3.16.1) Let $h(x_n), h(y_n) > 2$ *and* $h(x_n \wedge y_n) \neq 0$. *If* $y_j \wedge (x_n \vee y_n) = 0$ *holds* $(j = 1, \ldots, n - 1)$, *and in addition, there is* $z_n \in L \setminus 0$ *satisfying* $z_n \prec x_n, y_n$. *Then, for any* $p, q \in L \setminus (A(L) \cup 0)$, $x_n < p < x_n \vee y_n$ *and* $y_n < q < x_n \vee y_n$, *there exist* $s, t \in \{1, \ldots, n - 1\}$ *satisfying* $y_s \wedge p \neq 0$ *and* $y_t \wedge q \neq 0$.

Additionally, if there are $x_n \wedge y_n \neq 0$ *and* $h(x_n) = h(y_n) = 2$, *but* $y_j \wedge (x_n \vee y_n) = 0, (j = 1, \ldots, n - 1)$. *Then, for any* $p, q \in L \setminus (A(L) \cup 0), x_n < p < x_n \vee y_n$ *and* $y_n < q < x_n \vee y_n$, *there exist* $s, t \in \{1, \ldots, n - 1\}$ *satisfying* $y_s \wedge p \neq 0$ *and* $y_t \wedge q \neq 0$.

(3.16.2) Let $h(x_n), h(y_n) > 2$ *and* $h(x_n \wedge y_n) \neq 0$. *When* $y_j \wedge (x_n \vee \vee_{i=m+1}^{n} y_i) = 0, (j = 1, \ldots, m; 1 < m < n - 1)$, *and in addition, there exists* $z_n \in L \setminus 0$ *satisfying*

$z_n \prec x_n, y_n$. *Then, for any* $p, q, f \in L \setminus (A(L) \cup 0)$, *if the expressions (i), (ii) and (iii) hold*

(i) $\vee_{i=t}^{t_p} y_{j_i} \leq p < x_n \vee \vee_{i=m+1}^{n} y_i$ *where* $\{j_i \mid t \leq i \leq t_p\} \subseteq \{m+1, \ldots, n-1\}$,

(ii) $x_n \vee \vee_{i=t_x}^{t_q} y_{j_i} \leq q < x_n \vee \vee_{i=m+1}^{n} y_i$ *where* $\{j_i \mid t_x \leq i \leq t_q\} \subseteq \{m+1, \ldots, n-1\}$,

(iii) $\vee_{i=t_1}^{t_f} y_{j_i} \leq f < \vee_{i=m+1}^{n} y_i \vee x_n$ *where* $\{j_i \mid t_1 \leq i \leq t_f\} \subseteq \{m+1, \ldots, n\}$.

then, the following expressions hold:

there exists $s_p \in \{1, \ldots, n-1\} \setminus \{j_i \mid t \leq i \leq t_p\}$ *satisfying* $y_{s_p} \wedge p \neq 0$ *or* $x_n \wedge p \neq 0$*;*

there exists $s_q \in \{1, \ldots, n-1\} \setminus \{j_i \mid t_x \leq i \leq t_q\}$ *satisfying* $y_{s_q} \wedge q \neq 0$*;*

there exists $s_f \in \{1, \ldots, n\} \setminus \{j_i \mid t_1 \leq i \leq t_f\}$ *satisfying* $y_{s_f} \wedge f \neq 0$.

Additionally, let $x_n \wedge y_n \neq 0$ *and* $h(x_n) = h(y_n) = 2$. *When* $y_j \wedge (x_n \vee \vee_{i=m+1}^{n} y_i) = 0, (j = 1, \ldots, m; 1 < m < n-1)$. *Then, for any* $p, q, f \in L \setminus (A(L) \cup 0)$, *if the expressions (i), (ii) and (iii) hold, then the following expressions hold:*

there is $s_p \in \{1, \ldots, n-1\} \setminus \{j_i \mid t \leq i \leq t_p\}$ *satisfying* $y_{s_p} \wedge p \neq 0$ *or* $x_n \wedge p \neq 0$*;*

there is $s_q \in \{1, \ldots, n-1\} \setminus \{j_i \mid t_x \leq i \leq t_q\}$ *satisfying* $y_{s_q} \wedge q \neq 0$*;*

there is $s_f \in \{1, \ldots, n\} \setminus \{j_i \mid t_1 \leq i \leq t_f\}$ *satisfying* $y_{s_f} \wedge f \neq 0$.

Proof of Lemma 6. We will prove the needed results step by step.

Step 1. To prove item (3.8).

In view of the definition of height function in a lattice and $h(x) = n < \infty$, we may indicate that there is $y \in \mathcal{F}^{n-1}(x)$. It is easily seen $y \prec x$. So, $y \in \mathcal{C}(x) \setminus A(L)$ holds.

If $|\mathcal{C}(x) \setminus A(L)| = 1$. Considered the atomistic property of L, we receive $x = y \vee (\vee_{a \in A(L), a \not\leq y, a \leq x} a)$. Hence, $Cl_s([0, x])$ satisfies $S \leq S_y$ for any $S \in Cl_s([0, x]) \setminus S_x$. This implies that S_x is not a join of atoms in $[S_0, S_x]$, a contradiction to the geometry of $Cl_s(L)$.

In another word, $|\mathcal{C}(x) \setminus A(L)| \geq 2$ holds.

Step 2. To prove item (3.9).

It is easily found $x = \vee_{d \in \mathcal{F}^2(x)} d$ if $h(x) = 2$.

Utilizing Corollary 1, we may attain $x = \vee_{d \in \mathcal{F}^2(x)} d$ for any $x \in L$ and $h(x) = 3$.

Suppose that if $x \in L$ with $2 \leq h(x) \leq n-1$, then $x = \vee_{d \in \mathcal{F}^2(x)} d$.

Let $x \in L$ and $h(x) = n$. In view of item (3.8), we find $x = \vee_{y \in \mathcal{C}(x) \setminus A(L)} y$. We may easily decide $h(y) \leq n-1$ for any $y \in \mathcal{C}(x) \setminus A(L)$. Using the inductive on n, we obtain $y = \vee_{d \in \mathcal{F}^2(y)} d$ for any $y \in \mathcal{C}(x) \setminus A(L)$. On the other hand, for any $d \in \mathcal{F}^2(x)$, we find $d \prec x$ or $d \leq y$ for some $y \in \mathcal{C}(x) \setminus A(L)$.

Therefore, it follows $x = \vee_{y \in \mathcal{C}(x) \setminus A(L)} y = \vee_{d \in \mathcal{F}^2(x)} d$.

Step 3. To prove item (3.10).

Let $x, y \in L \setminus (A(L) \cup 0)$ and $x \wedge y \prec x, y$.

If $x \wedge y = 0$, then $x, y \in A(L)$ holds since $0 \prec x, y$. This is a contradiction to $x, y \in L \setminus (A(L) \cup 0)$.

Thus, $x \wedge y \neq 0$ is true.

Furthermore, by item (3.3), we may be assured $S_x \vee S_y = S_{x\vee y}$. Since $Cl_s(L)$ is geometric and $S_x, S_y \prec S_x \vee S_y$ holds, we confirm $S_x, S_y \prec S_{x\vee y}$. If $x \not\prec x \vee y$, then $S_x < S_b < S_{x\vee y}$ for some $b \in L$ and $x < b < x \vee y$. This causes a contradiction to $S_x \prec S_{x\vee y}$. Hence, $x \prec x \vee y$ holds. Similarly, $y \prec x \vee y$ holds.

Step 4. The result in item (3.11) can be produced by Corollary 1.

Step 5. To prove item (3.12).

Let $x \in L \setminus (A(L) \cup 0)$ with $3 \leq h(x)$.

If $p \wedge q = 0$ for any $p, q \in \mathcal{C}(x) \setminus A(L)$. Then, using (3) in Remark 1, item (3.8) and induction on $|\mathcal{C}(x) \setminus A(L)|$, we confirm $S_{y, y\in\mathcal{C}(x)\setminus A(L)} = \{y \mid y \in \mathcal{C}(x) \setminus A(L)\} \cup \{a \in A(L) \mid a \not\leq y$, for any $y \in \mathcal{C}(x) \setminus A(L)\} \in Cl_s(L)$ and $S_{y, y\in\mathcal{C}(x)\setminus A(L)} < S_x$, and $S \leq S_{y, y\in\mathcal{C}(x)\setminus A(L)}$ for any $S \in [S_0, S_x] \setminus S_x$. Thus, S_x is not the join of atoms. This causes a contradiction to the geometry of $Cl_s(L)$.

Therefore, there are at least two elements $y, z \in \mathcal{C}(x) \setminus A(L)$ satisfying $y \wedge z \neq 0$.

Step 6. To prove item (3.13).

From the given conditions, we follow $S_{x\wedge y} \prec S_x, S_y$. It is easily known $S_{x\wedge y} \leq S_x \wedge S_y$ by item (3.1).

We will demonstrate $S_x \wedge S_y = S_{x\wedge y}$.

Otherwise, $S_{x\wedge y} < S_x \wedge S_y$. If $S_x \wedge S_y = S_z$ for some $z \in L$, then $x \wedge y < z < x, y$. Using item (3.1), we may attain $S_{x\wedge y} < S_z < S_x, S_y$, a contradiction to $S_{x\wedge y} \prec S_x, S_y$. Thus, we find $S_x \wedge S_y = S_{z_i, i\in\mathcal{I}} \in Cl_s(L)$ and $2 \leq |\mathcal{I}|$. Applying the (2) of Definition 1, the (2) of Lemma 2 with Definition 2, we may obtain $A(x \wedge y) \subseteq A(z_{i_0})$ for some $i_0 \in \mathcal{I}$. Considering this result with the atomistic property of L and $2 \leq |\mathcal{I}|$, we follow $x \wedge y \leq z_{i_0} < \vee_{i\in\mathcal{I}} z_i \leq x, y$. And further, we find $S_{x\wedge y} < S_{z_{i_0}} \leq S_{\vee_{i\in\mathcal{I}} z_i} < S_x$, or $S_{x\wedge y} \leq S_{z_{i_0}} < S_{\vee_{i\in\mathcal{I}} z_i} \leq S_y$ since $S_x \neq S_y$. This causes a contradiction to $S_{x\wedge y} \prec S_x$ if $S_{x\wedge y} < S_{z_{i_0}} \leq S_{\vee_{i\in\mathcal{I}} z_i} < S_x$, and $S_{x\wedge y} \prec S_y$ if $S_{x\wedge y} \leq S_{z_{i_0}} < S_{\vee_{i\in\mathcal{I}} z_i} \leq S_y$.

In other words, $S_x \wedge S_y = S_{x\wedge y}$ is true.

Therefore, we may be assured $S_{x\wedge y} = S_x \wedge S_y \prec S_x, S_y$.

Furthermore, since $Cl_s(L)$ is geometric, we may point out $S_x, S_y \prec S_x \vee S_y$. Since $S_x \vee S_y = S_{x\vee y}$ holds by $x \wedge y \neq 0$ and item (3.3), we find $x, y \prec x \vee y$ according to the given condition $x \wedge y \prec x, y$ with item (3.10). That is, $S_x, S_y \prec S_z < S_{x\vee y}$ does not hold for any $z \in L$. Actually, if there is $p \in L \setminus (A(L) \cup 0)$ satisfying $p < x \vee y$ and $p \wedge x = 0$, then $S_{xp} \in Cl_s(L)$ holds. We may easily find $S_x < S_{xp} < S_{x\vee y}$, a contradiction to $S_x \prec S_{x\vee y}$. Therefore, for any $p \in L \setminus (A(L) \cup 0)$ and $p < x \vee y$, it has $p \wedge x \neq 0$. Analogously, $q \wedge y \neq 0$ holds for any $q \in L \setminus (A(L) \cup 0)$ and $q < x \vee y$.

Step 7. To prove item (3.14).

Let $y_j \in \mathcal{F}^2$, that is, y_j has height 2 in L, $(j = 1, 2, \ldots, n; n \geq 2)$. If $S \prec S_{y_1 \ldots y_n}$. Then, taken $S \prec S_{y_1 \ldots y_n}$ and the (3) in Remark 1 together, we attain $S = S_{z_1 \ldots z_m}$ for

$z_i \in L \setminus (A(L) \cup 0), (i = 1, \ldots, m)$. We may easily obtain $z_1 = y_{i_1}, \ldots, z_{n-1} = y_{i_{n-1}}$ in which $\{i_1, \ldots, i_{n-1}\} \subseteq \{1, \ldots, n\}$, $|\{i_1, \ldots, i_{n-1}\}| = n-1$ and $m \leq n$.

If $m = n$. Then $z_n \leq y_{i_n}$, $z_n \in L \setminus (A(L) \cup 0)$ and $S < S_{y_1 \ldots y_n}$ taken together causes $z_n < y_{i_n}$ in which $i_n = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{n-1}\}$. But, $h(y_{i_n}) = 2$ follows $z_n \in A(L) \cup 0$, a contradiction to $z_n \in L \setminus (A(L) \cup 0)$.

Therefore, $m \leq n-1$ is true. In addition, we may easily explore that if $p < q \leq m$, then $S_{z_1 \ldots z_p} < S_{z_1 \ldots z_p z_{p+1} \ldots z_q} \leq S_{z_1 \ldots z_m} \leq S_{y_{i_1} \ldots y_{i_{n-1}}} < S_{y_1 \ldots y_n}$. Hence, $n-1 \leq m$ is true.

That is to say, $S = S_{y_{i_1} \ldots y_{i_{n-1}}}$ holds.

Next, to prove the other part in item (3.14).

Suppose $S \prec S_{y_1 \ldots y_n}$ and $S = S_z$ for some $z \in L \setminus 0$. Since both of $y_{i_1} \notin A(L) \cup 0$ and $S_0 < S_{y_{i_1}} < S_{y_{i_1} y_{i_2}} < \ldots < S_{y_{i_1} \ldots y_{i_{n-1}}} < S_{y_{i_1} \ldots y_{i_{n-1}} z_n} < S_{y_1 \ldots y_n}$ are true. Additionally, according to the geometry of $Cl_s(L)$, we may point out $z \notin A(L) \cup 0$. Furthermore, we follow $z < y_{i_0}$ for one and only one $i_0 \in \{1, \ldots, n\}$ since $S_z \prec S_{y_1 \ldots y_n}$ and $y_i \wedge y_j = 0, (i \neq j; i, j = 1, \ldots, n; 2 \leq n)$. Thus, by item (3.1), we arrive at $S_z < S_{y_{i_0}} < S_{y_1 \ldots y_n}$. This is a contradiction to $S_z \prec S_{y_1 \ldots y_n}$.

Suppose $S = S_{z_1 \ldots z_m}$ with $2 \leq m$. Then, $z_j \in L \setminus (A(L) \cup 0), (j = 1, \ldots, m)$ hold according to the (3) of Remark 1.

Since $S_{z_1 \ldots z_m} < S_{y_1 \ldots y_n}$ infers that for any $i \in \{1, \ldots, m\}$, there is $z_i \leq y_{j_i}$ for some $j_i \in \{1, \ldots, n\}$.

If $z_{i_t} < y_{j_{i_1}}, (t = 1, 2, \ldots, m)$. Then $S_{z_1 \ldots z_m} < S_{y_{i_1}} < S_{y_1 \ldots y_n}$ holds according to Definition 2 and $2 \leq n$. This causes a contradiction to $S_{z_1 \ldots z_m} \prec S_{y_1 \ldots y_n}$.

Additionally, combining $z_j \in L \setminus (A(L) \cup 0), (j = 1, \ldots, m)$ and $y_p \wedge y_q = 0, (p \neq q; p, q = 1, \ldots, n)$, we decide that for any z_{j_0}, where $j_0 \in \{1, \ldots, m\}$, there is one and only one $y_{i_{j_0}}$ satisfying $z_{j_0} \leq y_{i_{j_0}}, (i_{j_0} \in \{1, \ldots, n\})$. If $z_{i_1} \neq z_{i_2}$ but $z_{i_1} \leq y_{j_{i_1}}, z_{i_2} \leq y_{j_{i_1}}$ and $z_{i_3} \leq y_{j_{i_3}}$ where $i_3 \notin \{i_1, i_2\}$, then $z_{i_1} \vee z_{i_2} \leq y_{j_{i_1}}$ and $0 \leq z_{i_3} \wedge (z_{i_1} \vee z_{i_2}) \leq y_{j_{i_1}} \wedge y_{j_{i_3}} = 0$. We may obtain $S_{z_1 \ldots z_m} < S_{(z_{i_1} \vee z_{i_2}) z_{i_3} \ldots z_{i_m}} \leq S_{y_1 \ldots y_n}$. This causes a contradiction to $S_{z_1 \ldots z_m} \prec S_{y_1 \ldots y_n}$. Thus, $z_i \leq y_i$ holds $(i = 1, \ldots, m)$. In addition, if $z_p \neq z_q$, then $y_p \neq y_q, (p, q \in \{1, \ldots, m\})$. Hence, We may assume $z_j \leq y_j, (j = 1, \ldots, m)$. So, $m \leq n$ is true.

If $m = n$. Since $z_j \leq y_j$ $(j = 1, \ldots, n)$ and $S \prec S_{y_1 \ldots y_n}$ follow that there is $i_0 \in \{1, \ldots, m\} = \{1, \ldots, n\}$ satisfying $z_{i_0} < y_{i_0}$. We determine that if there are more than two elements $i_1, i_2 \in \{1, \ldots, m\}$ satisfying $z_{i_1} < y_{i_1}$ and $z_{i_2} < y_{i_2}$, then $S_{z_1 \ldots z_{i_1 - 1} z_{i_1} \ldots z_{i_2} \ldots z_n} < S_{z_1 \ldots z_{i_1 - 1} y_{i_1} z_{i_1 + 1} \ldots z_{i_2 - 1} y_{i_2} z_{i_2 + 1} \ldots z_n} < S_{y_1 \ldots y_n}$ holds. This is a contradiction to $S \prec S_{y_1 \ldots y_n}$. Hence, there is one and only one $i_0 \in \{1, \ldots, m\}$ satisfying $z_{i_0} < y_{i_0}$. No matter to suppose $i_0 = n$.

When $m = n$ and $h(y_n) > 2$. We find $S_{y_1 \ldots y_{n-1} z} \in Cl_s(L)$ and $S_{y_1 \ldots y_{n-1} z} < S_{y_1 \ldots y_n}$ for any $z \prec y_n$. In addition, if there is $M \in Cl_s(L)$ satisfying $S_{y_1 \ldots y_{n-1} z} < M < S_{y_1 \ldots y_n}$, then it is easily to obtain $y_j \in M, (j = 1, \ldots, n-1)$ and $b \in M$ such that $z < b < y_n$. This causes a contradiction to $z \prec y_n$. Therefore, $S = S_{y_1 \ldots y_{n-1} z_n}$ holds where $z_n \prec y_n$.

When $m \leq n$ and $h(y_n) = 2$.

Then, we will demonstrate $m < n$ holds.

Otherwise, $z_n < y_n$ infers to $z_n \in A(L) \cup 0$, a contradiction to $z_n \in L \setminus (A(L) \cup 0)$. That is to say, this case will not happen actually.

Combining

$$p < q \leq m \Rightarrow S_{z_1 \ldots z_p} < S_{z_1 \ldots z_p z_{p+1}} \cdots z_q$$
$$\leq S_{z_1 \ldots z_m} \leq S_{y_1 \ldots y_m} \leq S_{y_1 \ldots y_m y_{m+1} \ldots y_{n-1}} \leq S_{y_1 \ldots y_n}$$

and $z_j < y_j, (j = 1, \ldots, m)$ with $S_{z_1 \ldots z_m} = S \prec S_{y_1 \ldots y_n}$ and the geometry of $Cl_s(L)$, we decide $n - 1 \leq m$.

If $m = n-1$. That is, $|\{1, \ldots, m\}| = n-1$ holds. No matter to suppose $\{1, \ldots m\} = \{1, \ldots, n-1\}$. Considered $z_j \leq y_j, (j = 1, \ldots, m)$ and $S = S_{z_1 \ldots z_m} \prec S_{y_1 \ldots y_n}$, we may confirm $z_j = y_j, (j = 1, \ldots, m; m = n-1)$.

We will demonstrate $h(y_n) = 2$.

Otherwise, $h(y_n) > 2$ is true. But, using the definition of height function in a lattice, we affirm the existence of $b \in L \setminus 0$ satisfying $b \prec y_n$ and $h(b) = h(y_n) - 1 \geq 2$. In addition, it is easily found $S_{z_1 \ldots z_m} < S_{z_1 \ldots z_m b} < S_{y_1 \ldots y_n}$. This follows a contradiction to $S_{z_1 \ldots z_m} \prec S_{y_1 \ldots y_n}$.

When $m = n-1$ and $h(y_n) = 2$. Since $S_{z_1 \ldots z_m} \leq S_{y_1 \ldots y_m y_{m+1} \ldots y_{n-1}} < S_{y_1 \ldots y_n}$ holds in light of $z_j \leq y_j, (j = 1, \ldots, m)$ and the given condition $S_{z_1 \ldots z_m} \prec S_{y_1 \ldots y_n}$. We may attain $S_{y_1 \ldots y_{n-1}} \leq S$. If $S_{y_1 \ldots y_{n-1}} < S \prec S_{y_1 \ldots y_n}$ holds, then there is $b \in S \setminus (A(L) \cup \{y_1, \ldots y_{n-1}\})$ satisfying $b < y_n$. However, $h(y_n) = 2$ and $b < y_n$ together causes $b \in A(L) \cup 0$, a contradiction to $b \in S \setminus (A(L) \cup \{y_1, \ldots, y_{n-1}\})$. Therefore, $S = S_{y_1 \ldots y_{n-1}}$ holds.

Step 8. To prove item (3.15).

Under the supposition of (α), $x \wedge y_{i_n} = 0$ will not happen since $w_n \neq 0, w_n \leq x$ and $w_n \prec y_{i_n}$. That is to say, there is item (3.15.1) and only item (3.15.1) to happen. According to the given conditions, items (3.5) and (3.14) with (2) in Definition 1, we affirm $S_{y_1 \ldots y_{n-1} w_n} \prec S_{y_1 \ldots y_n}, S_x$. Thus, under the supposition of (α), we may be assured $S_{y_{i_1} \ldots y_{i_{n-1}} w_n} = S_x \wedge S_{y_1 \ldots y_n} \prec S_x, S_{y_1 \ldots y_n}$. Furthermore, by the geometry of $Cl_s(L)$, we find $S_x, S_{y_1 \ldots y_n} \prec S_x \vee S_{y_1 \ldots y_n}$.

If the supposition of (β) happens. Then we find $S_{y_{j_1} \ldots y_{j_{n-1}}} = S_z \wedge S_{y_1 \ldots y_n} \prec S_z, S_{y_1 \ldots y_n}$ according to item (3.14) and the (2) in Definition 1. Thus, by the geometry of $Cl_s(L)$, we affirm $S_z, S_{y_1 \ldots y_n} \prec S_z \vee S_{y_1 \ldots y_n}$.

Step 8.1. To prove item (3.15.1).

Let $x \wedge y_{i_n} \neq 0$. We will prove the needed results as the following two parts.

Part I. We prove $S_{x \vee y_{i_n}} = S_x \vee S_{y_1 \ldots y_n}$.

Let $S_x \vee S_{y_1 \ldots y_n} = S_{z_1 \ldots z_m}$ for some $z_j \in L \setminus 0$.

The result in item (3.1) and $x, y_j \in L \setminus (A(L) \cup 0)$ together follows $z_i \notin A(L) \cup 0, (j = 1, \ldots, n; i = 1, \ldots, m)$. This implies $y_{i_j} \leq x \leq z_1$, $(j = 1, \ldots, n-1)$ and $y_{i_n} \leq z_2$. If $z_1 \neq z_2$, then $x \wedge y_{i_n} \leq z_1 \wedge z_2 = 0$. This follows $x \wedge y_{i_n} = 0$, a contradiction to $x \wedge y_{i_n} \neq 0$. Thus, we produce $z_1 = z_2$. According to item (3.1) and $S_x, S_{y_1 \ldots y_n} \leq S_{z_1} \leq S_{z_1 \ldots z_m} = S_x \vee S_{y_1 \ldots y_n}$, we confirm $S_x \vee S_{y_1 \ldots y_n} = S_{z_1}$. Therefore, we obtain $S_x, S_{y_1 \ldots y_n} \prec S_x \vee S_{y_1 \ldots y_n} = S_{z_1}$.

Part II. We will prove the other needed results in item (3.15.1).

In light of the result in Part I, $x \vee (y_1 \vee \ldots \vee y_n) = x \vee y_{i_n}$ and item (3.1), we find $S_{x \vee y_{i_n}} = S_{z_1}$. Combining this result with $S_x \prec S_x \vee S_{y_1 \ldots y_n} = S_z$ and $w_n \prec y_{i_n}$, we may easily affirm $x \prec x \vee y_{i_n}$. In addition, $S_{y_1 \ldots y_n} \prec S_{x \vee y_{i_n}} = S_{\vee_{j=1}^n y_j}$ illustrates that no element $S \in Cl_s(L)$ satisfies $S_{y_1 \ldots y_n} < S < S_{\vee_{j=1}^n y_j}$. According to item (3.1) and the (3) of Remark 1, we may state that for any $m \leq n$ and $\vee_{j=1}^m y_{i_{t_j}} \leq q < \vee_{j=1}^n y_j$, there exists $s \in \{1, \ldots, n\} \setminus \{i_{t_j} \mid j = 1, \ldots, m\}$ satisfying $q \wedge y_s \neq 0$ where $\{i_{t_j} \mid j = 1, \ldots, m\} \subset \{1, \ldots, n\}$.

Step 8.2. Item (3.15.2) may be verified by similar way to that for item (3.15.1).

Step 9. To prove item (3.16).

Considering the known conditions, item (3.1), the (2) in Definition 1 with Definition 2, we obtain $S_{y_1 \ldots y_{n-1}} < S_{y_1 \ldots y_{n-1} x_n}, S_{y_1 \ldots y_{n-1} y_n}$. Suppose that there are $S_2, S_3 \in Cl_s(L)$ satisfying $S_{y_1 \ldots y_{n-1}} < S_2 \prec S_{y_1 \ldots y_{n-1} x_n}$ and $S_{y_1 \ldots y_{n-1}} < S_3 \prec S_{y_1 \ldots y_{n-1} y_n}$. Then, using items (3.5) and (3.14), we may obtain the following four statements.

if $h(x_n) > 2$, then $S_2 = S_{y_1 \ldots y_{n-1} z_2}$ where $z_2 \prec x_n$;
if $h(x_n) = 2$, then $S_2 = S_{y_1 \ldots y_{n-1}}$;
if $h(y_n) > 2$, then $S_3 = S_{y_1 \ldots y_{n-1} z_3}$ where $z_3 \prec y_n$;
if $h(y_n) = 2$, then $S_3 = S_{y_1 \ldots y_{n-1}}$.

Therefore, if there is $S \in Cl_s(L)$ satisfying $S \prec S_{y_1 \ldots y_{n-1} x_n}, S_{y_1 \ldots y_{n-1} y_n}$, then there is $S = S_{y_1 \ldots y_{n-1} z_n}$ when $z_n = x_n \wedge y_n \notin A(L) \cup 0$ and $z_n \prec x_n, y_n$; or there is $S = S_{y_1 \ldots y_{n-1}}$ when $x_n \wedge y_n \in A(L) \cup 0$.

Step 9.1. To verify item (3.16.1).

Since $z_n \prec x_n$ and y_n taken together follows $z_n = x_n \wedge y_n$. Hence, in view of $h(x_n \wedge y_n) \neq 0$, there is $S_{y_1 \ldots y_{n-1}} \prec S_{y_1 \ldots y_{n-1} x_n}, S_{y_1 \ldots y_n}$ if $x_n \wedge y_n \in A(L)$, and in addition, there is $S_{y_1 \ldots y_{n-1} z_n} \prec S_{y_1 \ldots y_{n-1} x_n}, S_{y_1 \ldots y_n}$ if $z_n \in L \setminus (A(L) \cup 0)$. No matter which of the above cases happens, using the geometry of $Cl_s(L)$ and Lemma 1, we always obtain $S_{y_1 \ldots y_{n-1} x_n}, S_{y_1 \ldots y_n} \prec S_{y_1 \ldots y_{n-1} x_n} \vee S_{y_1 \ldots y_n}$. Utilizing the (3) in Remark 1 and Definition 2, we may easily gain $S_{y_1 \ldots y_{n-1} x_n} \vee S_{y_1 \ldots y_n} = S_{y_1 \ldots y_{n-1} (x_n \vee y_n)}$. Thus, we arrive at $S_{y_1 \ldots y_{n-1} x_n}, S_{y_1 \ldots y_n} \prec S_{y_1 \ldots y_{n-1} (x_n \vee y_n)}$. Therefore, for any $p, q \in L \setminus (A(L) \cup 0)$, $x_n < p < x_n \vee y_n$ and $y_n < q < x_n \vee y_n$, there are $s, t \in \{1, \ldots, n-1\}$ satisfying $y_s \wedge p \neq 0$ and $y_t \wedge q \neq 0$.

Analogously to the proof above, we may easily obtain the "additionally" part in item (3.16.1).

Step 9.2. To prove item (3.16.2).

Using item (3.16.1) and the induction on $n - (m+1)$, similarly to the proof in Step 9.1, we may easily obtain item (3.16.2). □

Remark 2. In the proof of Step 7 for Lemma 6, for the case "when $m = n-1$ and $h(y_n) = 2$", we may use item (3.5) to obtain the same result.

But, we think that the proof in Lemma 6 for this case is useful to prove the other cases.

Third, we will reveal under what conditions, $Cl_s(L)$ is geometric for an atomistic complete finite lattice L.

Lemma 7. *Let L be an atomistic complete finite lattice. If L satisfies items from (3.8) to (3.16), then $Cl_s(L)$ is a geometric lattice.*

Proof of Lemma 7. Applying the information in Subsection 2.1 and Lemma 3, we only need to prove that $Cl_s(L)$ is atomistic and semimodular.

According to Lemma 4, we may be assured $A(Cl_s(L)) = \{S_d \mid d \in \mathcal{F}^2\}$.

Step 1. We prove that every element in $Cl_s(L)$ is a join of atoms using the following Step 1.1 and Step 1.2.

Step 1.1. To prove: $S_x = \vee_{d \in \mathcal{F}^2(x)} S_d$ for any $x \in L \setminus (A(L) \cup 0)$.

Let $x \in \mathcal{F}^2$. The needed result is easily followed.

Let $x \in L \setminus (A(L) \cup 0)$ and $h(x) = 3$. By items (3.8) and (3.11), we find $2 \leq |\mathcal{F}^2(x)|$. Let $d_1, d_2 \in \mathcal{F}^2(x)$ and $d_1 \wedge d_2 \neq 0$. Then in light of item (3.3), we may obtain $S_{d_1} \vee S_{d_2} = S_{d_1 \vee d_2} = S_x$. Furthermore, we may decide $S_x = \vee_{d \in \mathcal{F}^2(x)} S_d$ since $S_d \leq S_x$ for any $d \in \mathcal{F}^2(x)$.

Suppose that there is $S_x = \vee_{d \in \mathcal{F}^2(x)} S_d$ for any $x \in L \setminus (A(L) \cup 0)$ and $h(x) \leq n-1$.

Let $x \in L \setminus (A(L) \cup 0)$ with $h(x) = n$. Let $\mathcal{C}(x) = \{y_i \mid i \in \mathcal{I}\}$, that is, $\mathcal{C}(x)$ is the set of elements covered by x in L.

In view of item (3.8), we produce $x = \vee_{i \in \mathcal{I}} y_i$. Since L satisfies item (3.12). We obtain $S_{y_1} \vee S_{y_2} = S_{y_1 \vee y_2} = S_x$ using item (3.3), where $y_1, y_2 \in \mathcal{C}(x) \setminus A(L), y_1 \neq y_2$ and $y_1 \wedge y_2 \neq 0$. On the other hand, by item (3.1), we confirm $S_y \leq S_x$ for any $y \in \mathcal{C}(x) \setminus A(L)$.

Moreover, combining the above results, we attain $S_x = \vee_{y \in \mathcal{C}(x)} S_y$.

Considered $y \in \mathcal{C}(x)$ with $h(x) = n$, we find $h(y) \leq n-1$. Using inductive supposition, we will obtain $S_y = \vee_{d \in \mathcal{F}^2(y)} S_d$. Additionally, we may easily find that if $d \in \mathcal{F}^2(x)$, then $d \in \mathcal{C}(x) \setminus A(L)$ or $d \leq y_d$ for some $y_d \in \mathcal{C}(x)$. That is to say, $d \in \mathcal{F}^2(y)$ holds for some $y \in \mathcal{C}(x) \setminus A(L)$. Therefore, we arrive at $S_x = \vee_{d \in \mathcal{F}^2(x)} S_d$.

Step 1.2. To prove: $S_{x_i, i \in T} \in Cl_s(L)$ is a join of atoms where $|T| \geq 2$.

Let $S = S_{x_t, t \in \mathcal{T}} \in Cl_s(L)$ with $2 \leq |\mathcal{T}|$. Using (2) of Definition 1 and item (3.2), we may find $S = \vee_{t \in \mathcal{T}} S_{x_t}$. For every S_{x_t}, applying with Step 1.1, we may obtain $S_{x_t} = \vee_{d_j \in \mathcal{F}^2(x_t)} S_{d_j}$ where $\mathcal{F}^2(x_t) = \{d_j \mid j \in \mathcal{J}_t\}, (t \in \mathcal{T})$. Therefore, we attain $S = \vee_{t \in \mathcal{T}} \vee_{j \in \mathcal{J}_t} S_{d_j}$.

Step 2. We prove that $Cl_s(L)$ is semimodular.

Let $S_2, S_3 \in Cl_s(L), S_2||S_3$ and $S_2 \wedge S_3 \prec S_2, S_3$. According to the (3) of Remark 1, we may point that there are $S_2 = S_x$ or $S_2 = S_{x_j, j \in \mathcal{J}}$, and in addition, $S_3 = S_y$ or $S_3 = S_{y_i, i \in \mathcal{I}}$, for some $x, y \in L \setminus 0$ and $x_j, y_i \in L \setminus (A(L) \cup 0), (j \in \mathcal{J}, i \in \mathcal{I})$ with $|\mathcal{J}|, |\mathcal{I}| \geq 2$. Based on this statement, we will divide different cases to prove $S_2, S_3 \prec S_2 \vee S_3$ by the following Steps 2.1, 2.2 and 2.3.

Step 2.1. Assume $S_2 = S_x$ and $S_3 = S_y$ for some $x, y \in L \setminus 0$.

Since $S_0 \leq S_2 \wedge S_3 \prec S_2, S_3$ and $S_2 \wedge S_3 \in Cl_s(L)$, we may affirm $x, y \in L \setminus (A(L) \cup 0)$.

We will distinguish two cases to fulfill the proof.

Case 1. Suppose $S_2 \wedge S_3 = S_0$.

Then, $S_x, S_y \in A(Cl_s(L))$ holds. Thus, by item (3.4), we believe $x, y \in \mathcal{F}^2$.

When $x \wedge y = 0$. Using item (3.2), we obtain $S_x, S_y < S_{xy} = S_x \vee S_y$. If $S_x < S < S_{xy}$ holds for some $S \in Cl_s(L)$, then $S = S_{xp}$ holds where $p \in L \setminus (A(L) \cup 0)$ and $p < y$. However, $h(y) = 2$ and $p < y$ taken together follows $p \in A(L) \cup 0$. This is a contradiction to $p \in L \setminus (A(L) \cup 0)$. Thus, $S_x \prec S_{xy}$ holds. Analogously, $S_y \prec S_{xy}$ is true.

When $x \wedge y \neq 0$. Using item (3.3), we affirm $S_x \vee S_y = S_{x \vee y}$. In light of $x, y \in \mathcal{F}^2$ and $S_2 \wedge S_3 = S_0$, we find $x \wedge y \in A(L)$. So, $x \wedge y \prec x, y$ is followed. Using item (3.10), we may affirm $x, y \prec x \vee y$. Moreover, $h(x \vee y) = h(x) + 1 = 3$ holds. Hence, it follows $S_x, S_y \prec S_{x \vee y}$. Therefore, we may obtain $S_x, S_y \prec S_x \vee S_y$.

Case 2. Suppose $S_x \wedge S_y \neq S_0$.

Then, $3 \leq h(x), h(y)$ hold since $S_x \wedge S_y \prec S_x, S_y$.

By item (3.1), it is easily to find $S_{x \wedge y} \leq S_x \wedge S_y$.

We will demonstrate $S_x \wedge S_y = S_{x \wedge y}$.

Otherwise, $S_{x \wedge y} < S_x \wedge S_y$ holds. If $S_x \wedge S_y = S_z$ for some $z \in L$, then $x \wedge y < z < x, y$. According to item (3.1), we may achieve $S_{x \wedge y} < S_z < S_x, S_y$, a contradiction to $S_{x \wedge y} \prec S_x, S_y$. That is to say, we attain $S_x \wedge S_y = S_{z_i, i \in \mathcal{I}} \in Cl_s(L)$ and $2 \leq |\mathcal{I}|$. By the (2) of Definition 1, the (2) of Lemma 2, and Definition 2, we may obtain $A(x \wedge y) \subseteq A(z_{i_0})$ for some $i_0 \in \mathcal{I}$. Considering this result with the atomistic property of L and $2 \leq |\mathcal{I}|$, we provide $x \wedge y \leq z_{i_0} < \vee_{i \in \mathcal{I}} z_i \leq x, y$, and further, $S_{x \wedge y} \leq S_{z_{i_0}} < S_{\vee_{i \in \mathcal{I}} z_i} < S_x$, or $S_{x \wedge y} \leq S_{z_{i_0}} < S_{\vee_{i \in \mathcal{I}} z_i} \leq S_y$ since $S_x \neq S_y$. No matter which of the above cases to happen, it causes a contradiction to $S_{x \wedge y} \prec S_x$ or $S_{x \wedge y} \prec S_y$.

In other words, $S_x \wedge S_y = S_{x \wedge y}$ is real.

In fact, $S_x \wedge S_y \neq S_0$ and $S_x \wedge S_y = S_{x \wedge y}$ taken together infers to $h(x \wedge y) \geq 2$.

Moreover, $x \wedge y \prec x, y$ holds according to $S_{x \wedge y} = S_x \wedge S_y \prec S_x, S_y$. Combining with item (3.10), we may get $x, y \prec x \vee y$. Hence, since L satisfies item (3.13) and there are $S_{x \wedge y} \prec S_x, S_y$, we may determine $S_x, S_y \prec S_{x \vee y}$. Additionally, $S_0 \neq S_x \wedge S_y$ and $S_x \wedge S_y = S_{x \wedge y}$ follow $x \wedge y \neq 0$. According to item (3.3), we find $S_{x \vee y} = S_x \vee S_y$.

Therefore, we decide $S_x, S_y \prec S_x \vee S_y$.

Step 2.2. Assume $S_2 = S_x$ for some $x \in L \setminus 0$ and $S_3 = \{y_j \mid j \in \mathcal{Y}\} \cup \{a \in A(L) \mid a \nleq y_j, \forall j \in \mathcal{Y}\} \in Cl_s(L)$ with $2 \leq |\mathcal{Y}|$.

Combining items (3.5) and (3.14) with $S_2 \wedge S_3 \prec S_3$, we may state that there is $\mathcal{Y}_3 \subset \mathcal{Y}$ satisfying $|\mathcal{Y}_3| + 1 = |\mathcal{Y}|$ and $S_2 \wedge S_3 \geq \{y_j \mid j \in \mathcal{Y}_3\} \cup \{a \in A(L) \mid a \nleq y_j, \forall j \in \mathcal{Y}_3\}$. According to $S_2 \wedge S_3 \prec S_x$, we may indicate $x \geq \vee_{j \in \mathcal{Y}_3} y_j$. No matter to assume $\mathcal{Y} = \{1, 2, \ldots, n\}$ and $\mathcal{Y}_3 = \{1, 2, \ldots, n-1\}$. In view of item (3.14) and $S_2 \wedge S_3 \prec S_{y_1 \ldots y_n} = S_3$, we may obtain that

if $h(y_n) = 2$, then $S_2 \wedge S_3 = S_{y_1 \ldots y_{n-1}}$;

if $h(y_n) \geq 3$, then $S_2 \wedge S_3 = S_{y_1 \ldots y_{n-1} z_n}$ where $z_n \prec y_n$ and $z_n \in L \setminus (A(L) \cup 0)$.

Suppose $h(y_n) = 2$. Then $S_{y_1 \ldots y_{n-1}} \prec S_x, S_{y_1 \ldots y_n}$ produce that x, which $x = \vee_{j=1}^{n} y_j$, satisfies: for any $p \in L \setminus (A(L) \cup 0)$ with $\vee_{j=1}^{m} y_{i_j} \leq p < x$, there is $y_s \wedge p \neq 0$ for some $s \in \{1, \ldots, n-1\} \setminus \{i_j \mid j = 1, \ldots, m\}$ where $\{i_j \mid j = 1, \ldots, m\} \subseteq \{1, \ldots, n-1\}$. We will use the following two statuses to fulfill the proof.

Status 1. When $x \wedge y_n = 0$.

Then, $S_{xy_n} \in Cl_s(L)$ is obtained from the (2) of Definition 1. Hence, it follows $S_x, S_{y_1 \ldots y_n} \leq S_{xy_n}$. Moreover, we may be assured $S_x \vee S_{y_1 \ldots y_n} \leq S_{xy_n}$.

Considering the (3) of Remark 1 and $S_{y_1} < S_x < S_x \vee S_{y_1 \ldots y_n}$, we may suppose $S_x \vee S_{y_1 \ldots y_n} = S_b$ for some $b \in L \setminus (A(L) \cup 0)$. Evidently, this supposition will cause $S_{xy_n} \leq S_b$. Thus, we decide $S_{xy_n} = S_x \vee S_{y_1 \ldots y_n} = S_b$. This is a contradiction to $x \neq y_n$ and $|\{b\}| = 1$.

On the other hand, we may also suppose $S_x \vee S_{y_1 \ldots y_n} = S_{z_1 \ldots z_m}$ where $2 \leq m$ according to the (3) of Remark 1. This supposition will cause $x \leq z_{t_1}$ and $y_n \leq z_{t_2}$ in which $z_{t_1}, z_{t_2} \in \{z_1, \ldots, z_m\}$. Considered both of the (1) and (2) in Lemma 2 with Definition 2, we may determine $S_{xy_n} \leq S_{z_{t_1} z_{t_2}} \leq S_{z_1 \ldots z_m}$. Thus, it follows $S_{xy_n} = S_{z_{t_1} z_{t_2}} = S_x \vee S_{y_1 \ldots y_n}$. If $z_{t_1} = z_{t_2}$, then $S_{xy_n} = S_{z_{t_1}}$. This expression transfers to the above case. But the above case shows that this expression is wrong. In other words, $z_{t_1} \neq z_{t_2}$ is true.

Therefore, we obtain $S_x \vee S_{y_1 \ldots y_n} = S_{xy_n}$.

If $S_x < S < S_{xy_n}$ for some $S \in Cl_s(L)$. Then $x \in S$ and $b_n \in S$, where $b_n \in L \setminus (A(L) \cup 0)$ and $b < y_n$. However, $h(y_n) = 2$ follows the non-existence of b_n. Thus, there does not exist $S \in Cl_s(L)$ satisfying $S_x < S < S_{xy_n}$. That is to say, $S_x \prec S_{xy_n}$ holds.

If $S_{y_1 \ldots y_n} < S < S_{xy_n}$ for some $S \in Cl_s(L)$. Then $y_n \in S$, $\vee_{t=1}^{u} y_{j_t} \leq q \in S, \{j_t \mid t = 1, \ldots, u\} \subseteq \{1, \ldots, n-1\}, 2 \leq |\{j_t \mid t = 1, \ldots, u\}|$ and $A(y_j) \subseteq A(z_{j_p})$ for $z_{j_p} \in S, (j \in \{1, \ldots, n-1\} \setminus \{j_t \mid t = 1, \ldots, u\}$ and $|\{j_p \mid p = 1, \ldots, v\}| \leq |\{1, \ldots, n-1\} \setminus \{j_t \mid t = 1, \ldots, u\}|)$. In addition, we easily find $(\vee_{t=1}^{u} y_{j_t}) \wedge z_{j_p} = 0$ for every $z_{j_p} \in S$, $(p = 1, \ldots, v)$. However, we already know that there exists $s \in \{1, \ldots, n-1\} \setminus \{j_t \mid t = 1, \ldots, u\}$ satisfying $(\vee_{t=1}^{u} y_{j_t}) \wedge y_s \neq 0$. That is to say, it does not exist $S \in Cl_s(L)$ satisfying $S_{y_1 \ldots y_n} < S < S_{xy_n}$. Thus, there is $S_{y_1 \ldots y_n} \prec S_{xy_n} = S_x \vee S_{y_1 \ldots y_n}$.

Status 2. When $x \wedge y_n \neq 0$.

Actually, using item (3.15.2), we will attain $S_{x\vee y_n} = S_x \vee S_{y_1\ldots y_n}$ and $S_x, S_{y_1\ldots y_n} \prec S_{x\vee y_n}$.

Suppose $h(y_n) \neq 2$. Then, $S_2 \wedge S_3 = S_{y_1\ldots y_{n-1}z_n}$ is true where $z_n \prec y_n$. Additionally, $S_2 \wedge S_3 \prec S_x$ follows $x = \vee_{j=1}^{n-1} y_j \vee z_n$. So, $z_n = x \wedge y_n \in L \setminus (A(L) \cup 0)$ holds. This implies $x \wedge y_n \neq 0$.

We will prove $S_x \vee S_{y_1\ldots y_n} = S_{x\vee y_n}$.

It is easily found $S_x, S_{y_1\ldots y_n} \leq S_{x\vee y_n}$ since item (3.1). Thus, we obtain $S_x \vee S_{y_1\ldots y_n} \leq S_{x\vee y_n}$. Assume $S_x \vee S_{y_1\ldots y_n} = S_{b_1\ldots b_m}$ for $b_i \in L \setminus (A(L) \cup 0), (i = 1, \ldots, m)$. By item (3.1), it follows $y_j \leq x < b_1, (j = 1, \ldots, n-1)$ and $y_n \leq b_t$ for some $t \in \{1, \ldots, m\}$. If $t \neq 1$, then $0 \leq x \wedge y_n \leq b_1 \wedge b_t = 0$. So, $x \wedge y_n = 0$ is followed. This causes a contradiction to $x \wedge y_n \neq 0$. Moreover, we gain $y_j \leq x \leq b_1$ and $y_n \leq b_1$, $(j = 1, \ldots, n-1)$. Thus, $m = 1$ holds. In addition, according to item (3.1) and $x \vee y_n \leq b_1$, we may present $S_x \vee S_{y_1\ldots y_n} \leq S_{x\vee y_n} \leq S_{b_1}$. Hence, $S_{x\vee y_n} = S_x \vee S_{y_1\ldots y_n}$ holds.

Next, according to $S_{y_1\ldots y_{n-1}z_n} \prec S_x, S_{y_1\ldots y_n}$, we may indicate that for any $\vee_{j=1}^{t} y_{i_j} \leq p < x$ or $\vee_{j=1}^{t} y_{i_j} \vee z_n \leq q < x$ where $\{i_j \mid j = 1, \ldots, t\} \subseteq \{1, \ldots, n-1\}$, there exists $s \in \{1, \ldots, n-1\} \setminus \{i_j \mid j = 1, \ldots, t\}$ satisfying $y_s \wedge p \neq 0$ or $z_n \wedge p \neq 0$ or $y_s \wedge q \neq 0$. Using items (3.15.1) and (3.1), it follows $S_2, S_3 \prec S_2 \vee S_3$.

Step 2.3. Let $S_2 = S_{x_1 x_2 \ldots x_m}$ and $S_3 = S_{y_1 y_2 \ldots y_n}$. By item (3.5) and $S_2 \wedge S_3 \prec S_2, S_3$, we follow $S_2 \wedge S_3 \geq S_{y_{i_1}\ldots y_{i_{n-1}}} = S_{x_{j_1}\ldots x_{j_{m-1}}} = S_{\{y_1,\ldots,y_n\}\cap\{x_1,\ldots,x_m\}}$ and $|\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\}| = n-1$. No matter to suppose $\{y_1, \ldots, y_n\} \cap \{x_1, \ldots, x_m\} = \{y_1, y_2, \ldots, y_{n-1}\}$. Thus, we may reveal $n = m, x_m \neq y_n$, and $S_2 = S_{y_1 y_2 \ldots y_{n-1} x_n}$, $S_3 = S_{y_1 y_2 \ldots y_{n-1} y_n}$.

We will demonstrate

$$h(x_n) = 2 \text{ if and only if } h(y_n) = 2.$$

In fact, if $h(y_n) = 2$, then by item (3.14) and $S_2 \wedge S_3 \prec S_{y_1\ldots y_n}$, there is $S_2 \wedge S_3 = S_{y_1\ldots y_{n-1}}$. Meanwhile, $S_{y_1\ldots y_{n-1}} = S_2 \wedge S_3 \prec S_2 = S_{y_1\ldots y_{n-1}x_n}$ follows $h(x_n) = 2$ by item (3.14).

Similarly, if $h(x_n) = 2$, then $h(y_n) = 2$.

Additionally, if $h(y_n) = 2$, then according to items (3.5) and (3.14) and $h(x_n) = 2$, we obtain $S_2 \wedge S_3 = S_{y_1\ldots y_{n-1}} \prec S_{y_1\ldots y_{n-1}x_n}, S_{y_1\ldots y_{n-1}y_n}$ holds.

When $h(y_n) = 2$ and $x_n \wedge y_n = 0$. We will prove $S_{y_1\ldots y_n x_n} = S_{y_1\ldots y_n} \vee S_{y_1\ldots y_{n-1}x_n} = S_2 \vee S_3$.

Suppose $S \in Cl_s(L)$ and $S_{y_1\ldots y_{n-1}x_n} < S < S_{y_1\ldots y_n x_n}$. Then, by $y_p \wedge y_q = 0, x_n \wedge y_p = 0, (p \neq q; p, q = 1, \ldots, n)$, it follows $y_j \in S, (j = 1, \ldots, n-1)$ and $x_n \in S$. Additionally, by the supposition, we decide that there is $z \in L \setminus (A(L) \cup 0)$ with $z < y_n$ satisfying $z \wedge y_j = z \wedge x_n = 0, (j = 1, \ldots, n-1)$. This implies $2 \leq h(z) < h(y_n)$, a contradiction to $h(y_n) = 2$. Moreover, $S_{y_1\ldots y_{n-1}x_n} \prec S_{y_1\ldots y_n x_n}$ is real. Analogously, $S_{y_1\ldots y_n} \prec S_{y_1\ldots y_n x_n}$ is true. Hence, $S_{y_1\ldots y_n} \vee S_{y_1\ldots y_{n-1}x_n} = S_{y_1\ldots y_n x_n}$ is true and $S_{y_1\ldots y_n}, S_{y_1\ldots y_{n-1}x_n} \prec S_{y_1\ldots y_n} \vee S_{y_1\ldots y_{n-1}x_n}$ holds.

When $h(y_n) = 2$ and $x_n \wedge y_n \neq 0$.

Since $h(x_n) = h(y_n) = 2$ and $x_n \wedge y_n \neq 0$ taken together follows $x_n \wedge y_n \prec x_n, y_n$ and $x_n \wedge y_n \in A(L)$.

We prove $S_{y_1 \dots y_{n-1}x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_{n-1}(x_n \vee y_n)}$ if $y_j \wedge (x_n \vee y_n) = 0, (j = 1, \dots, n-1)$.

Using the (3) in Remark 1, item (3.1) and Definition 1, it is easily to obtain $S_{y_1 \dots y_{n-1}x_n} \vee S_{y_1 \dots y_n} = S_{z_1 \dots z_m}, (z_i \in L \setminus (A(L) \cup 0); i = 1, \dots, m)$. According to $z_t \wedge z_s = 0, (t, s = 1, \dots, m; t \neq s)$, we follow $y_j = z_j, (j = 1, \dots, n-1)$. In view of $x_n \wedge y_n \neq 0$, we attain $x_n, y_n \leq z_n$. And further, we gain $x_n \vee y_n \leq z_n$. Moreover, we decide $S_{y_1 \dots y_{n-1}(x_n \vee y_n)} \leq S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1}x_n}$. On the other hand, combining $x_n, y_n \leq x_n \vee y_n$ with items (3.1) and (3.2), we may easily arrive at $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1}x_n} \leq S_{y_1 \dots y_{n-1}(x_n \vee y_n)}$. So,

$$S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1}x_n} \leq S_{y_1 \dots y_{n-1}(x_n \vee y_n)}$$

is followed.

Combining the above, we affirm $S_{y_1 \dots y_{n-1}x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_{n-1}(x_n \vee y_n)}$.

Using the "Additionally" part in item (3.16.1), we obtain $S_{y_1 \dots y_{n-1}x_n}, S_{y_1 \dots y_n} \prec S_{y_1 \dots y_{n-1}(x_n \vee y_n)}$. Furthermore, we find $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1}x_n} \prec S_{y_1 \dots y_{n-1}x_n} \vee S_{y_1 \dots y_n}$.

If $y_j \wedge (x_n \vee \vee_{i=m+1}^{n} y_i) = 0, (j = 1, \dots, m)$. Then, using induction on $n-(m+1)$ and the above proof for the case of $y_j \wedge (x_n \vee y_n) = 0, (j = 1, \dots, n-1)$, we may gain $S_{y_1 \dots y_{n-1}x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_m(x_n \vee \vee_{i=m+1}^{n} y_i)}$. Applying the "Additionally" part in item (3.16.2), we gain $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1}x_n} \prec S_{y_1 \dots y_{n-1}x_n} \vee S_{y_1 \dots y_n}$.

When $h(y_n) > 2$. Applying with the result above, we find $h(x_n) > 2$. In virtue of $S_2 \wedge S_3 \prec S_{y_1 \dots y_n} = S_3$ and item (3.14), we obtain $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1}z_n}$ in which $z_n \prec y_n$ and $z_n \in L \setminus (A(L) \cup 0)$. At the same time, we may attain $z_n \prec x_n$ since $S_2 \wedge S_3 \prec S_{y_1 \dots y_{n-1}x_n}$. This follows $z_n = x_n \wedge y_n \in L \setminus (A(L) \cup 0)$. Furthermore, we reveal $S_{y_1 \dots y_{n-1}z_n} \prec S_{y_1 \dots y_{n-1}x_n}, S_{y_1 \dots y_n}$. Thus, for any $\vee_{i=1}^{m_x} y_{j_i} \leq p < x_n, \vee_{i=1}^{m_y} y_{k_i} \leq q < y_n$, and $\{j_i \mid i = 1, \dots m_x\}, \{k_i \mid i = 1, \dots, m_y\} \subseteq \{1, \dots, n-1\}$, there is $s_x \in \{1, \dots, n-1\} \setminus \{j_i \mid i = 1, \dots m_x\}, s_y \in \{1, \dots, n-1\} \setminus \{k_i \mid i = 1, \dots, m_y\}$ satisfying $y_{s_x} \wedge p \neq 0, y_{s_y} \wedge q \neq 0$. Owing to items (3.16.2) and (3.1), we confirm $S_2, S_3 \prec S_2 \vee S_3 = S_{y_1 \dots y_m(x_n \vee \vee_{j=m+1}^{n} y_j)}$. □

Combining Lemma 6 with Lemma 7, we may express the following theorem.

Theorem 2. *Let L be an atomistic complete finite lattice with $|\mathcal{F}^2| \geq 2$. Then $Cl_s(L)$ is a geometric lattice if and only if L satisfies items from (3.8) to (3.16).*

4. Conclusion

To sum up our results, we make the following remarks.

(1) Though some of conditions in items from (3.8) to (3.16) seem to be complex, they are actually expressed in a detailed and applicable way. In addition they complete the check process, since items from (3.8) to (3.16) are suitable for finite cases. Additionally, from Lemma 7, or from the results of Section 2, we can confirm that items from (3.8) to (3.16) are necessary and essential when we decide the geometry of $Cl_s(L)$ for an atomistic complete finite lattice L.

(2) It is well known that an atomistic complete lattice is finite or infinite. Theorem 1 and Theorem 2 together answer the open problem of S.Radeleczki for finite cases. In fact, Theorem 1 is also true for infinite atomistic complete lattice. However, many preparatory works for Theorem 2 of this paper are proved with inductive method. This illustrates that Theorem 2 cannot be directly generalized to the infinite case. Even though, we may hope that the results of this paper will assist the solution of S.Radeleczki's open problem for infinite cases. We intend to pursue this line of research in the future.

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