CHARACTERIZATIONS OF ATOMISTIC COMPLETE FINITE LATTICES RELATIVE TO GEOMETRIC ONES

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Abstract. S. Radeleczki in 2002 raised the open problem of the characterization of those atomistic complete lattices whose classification lattices are geometric. This paper solves the problem for the finite cases.

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1. INTRODUCTION

S. Radeleczki et al. pointed out [1,2,6,7] that the notion of the classification system can be applied in concept lattices. The dual of this notion is introduced by R. Wille [3, 4, 8]. In addition, S. Radeleczki et al. [1, 2, 6, 7] apply and study the properties of concept lattices in the process of construction of classification systems. Since any partition lattice is a particular geometric lattice, an open problem was arisen naturally (S. Radeleczki [7]): Characterize those atomistic complete lattices whose classification lattices are geometric. The solution of this open problem might be useful in the study of classification systems and concept lattices. Hence, this paper will characterize atomistic complete finite lattices whose classification lattices are geometric and answer the open problem for the finite case.

This paper is organized as follows. Section 2 presents some basic information relative to geometric lattices and classification systems. In Section 3, we describe certain properties on atomistic complete finite lattices related to geometric lattices. Afterwards, it answers the open problem suggested by S. Radeleczki [7] for the finite case.

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2. Preliminaries

Some basic notions and results related to posets, lattices and classification systems are presented in this section.

2.1. Posets and lattices

We review some basic properties and notations of posets and lattices in this subsection. For more detail about posets and lattices, please refer to [5].

Lemma 1. (1) A lattice $L$ is semimodular if and only if $x \land y < x$ implies that $y < x \lor y$.

(2) A lattice $L$ is called geometric if and only if $L$ is complete, atomistic (that is, every element of $L$ is a join of atoms), all atoms are compact, and $L$ is semimodular.

(3) Any interval $[y, x] = \{a \in L \mid y \leq a \leq x\}$ of a geometric lattice $L$ is again a geometric lattice.

We also need the following statements from [5].

(1) A finite lattice $L$ is semimodular if for all $x, y \in L$: the property which $x$ and $y$ cover $x \land y$ implies that $x \lor y$ covers $x$ and $y$.

(2) A finite lattice is geometric if it is semimodular and every element is a join of atoms.

(3) A finite semimodular lattice is characterized by the following property $L$ is semimodular if and only if for all $x, y \in L$, all maximal chains between elements $x, y$ have the same length, and the height function $h$ of $L$ satisfies $h(x) + h(y) \geq h(x \land y) + h(x \lor y)$.

Some notations 1. Let $(P, \leq)$ be a poset. In this paper, if there is no confusion in the text, then we use the notation $P$ in stead of $(P, \leq)$. If $P$ has the greatest element $1$, then the height $h(1)$ is sometimes denoted by $h(P)$. In $P$, if $y$ covers $x$, it is in notation $x < y$; if $y$ does not cover $x$, it is in notation $x \not< y$; if $y \leq x$ and $y \not= x$, it is in notation $y < x$; if $y$ is not less than $x$, it is denoted by $y \not< x$; if $x$ and $y$ are incomparable, it is denoted by $x||y$.

2.2. Classification systems

Let us recall some information of classification systems from [7]. For more detail for classification systems, please see [7].

Definition 1. (1) A nonzero element $p$ of a lattice $L$ is called completely join-irreducible if for any system of elements $x_i \in L \ (i \in I)$, the equality $p = \lor\{x_i : i \in I\}$ implies $p = x_{i_0}$ for some $i_0 \in I$. If any nonzero element of $L$ is a join of completely join-irreducible elements, then $L$ is called a CJ-generated lattice.
(2) Let \( L \) be a complete lattice. A set \( S = \{a_j \mid j \in J\} (J \neq \emptyset) \) of nonzero elements of \( L \) is called a classification system of \( L \) if the following conditions are satisfied:

(2.1) \( a_i \land a_j = 0 \), for all \( i \neq j \), where 0 is the least element in \( L \).

(2.2) \( x = \bigvee_{j \in J} (x \land a_j) \), for all \( x \in L \).

Some notations 2. (1) Let \( L \) be a CJ-generated complete lattice. The set of all completely join-irreducible elements of \( L \) is denoted by \( J(L) \). For \( a \in L \), let \( J(a) = \{p \in J(L) \mid p \leq a\} \) and set \( \lor \emptyset = 0 \).

(2) Let \( A(L) \) denote the set of atoms of a lattice \( L \).

**Lemma 2.** (1) If \( S = \{a_i \mid i \in I\} \) is a classification system of \( L \), then \( \pi_S = \{J(a_i) \mid i \in I\} \) is a partition induced by \( S \) on \( J(L) \).

(2) If \( L \) is a CJ-generated lattice, then any classification system \( S = \{a_i \mid i \in I\} \) of \( L \) is determined by the partition \( \pi_S \), induced by \( S \) on \( J(L) \).

(3) Any atomistic lattice \( L \) is a CJ-generated lattice with \( J(L) = A(L) \).

Let \( L \) be an atomistic complete lattice. For any \( x \in L \setminus \{0\} \), the set \( S_x = \{x\} \cup \{a \in A(L) \mid a \land x = 0\} \) is a classification system of \( L \).

(4) Let \( L \) be a complete lattice and \( 1 \) be the greatest element in \( L \). Then \( S = \{1\} \) is a classification system.

**Definition 2.** Let \( L \) be a CJ-generated complete lattice and let \( S_p \) and \( S_q \) be two classification systems of \( L \). We say that the system \( S_p \) is finer than \( S_q \) and we write \( S_p \subseteq S_q \), if the partition \( \pi_{S_p} \) induced by \( S_p \) refines the partition \( \pi_{S_q} \) induced by \( S_q \), that is, if \( \pi_{S_p} \leq \pi_{S_q} \).

**Lemma 3.** (1) Let \( Cl_s(L) \) denote the set of all classification systems of a CJ-generated complete lattice \( L \). Then, \((Cl_s(L), \subseteq)\) is a complete lattice.

\((Cl_s(L), \subseteq)\) or \( Cl_s(L) \) for short, is called the classification lattice of the lattice \( L \).

(2) The least element of the lattice \( Cl_s(L) \), that is, the finest classification system of \( L \), is the same as \( S_0 = \wedge\{S \mid S \in Cl_s(L)\} \).

**Remark 1.** Let \( L \) be an atomistic complete lattice.

(1) In this paper, \( S_0 \) stands for \( \{a \mid a \in A(L)\} \), \( S_1 \) is \( \{1\} \), and \( S_x \) represents \( \{x\} \cup \{a \in A(L) \mid a \not\leq x\} \) for any \( x \in L \setminus \{0\} \).

(2) For any \( x \in L \setminus \{0\} \), using the (3) in Lemma 1, we may easily know that \( S_x = \{x\} \cup \{a \in A(L) \mid a \land x = 0\} \) is a classification system of \( L \). In addition, in light of the (2) in Lemma 3, \( S_0 = \{a \mid a \in A(L)\} \in Cl_s(L) \) holds, and \( S_a = S_0 \) is true for any \( a \in A(L) \).

(3) Let \( S \in Cl_s(L) \). Then according to the (2) of Definition 1, there is \( S = S_x \) for some \( x \in L \setminus \{0\} \), or \( S = \{x_i \in L \setminus (A(L) \cup \{0\}) \mid i \in I\} \cup \{a \in A(L) \mid a \not\leq x_i, i \in I\} \) where \( x_p \land x_q = 0, (p \neq q; p, q \in I) \) and \( |I| \geq 2 \).
For convenient, if \( S = \{ x_i \in L \setminus (A(L) \cup 0) \mid i \in I \} \cup \{ a \in A(L) \mid a \not\leq x_i, i \in I \} \in Cl_s(L) \), then it is denoted by \( S_{x_i, i \in I} \). And further, when \( I = \{1, 2, \ldots, n\} \), \( S_{x_i, i \in I} \) is sometimes denoted as \( S_{x_1, x_2, \ldots, x_n} \).

(4) Let \( S = S_{x_1, x_2, \ldots, x_n} \). By Definition 1 and the (3) above, it is easily to find \( S = S_{x_1, x_2, \ldots, x_n} \) where \( i_1, i_2, \ldots, i_n \) is an arbitrary permutation of \( 1, 2, \ldots, n \).

(5) Let \( x \in L \setminus (A(L) \cup 0) \). By the discussion beyond, we easily obtain \( Cl_s([0, x]) = [S_0, S_1] \) in which \( [S_0, S_1] \) is an interval in \( Cl_s(L) \) and \( [0, x] \) is an interval in \( L \).

**Some notations 3.** Let \( L \) be an atomistic complete finite lattice.

(1) Let \( \mathcal{F}^k \) denote \( \{ x \in L \mid x \text{ has height } k \text{ in } L \} \), \( (k = 2, \ldots, h(L)) \).

(2) For any \( x \in L \) and \( l \leq h(x) \), let \( \mathcal{F}^l(x) = \{ y \in L \mid y \in \mathcal{F}^l \text{ and } y \leq x \} \) and \( \mathcal{C}(x) = \{ y \in L \mid x \text{ covers } y \text{ in } L \} \). \( \mathcal{F}^1 \) is also in notation \( A(x) = \{ a \in A(L) \mid a \leq x \} \) for any \( x \in L \).

### 3. Answer to the open problem

In this section, we will discover the characterizations of atomistic complete finite lattices whose classification lattices are geometric lattices. Therefore, the open problem, which is repeated in Section 1, will be solved for finite cases.

**Theorem 1.** Let \( L \) be an atomistic complete finite lattice. If \( |\mathcal{F}^2| = 1 \), i.e. \( L \) has only one element of height 2. Then \( Cl_s(L) \) is geometric.

**Proof of Theorem 1.** It is easily to find \( Cl_s(L) = \{ S_0, S_1 \}. \) Thus, \( Cl_s(L) \) is geometric.

In what follows, \( L \) always stands for an atomistic complete finite lattice with at least two elements of height 2. That is to say, \( \mathcal{F}^2 = \{ x \in L \mid x \text{ has height } 2 \text{ in } L \} \) satisfies \( |\mathcal{F}^2| \geq 2 \).

First, we may deal with some properties related to \( Cl_s(L) \).

**Lemma 4.** \( Cl_s(L) \) possesses the following properties.

(3.1) Let \( x, y \in L \setminus (A(L) \cup 0) \). Then \( x \leq y \Leftrightarrow S_x \leq S_y \). Further, \( x < y \Leftrightarrow S_x < S_y \).

(3.2) Let \( x, y \in L \setminus (A(L) \cup 0) \). If \( x \land y = 0 \), then \( S_x \lor S_y = S_{xy} \).

Furthermore, let \( x_i \in L \setminus (A(L) \cup 0), (j \in \mathcal{F}; |\mathcal{F}| \geq 2) \). If \( x_i \land x_j = 0, (i, j \in \mathcal{F} ; i \neq j) \), then \( \cup_{j \in \mathcal{F}} S_{x_j} = S_{x_i, j \in \mathcal{F}} \).

(3.3) Let \( x, y \in L \setminus (A(L) \cup 0) \). If \( x \land y \neq 0 \), then \( S_x \lor S_y = S_{x \lor y} \).

(3.4) \( A(Cl_s(L)) = \{ S_d \mid d \in \mathcal{F}^2 \} \).

(3.5) Let \( S = S_{x_i, i \in I} \) with \( 2 \leq |I| \). Let \( S' \in Cl_s(L) \). Then the following items (3.5.1), (3.5.2) and (3.5.3) are true.

(3.5.1) If \( S' = S_y \prec S \) holds for some \( y \in L \setminus (A(L) \cup 0) \), then there are \( I = \{1, 2\} \), \( y = x_1 \) and \( h(x_2) = 2 \).

(3.5.2) If \( S' \neq S_y \) for any \( y \in L \setminus (A(L) \cup 0) \), \( S' \prec S \), and \( h(x_n) = 2 \) with \( x_n \notin S' \).
in which \( n \in I \setminus I_1 \) for a \( I_1 \subseteq I \) with \( |I| = |I_1| + 1 \). Then \( S' = S_{x_1,j \in I_1} \).

(3.5.3) Let \( S' = S_{x_1,j \in I_1} \) where \( I_1 \subseteq I \) with \( |I| = |I_1| + 1 \). If \( S' \) satisfies \( h(x_n) = 2 \) where \( n \in I \setminus I_1 \). Then \( S' < S \) holds.

Proof of Lemma 4. We will demonstrate all the results step by step.

Step 1. From Definition 2 with the atomistic property of \( L \), we may easily obtain item (3.1).

Step 2. We verify item (3.2) using steps 2.1 and 2.2 as follows.

Step 2.1. To prove: for \( x, y \in L \setminus (A(L) \cup 0) \),

\[ x \wedge y = 0 \Rightarrow S_x \vee S_y = S_{xy}. \]

If \( x \wedge y = 0 \), then \( S_{xy} \in Cl_s(L) \) and \( S_x, S_y < S_{xy} \) according to Definition 2.

Let \( S \in Cl_s(L) \) satisfy \( S_x, S_y < S \). Using \( x \wedge y = 0 \) and Definition 2, we may find out \( A(x) \subseteq A(x_j) \) and \( A(y) \subseteq A(y_k) \) for some \( x_j, y_k \in S \). This causes \( S_{xy} \subseteq S \).

Considering \( S_x, S_y < S \) with \( S_{xy} \subseteq S \), we may be assured \( S_x \vee S_y = S_{xy} \).

Step 2.2. To prove: for \( x_j \in L \setminus (A(L) \cup 0) \), \( (j \in \mathcal{J}; |\mathcal{J}| \geq 2) \),

\[ x_j \wedge x_j = 0, (i, j \in \mathcal{J}; i \neq j) \Rightarrow \vee_{j \in \mathcal{J}} S_{x_j} = S_{x_j, j \in \mathcal{J}}. \]

Using induction on \(|\mathcal{J}|\) and item (3.1), we may obtain \( \vee_{j \in \mathcal{J}} S_{x_j} \leq S, j \in \mathcal{J} \) if \( x_j \wedge x_j = 0, (i, j \in \mathcal{J}; i \neq j) \).

By the (3) in Remark 1, \( \vee_{j \in \mathcal{J}} S_{x_j} = S_y \) holds for some \( y \in L \setminus (A(L) \cup 0) \), or \( \vee_{j \in \mathcal{J}} S_{x_j} = S_{z_1...z_m} \) holds for some \( z_1,...,z_m \in L \setminus (A(L) \cup 0) \) with \( 2 \leq m \).

Suppose that \( \vee_{j \in \mathcal{J}} S_{x_j} = S_y \) holds. By item (3.1) and \( S_{z_1,...z_m} \leq S_y \) is true by Definition 2. So, it follows \( S_y = S_{z_1,...z_m} \) with \(|\mathcal{J}| \geq 2 \), which is a contradiction to the (2) of Definition 1.

Suppose that \( \vee_{j \in \mathcal{J}} S_{x_j} = S_{z_1...z_m} \) holds for \( 2 \leq m \). Using item (3.1) and \( S_{x_j} \leq S_{z_1,...z_m} \) \( (j \in \mathcal{J}) \), we point out \( x_j \leq z_{j_i}, (j \in \mathcal{J}; j_i \in \{1,...,m\}) \). Utilizing \( z_{j_i} \wedge z_{j_i} = 0, (j \in \mathcal{J}; j_i \in \{1,...,m\}) \) and the result in the above case, we follow that if \( z_{j_i} \neq z_{j_i} \), then \( x_j \neq x_i, (i, j \in \mathcal{J}) \), where \( x_j \leq z_{j_i} \) and \( x_i \leq z_{j_i} \). This implies \( A(x_j) \subseteq A(z_{j_i}), (j \in \mathcal{J}; j_i \in \{1,...,m\}) \) since \( L \) is atomistic. Combining items (1) and (2) in Lemma 2 with Definition 2, \( \pi \sigma_{x_j, j \in \mathcal{J}} \leq \pi \sigma_{z_1...z_m} \) holds. Hence, \( \pi \sigma_{x_j, j \in \mathcal{J}} \leq \pi \sigma_{z_1...z_m} \) is true. Additionally, we decide \( S_{x_j} \leq S_{x_j, j \in \mathcal{J}} \) according to \( A(x_j) \subseteq A(x_j), (j \in \mathcal{J}) \), the definitions of \( S_{x_j, j \in \mathcal{J}} \) and \( S_{x_j} \), and Definition 2. Thus, it follows \( \vee_{j \in \mathcal{J}} S_{x_j} \leq S_{x_j, j \in \mathcal{J}} \leq S_{z_1...z_m} \). Therefore, we may be assured \( S_{x_j, j \in \mathcal{J}} \leq S_{x_j} \).

Step 3. To prove item (3.3).

Let \( x \wedge y \neq 0 \). Then \( \{x, y\} \notin S \) is true for any \( S \in Cl_s(L) \). If \( S_x \vee S_y = S_{z_1...z_n} \) for any \( z_j \in L \setminus (A(L) \cup 0), (j = 1, 2, ..., n; 2 \leq n) \). Then, the properties of \( x \leq y \leq z_2 \) and \( z_1 \neq z_2 \) taken together follows \( x \wedge y = 0 \). This is a contradiction to \( x \wedge y \neq 0 \). Hence, we confirm \( x, y \leq z_1 \). However, \( S_x, S_y \leq S_{z_1} < S_{z_1...z_n} = S_x \vee S_y \) follows \( S_x \vee S_y \leq S_{z_1} < S_{z_1...z_n} = S_x \vee S_y \), a contradiction. Thus, \( S_x \vee S_y = S_z \) is true for some \( z \in L \setminus (A(L) \cup 0) \). Then, we attain \( S_x, S_y \leq S_x \vee S_y \). Furthermore, we obtain \( x, y \leq z \) using item (3.1). Moreover, \( x \vee y \leq z \) holds. So, we determine \( S_x \vee y \leq z \).
On the other hand, in view of \(x, y \leq x \lor y\) and item (3.1), we obtain \(S_x, S_y \leq S_{x \lor y}\). Furthermore, we attain \(S_x \lor S_y \leq S_{x \lor y}\).

Therefore, we have demonstrated \(S_x \lor S_y = S_{x \lor y}\).

Step 4. To prove item (3.4).

Let \(d \in \mathcal{F}^2\). According to Remark 1, we obtain \(S_d \in C_l(L)\). By item (3.1), it has \(S_0 < S_d\). If \(S_0 < S \leq S_d\) holds for some \(S \in C_l(L)\). Then we obtain \(x \in S\) and \(x \in L \setminus (A(L) \cup 0)\). So, \(2 \leq h(x)\) holds. Furthermore, we find the existence of \(d_x \in \mathcal{F}^2(x)\). Thus, we may obtain \(S_0 < S_{d_x} \leq S \leq S_d\). Combining \(S_0 < S\) and \(S_0 < S_{d_x} \leq S\), we find \(S = S_{d_x}\). On the other hand, taking \(S \leq S_d, S_{d_x} \leq S\) with \(h(d_x) = h(d) = 2\) together, we follow \(d_x = d\). Therefore, we attain \(S_0 < S_d\). Hence, \(\{S_d \mid d \in \mathcal{F}^2\} \subseteq A(C_l(L))\) is true.

If \(A(C_l(L)) \setminus \{S_d \mid d \in \mathcal{F}^2\} \neq \emptyset\). Then, there is \(S \in A(C_l(L)) \setminus \{S_d \mid d \in \mathcal{F}^2\}\). This causes \(y \in \mathcal{F}^2(L) \cup 0\). So, \(2 \leq h(y)\) is real. Considered the definition of height function of a lattice in [5], we find \(d_y \in \mathcal{F}^2\) with \(d_y \leq y\). Moreover, we obtain \(S_{d_y} \leq S\). Hence, \(S_0 < S_{d_y}\) is known according to the above proof. Hence, it follows \(S = S_{d_y}\). This is a contradiction to \(S \neq \{S_d \mid d \in \mathcal{F}^2\}\). In other words, \(A(C_l(L)) = \{S_d \mid d \in \mathcal{F}^2\}\) is true.

Step 5. To prove item (3.5).

Combining \(S \in C_l(L)\) with Definition 1, we confirm \(\{x_i \mid i \in T\} \cup \{a \in A(L) \mid a \not\leq x_i, i \in T\} \in C_l(L)\) for any \(T \subseteq I\).

Step 5.1. This step verifies items (3.5.1) and (3.5.2).

If \(S' \prec S\). Then, by the (3) in Remark 1, we may find \(S' = S_y\) for some \(y \in L \setminus (A(L) \cup 0)\) or \(S' = S_{z_1z_2...z_m}\) with \(2 \leq m\) and \(z_i \in L \setminus (A(L) \cup 0), (i = 1, ..., m)\).

Suppose \(S' = S_y\) for some \(y \in L \setminus (A(L) \cup 0)\). Then, \(A(y) \subseteq A(x_i)\) holds for some \(i_0 \in I\). According to \(x_i \land x_j = 0, (i \neq j; i, j \in I)\), we confirm that there is one and only one \(i_0 \in I\) satisfying \(A(y) \subseteq A(x_{i_0})\). In addition, owing to the atomistic property of \(C_l(L)\), we may find \(y \leq x_{i_0}\). And further, we obtain \(S_y \leq S_{x_{i_0}}\) by item (3.1).

Under the assumption \(S' \prec S\) and the closest result above, we may infer to \(|I| = 2\). Otherwise, \(3 \leq |I|\) will follow \(S' = S_y \leq S_{x_{i_0}} \leq S_{x_{i_0}x_1} < S, x_i \in S\), a contradiction to \(S' \prec S\).

We may assume \(I = \{1, 2\}\) and \(x_{i_0} = x_1\). In view of \(S' = S_y \leq S_{x_1} < S, S_{x_1} \in C_l(L)\) and \(S' \prec S\), we find \(S_y = S_{x_1}\). Thus, \(y = x_1\) holds. Since \(x_2 \in L \setminus (A(L) \cup 0)\) causes \(h(x_2) \geq 2\). Assume \(h(x_2) > 2\). By the definition of height function in a lattice, we obtain \(t < x_2\) and \(h(t) = 2\). Additionally, \(0 \leq t \land x_1 \leq x_2 \land x_1 = 0\) yields \(t \land x_1 = 0\). So, \(S_{x_1} \in C_l(L)\) holds. We may easily determine \(S_{x_1} < S_{x_1} < S\), a contradiction to \(S' = S_{x_1} < S\). Therefore, we confirm \(h(x_2) = 2\).

Suppose \(S' = S_{z_1z_2...z_m}\) with \(2 \leq m\). No harm to assume \(I = \{1, 2, ..., n\}\). Taking \(S' = S_{z_1...z_m} < S = S_{x_1...x_n}\) and \(x_i \land x_j = 0, (i \neq j; i, j = 1, ..., n)\) together, we find \(z_{i\alpha} \leq x_{j\alpha}, (\alpha_1 \in I_1), ..., z_{i\alpha} \leq x_{j\alpha}, (\alpha_i \in I_i)\) in which \(\{i_{\alpha_1}, | \alpha_1 \in I_1\} \cup \{i_{\alpha_2}, | \alpha_2 \in I_2\} \cup \ldots \cup \{i_{\alpha_\ell}, | \alpha_\ell \in I_\ell\} = \{1, ..., m\}\).
\( \alpha_k \in I_k \cap \{ \alpha_l \mid \alpha_l \in I_l \} = \emptyset, (k \neq l; k, l = 1, \ldots, p); \{ j_1, j_2, \ldots, j_p \} \subseteq \{ 1, \ldots, n \} \) and \( \beta \neq j; (\alpha \neq \beta; \alpha, \beta = 1, \ldots, p) \). Thus, by the (4) of Remark 1, we may be assured \( S_{z_1 \cdots z_m} = S_{z_{l_1} \cdots z_{l_p}, l_p \in I_p} \leq S_{x_{j_1} \cdots x_{j_p}} < S \). Since \( S_{z_1 \cdots z_m} \prec S \), we obtain \( S' = S_{x_{j_1} \cdots x_{j_p}} < S \). In addition, we attain \( \{ j_1, \ldots, j_p \} \subseteq \{ 1, \ldots, n \} \).

If \( p < n - 1 \). Then, according to the (4) of Remark 1 and \( \{ j_1, \ldots, j_p \} \subseteq \{ 1, \ldots, n \} \), we may confirm \( S_{x_{j_1} \cdots x_{j_p}} \prec S_{x_{j_1} \cdots x_{j_p+1} \cdots x_{j_{p+1}}} < S_{x_{j_1} \cdots x_{j_n}} = S_{x_1 \cdots x_n} = S \). Hence, in view of this result with \( x_j \) obtained in a lattice. If \( L \) be an atomistic complete finite lattice with height 3. Then \( \mathcal{L} \).

Part I. When \( \mathcal{L} \).

Proof of Lemma 5. We will prove with two parts.

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Therefore, we confirm \( S' \prec S \).}

Second, we will deal with some properties related to \( \mathcal{C}_L(L) \) if \( \mathcal{C}_L(L) \) is geometric for an atomistic complete finite lattice \( L \).

**Lemma 5.** Let \( L \) be an atomistic complete finite lattice with height 3. Then \( \mathcal{C}_L(L) \) is geometric if and only if

(3.6) \(|\mathcal{F}^2| \geq 2\), that is, \( L \) has at least two elements of height 2.

(3.7) If \( d_1, d_2 \in \mathcal{F}^2 \), then \( d_1 \wedge d_2 \neq 0 \).

**Proof of Lemma 5.** We will prove with two parts.

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**Proof of Lemma 5.** We will prove with two parts.

Part I. When \( \mathcal{C}_L(L) \) is geometric.

\( h(L) = 3 \) compels that there is \( d \in \mathcal{F}^2 \) according to the definition of height function in a lattice. If \( \mathcal{F}^2 = \{d\} \). Then, we may obtain \( \mathcal{C}_L(L) = \{S_0, S_1, S_d\} \) with
$S_0 < S_d < S_1$. It is easily found that $Cl_4(L)$ is not geometric. This is a contradiction to the known supposition. Thus, it should have $|F^2| \geq 2$. That is, item (3.6) is true.

Suppose that $d_1 \land d_2 = 0$ holds for any $d_1, d_2 \in F^2 = \{d_j \mid j \in J\}$. Then, this causes $S_{d_t, t \in T} \subseteq Cl_4(L) < S_1$ for any $T \subseteq J$ in virtue of the (3) of Remark 1 and the (2) in Definition 1. In addition, we may easily find $Cl_4(L) = \{S_0, S_1, S_{d_t, t \in T}, T \subseteq J\}$. This causes $S_0 < S_{d_t, t \in T} \leq S_{d_t, j \in J} < S_1$. So, $Cl_4(L)$ is not geometric since $S_1$ is not the join of atoms in $Cl_4(L)$. This is a contradiction to the geometry of $Cl_4(L)$.

Therefore, there are $d_6, d_7 \in F^2$ satisfying $d_6 \neq d_7$ and $d_6 \land d_7 \neq 0$.

If there is $d_3 \in F^2$ satisfying $d_3 \land d_7 = 0$. Then, we obtain $S_0 < S_{d_t,} < S_{d_3, d_7} < S_1$ and $S_0 < S_{d_3,} < S_1$ with $S_{d_3,} = S_{d_3, d_7}$. In addition, $S_{d_7,} = S_{d_3,} \land T = S_1$ holds according to $d_7 \land d_6 \neq 0$ and item (3.3). This follows $S_{d_7,} = S_{d_7,} \land d_6$. So, $Cl_4(L)$ is not semimodular. This follows a contradiction to the geometry of $Cl_4(L)$. In other words, $d_7 \land d_j \neq 0$ holds for any $j \in J$. Analogously, $d_6 \land d_j \neq 0$ holds for any $j \in J$.

Moreover, $S_0 < S_{d_7,} < S_1$ holds. Hence, this maximal chain $\{S_0, S_{d_7,}, S_1\}$ in $Cl_4(L)$ has length 3. If there are $d_4, d_5 \in F^2$ satisfying $d_4 \land d_5 = 0$. Then it causes $S_0 < S_{d_4,} < S_{d_4, d_5} < S_1$. So, there is a maximal chain in $Cl_4(L)$ with length at least 4. Moreover, there are two maximal chains with different lengths in $Cl_4(L)$. This follows a contradiction to the geometry of $Cl_4(L)$.

Summing up, we obtain $x \land y \neq 0$ for any $x, y \in F^2$. That is, item (3.7) is true.

Part II. When $L$ satisfies items (3.6) and (3.7).

Let $d_j$ has height 2 in $L$, $(j \in J)$, that is, $F^2 = \{d_j \mid j \in J\}$.

Under the suppositions of items (3.6) and (3.7) with $h(L) = 3$, we may easily decide that $Cl_4(L)$ is $\{S_0, S_1, S_{d_j}, j \in J\}$ in which $S_0 < S_{d_j,} < S_1, S_{d_j,} = d_{i,j}$. Therefore, $Cl_4(L)$ is geometric. □

Considering the (3) in Lemma 1 with Lemma 5, we may easily express the following corollary.

**Corollary 1.** Let $L$ be an atomistic complete finite lattice and $x \in L \setminus (A(L) \cup 0 \cup F^2)$ with $h(x) = 3$. If $Cl_4(L)$ is geometric, then there are $|F^2(x)| \geq 2$ and $d_1 \land d_2 \neq 0$ for any $d_1, d_2 \in F^2(x)$ and $x = \lor_{d \in F^2(x)} d$.

**Lemma 6.** Let $L$ be an atomistic complete finite lattice. If $Cl_4(L)$ is geometric, then $L$ satisfies the following properties.

(3.8) $|C(x) \setminus A(L)| \geq 2$ for any $x \in L \setminus (A(L) \cup 0)$.

(3.9) $x = \lor_{d \in F^2(x)} d$ for any $x \in L \setminus (A(L) \cup 0)$.

(3.10) Let $x, y \in L \setminus (A(L) \cup 0)$. If $x \land y \neq x, y$, then $x, y \neq x \lor y$.

(3.11) Let $x \in L$ with $h(x) = 3$. Then, there is $d_1 \land d_2 \neq 0$ for any $d_1, d_2 \in F^2(x)$ with $d_1 \neq d_2$.

(3.12) Let $x \in L \setminus (A(L) \cup 0)$. If $3 \leq h(x)$, then there are $y, z \in C(x) \setminus A(L)$ satisfying $y \neq z$ and $y \lor z \neq 0$. 
(3.13) Let \( x, y \in L \setminus (A(L) \cup 0) \). If \( 2 \leq h(x \land y), x \land y \neq 0, x \land y < x, x \land y < y \) and \( z_x \land (x \land y) \neq 0, z_y \land (x \land y) \neq 0 \) for any \( z_x, z_y \in L \setminus (A(L) \cup 0) \) with \( z_x < x, z_y < y \). Then, \( p \land x \neq 0 \) and \( q \land y \neq 0 \) for any \( p, q < x \lor y \) and \( p, q \in L \setminus (A(L) \cup 0) \).

(3.14) Let \( y_j \in L \setminus (A(L) \cup 0), (j = 1, 2, \ldots, n; 2 \leq n) \), \( y_i \land y_j = 0 \), \( i \neq j; i, j = 1, 2, \ldots, n \).

If \( y_j \in \mathcal{F}^2, (j = 1, \ldots, n) \), then \( S < S_{y_1 \ldots y_n} \Rightarrow S = S_{y_1 \ldots y_{i-1}} \), for some \( \{i_1, \ldots, i_{n-1}\} \subseteq \{1, \ldots, n\} \) and \( |\{i_1, \ldots, i_{n-1}\}| = n - 1 \).

If \( S < S_{y_1 \ldots y_{i-1}} \). We obtain that if there is \( y_i \notin \mathcal{F}^2 \) for some \( i \in \{1, \ldots, n\} \), then \( S = S_{y_1 \ldots y_{i-1} y_n} \) holds where \( z_n < y_i \) and \( \{1, \ldots, i\} = \{1, \ldots, n\} \); or if \( y_j \in \mathcal{F}^2 \) for some \( j \in \{1, \ldots, n\} \), then \( S = S_{y_1 \ldots y_{j-1}} \) where \( \{j, \ldots, j-1\} = \{1, \ldots, n\} \setminus \{j\} \).

(3.15) Let \( y_i \in L, (i = 1, 2, \ldots, n) \) satisfy \( y_i \land y_j = 0 \), \( i \neq j; i, j = 1, 2, \ldots, n; 2 \leq n \).

(a) Setting \( x = \lor_{i=1}^m y_i \lor w_n \) and \( w_n < y_i \) with \( h(w_n) \geq 2 \) and \( \{i_1, \ldots, i_n\} = \{1, \ldots, n\} \). Suppose that for any \( m \leq n - 1 \) and \( p \in L \) with \( \lor_{i=1}^m y_i \lor w_n \leq q < x \) or \( \lor_{i=1}^m y_i \lor u_n \leq q < x \), where \( \{i_j \mid j = 1, \ldots, m\} \subseteq \{i_1, \ldots, i_{n-1}\} \), there exists \( s \in \{i_1, \ldots, i_{n-1}\} \setminus \{i_j \mid j = 1, \ldots, m\} \) satisfying \( p \land y_s \neq 0 \), \( p \land w_n \neq 0 \), or \( q \land y_s \neq 0 \). Then the following property (3.15.1) is true.

(b) Setting \( z = \lor_{i=1}^m y_i \lor h(y_n) = 2 \) and \( \{j_1, \ldots, j_n\} = \{1, \ldots, n\} \). Suppose that for any \( m \leq n - 1 \) and \( p \in L \) with \( \lor_{j=1}^m y_j \leq q < z \) where \( \{i_j \mid j = 1, \ldots, m\} \subseteq \{j_1, \ldots, j_{n-1}\} \), there exists \( s \in \{j_1, \ldots, j_{n-1}\} \setminus \{i_j \mid j = 1, \ldots, m\} \) satisfying \( p \land y_s \neq 0 \). Then the following property (3.15.2) is real.

(3.15.1) If \( x \land y_i \neq 0 \). Then \( x \land z \lor y_i = \lor_{i=1}^m y_i \) holds, and in addition, for any \( m \leq n \) and \( \lor_{j=1}^m y_j \leq q \land y_i \) where \( \{i_j \mid j = 1, \ldots, m\} \subseteq \{1, \ldots, n\} \), there exists \( s \in \{1, \ldots, n\} \setminus \{i_j \mid j = 1, \ldots, m\} \) satisfying \( q \land y_s \neq 0 \).

(3.15.2) If \( z \land y_j \neq 0 \). Then \( z \land \lor \land y_j = \lor_{j=1}^m y_j \) holds, and in addition, for any \( m \leq n \) and \( \lor_{j=1}^m y_j \leq q \land y_i \) where \( \{i_j \mid j = 1, \ldots, m\} \subseteq \{1, \ldots, n\} \), there exists \( s \in \{1, \ldots, n\} \setminus \{i_j \mid j = 1, \ldots, m\} \) satisfying \( q \land y_s \neq 0 \).

(3.16) Let \( y_j \in L \setminus (A(L) \cup 0), (j = 1, 2, \ldots, n; 2 \leq n) \), \( x_n \in L \setminus (A(L) \cup 0) \) and \( x_n \neq y_n \) satisfies \( y_i \land y_j = 0 \), \( i \neq j; i, j = 1, 2, \ldots, n \) and \( y_j \land x_n = 0 \), \( j = 1, 2, \ldots, n - 1 \). Then the following statements hold.

(3.16.1) Let \( h(x_n), h(y_n) > 2 \) and \( h(x_n \land y_n) \neq 0 \). If \( y_j \land (x_n \land y_n) = 0 \) holds \( (j = 1, \ldots, n - 1) \), and in addition, there is \( z_n \in L \setminus 0 \) satisfying \( z_n < x_n, y_n \). Then, for any \( p, q \in L \setminus (A(L) \cup 0) \), \( x_n < p < x_n \land y_n \) and \( y_n < q < x_n \land y_n \), there exists \( s, t \in \{1, \ldots, n - 1\} \) satisfying \( z_n \land p \neq 0 \) and \( z_n \land q \neq 0 \).

Additionally, if there are \( x_n \land y_n \neq 0 \) and \( h(x_n) = h(y_n) = 2 \), but \( y_j \land (x_n \land y_n) = 0 \), \( (j = 1, \ldots, n - 1) \). Then, for any \( p, q \in L \setminus (A(L) \cup 0) \), \( x_n < p < x_n \land y_n \) and \( y_n < q < x_n \land y_n \), there exists \( s, t \in \{1, \ldots, n - 1\} \) satisfying \( z_n \land p \neq 0 \) and \( z_n \land q \neq 0 \).

(3.16.2) Let \( h(x_n), h(y_n) > 2 \) and \( h(x_n \land y_n) \neq 0 \). Then \( y_j \land (x_n \land y_n) = 0 \), \( (j = 1, \ldots, m; 1 < m < n - 1) \), and in addition, there exists \( z_n \in L \setminus 0 \) satisfying
$z_n \prec x_n, y_n$. Then, for any $p, q, f \in L \setminus (A(L) \cup 0)$, if the expressions (i), (ii) and (iii) hold

(i) $\forall_{i=t} y_j \leq p < x_n \lor \forall_{i=m+1} y_i$ where $\{j_i \mid t \leq i \leq t_p\} \subseteq \{m + 1, \ldots, n - 1\}$,

(ii) $x_n \lor \forall_{i=t} x_j \leq q < x_n \lor \forall_{i=m+1} y_i$ where $\{j_i \mid t_x \leq i \leq t_q\} \subseteq \{m + 1, \ldots, n - 1\}$,

(iii) $\forall_{i=t} y_j \leq f < \forall_{i=m+1} y_i \lor x_n$ where $\{j_i \mid t_1 \leq i \leq t_f\} \subseteq \{m + 1, \ldots, n\}$.

then, the following expressions hold:

there exists $s_p \in \{1, \ldots, n - 1\} \setminus \{j_i \mid t \leq i \leq t_p\}$ satisfying $y_{s_p} \land p \neq 0$ or $x_n \land p \neq 0$;

there exists $s_q \in \{1, \ldots, n - 1\} \setminus \{j_i \mid t \leq i \leq t_q\}$ satisfying $y_{s_q} \land q \neq 0$;

Additionally, let $x_n \land y_n \neq 0$ and $h(x_n) = h(y_n) = 2$. Then, for any $p, q, f \in L \setminus (A(L) \cup 0)$, if the expressions (i), (ii) and (iii) hold, then the following expressions hold:

there is $s_p \in \{1, \ldots, n - 1\} \setminus \{j_i \mid t \leq i \leq t_p\}$ satisfying $y_{s_p} \land p \neq 0$ or $x_n \land p \neq 0$;

there is $s_q \in \{1, \ldots, n - 1\} \setminus \{j_i \mid t \leq i \leq t_q\}$ satisfying $y_{s_q} \land q \neq 0$;

there is $s_f \in \{1, \ldots, n\} \setminus \{j_i \mid t_1 \leq i \leq t_f\}$ satisfying $y_{s_f} \land f \neq 0$.

Proof of Lemma 6. We will prove the needed results step by step.

Step 1. To prove item (3.8).

In view of the definition of height function in a lattice and $h(x) = n < \infty$, we may indicate that there is $y \in \mathcal{F}^{n-1}(x)$. It is easily seen $y \prec x$. So, $y \in \mathcal{C}(x) \setminus A(L)$ holds.

If $|\mathcal{C}(x) \setminus A(L)| = 1$. Considered the atomistic property of $L$, we receive $x = y \lor (\lor_{a \in A(L), a \not\leq y, a \leq x} a)$. Hence, $Cl_s([0, x])$ satisfies $S \leq S_p$ for any $S \in Cl_s([0, x]) \setminus S_X$. This implies that $S_X$ is not a join of atoms in $[S_0, S_2]$, a contradiction to the geometry of $Cl_s(L)$.

In another word, $|\mathcal{C}(x) \setminus A(L)| \geq 2$ holds.

Step 2. To prove item (3.9).

It is easily found $x = \lor_{d \in \mathcal{F}^2(x)} d$ if $h(x) = 2$.

Utilizing Corollary 1, we may attain $x = \lor_{d \in \mathcal{F}^2(x)} d$ for any $x \in L$ and $h(x) = 3$.

Suppose that if $x \in L$ with $2 \leq h(x) \leq n - 1$, then $x = \lor_{d \in \mathcal{F}^2(x)} d$.

Let $x \in L$ and $h(x) = n$. In view of item (3.8), we find $x = \lor_{y \in \mathcal{C}(x) \setminus A(L)} y$. We may easily decide $h(y) \leq n - 1$ for any $y \in \mathcal{C}(x) \setminus A(L)$. Using the inductive on $n$, we obtain $y = \lor_{d \in \mathcal{F}^2(y)} y$ for any $y \in \mathcal{C}(x) \setminus A(L)$. On the other hand, for any $d \in \mathcal{F}^2(x)$, we find $d \prec x$ or $d \leq y$ for some $y \in \mathcal{C}(x) \setminus A(L)$.

Therefore, it follows $x = \lor_{y \in \mathcal{C}(x) \setminus A(L)} y = \lor_{d \in \mathcal{F}^2(x)} d$.

Step 3. To prove item (3.10).

Let $x, y \in L \setminus (A(L) \cup 0)$ and $x \land y \prec x, y$.

If $x \land y = 0$, then $x, y \in A(L)$ holds since $0 \prec x, y$. This is a contradiction to $x, y \in L \setminus (A(L) \cup 0)$.  

Thus, \( x \land y \neq 0 \) is true.

Furthermore, by item (3.3), we may be assured \( S_x \lor S_y = S_{x \lor y} \). Since \( Cl_s(L) \) is geometric and \( S_x, S_y \prec S_x \lor S_y \), we confirm \( S_x, S_y \prec S_{x \lor y} \). If \( x \neq x \lor y \), then \( S_x < S_y < S_{x \lor y} \) for some \( b \in L \) and \( x < b < x \lor y \). This causes a contradiction to \( S_x < S_{x \lor y} \). Hence, \( x < x \lor y \) holds. Similarly, \( y < x \lor y \) holds.

Step 4. The result in item (3.11) can be produced by Corollary 1.

Step 5. To prove item (3.12).

Let \( x \in L \setminus (A(L) \cup 0) \) with \( 3 \leq h(x) \).

If \( p \land q = 0 \) for any \( p, q \in \mathcal{C}(x) \setminus A(L) \). Then, using (3) in Remark 1, item (3.8) and induction on \( |\mathcal{C}(x) \setminus A(L)| \), we confirm \( S_{y, y \in \mathcal{C}(x) \setminus A(L)} = \{ y \mid y \in \mathcal{C}(x) \setminus A(L) \} \cup \{ a \in A(L) \mid a \not\leq y \text{ for any } y \in \mathcal{C}(x) \setminus A(L) \} \in Cl_s(L) \) and \( S_{y, y \in \mathcal{C}(x) \setminus A(L)} < S_x \), and \( S \leq S_{y, y \in \mathcal{C}(x) \setminus A(L)} \) for any \( S \in [S_0, S_x] \setminus S_x \). Thus, \( S_x \) is not the join of atoms. This causes a contradiction to the geometry of \( Cl_s(L) \).

Therefore, there are at least two elements \( y, z \in \mathcal{C}(x) \setminus A(L) \) satisfying \( y \land z \neq 0 \).

Step 6. To prove item (3.13).

From the given conditions, we follow \( S_{x \land y} < S_x, S_y \). It is easily known \( S_{x \land y} \leq S_x \land S_y \) by item (3.1).

We will demonstrate \( S_x \land S_y = S_{x \land y} \).

Otherwise, \( S_{x \land y} \prec S_x \land S_y \). If \( S_x \land S_y = S_z \) for some \( z \in L \), then \( x \land y < z < x, y \). Using item (3.1), we may attain \( S_{x \land y} < S_z < S_x, S_y \), a contradiction to \( S_{x \land y} \prec S_x, S_y \). Thus, we find \( S_x \land S_y = S_{z_1, i \in I} \in Cl_s(L) \) and \( 2 \leq |I| \). Applying the (2) of Definition 1, the (2) of Lemma 2 with Definition 2, we may obtain \( A(x \land y) \leq A(z_{i_0}) \) for some \( i_0 \in I \). Considering this result with the atomistic property of \( L \) and \( 2 \leq |I| \), we follow \( x \land y \leq z_{i_0} < \forall_{i \in I} z_i \leq x, y \). And further, we find \( S_{x \land y} < S_{z_{i_0}} \leq S_{\forall_{i \in I} z_i} \leq S_x \), or \( S_{x \land y} \leq S_{z_{i_0}} \leq S_{\forall_{i \in I} z_i} \leq S_y \) since \( S_x \neq S_y \). This causes a contradiction to \( S_{x \land y} < S_x \) if \( S_{x \land y} < S_{z_{i_0}} \leq S_{\forall_{i \in I} z_i} \leq S_x, \) and \( S_{x \land y} \prec S_y \) if \( S_{x \land y} \leq S_{z_{i_0}} \prec S_{\forall_{i \in I} z_i} \leq S_y \).

In other words, \( S_x \land S_y = S_{x \land y} \) is true.

Therefore, we may be assured \( S_{x \land y} = S_x \land S_y < S_x, S_y \).

Furthermore, since \( Cl_s(L) \) is geometric, we may point out \( S_x, S_y < S_x \lor S_y \). Since \( S_x \lor S_y = S_{x \lor y} \) holds by \( x \land y \neq 0 \) and item (3.3), we find \( x, y < x \lor y \) according to the given condition \( x \land y < x, y \) with item (3.10). That is, \( S_x, S_y \prec S_{x \lor y} \) does not hold for any \( z \in L \). Actually, if there is \( p \in L \setminus (A(L) \cup 0) \) satisfying \( p < x \lor y \) and \( p \land x = 0 \), then \( S_{xp} \in Cl_s(L) \) holds. We may easily find \( S_x < S_{xp} < S_{x \lor y}, \) a contradiction to \( S_x < S_{x \lor y} \). Therefore, for any \( p \in L \setminus (A(L) \cup 0) \) and \( p < x \lor y \), it has \( p \land x \neq 0 \). Analogously, \( q \land y \neq 0 \) holds for any \( q \in L \setminus (A(L) \cup 0) \) and \( q < x \lor y \).

Step 7. To prove item (3.14).

Let \( y_j \in F^2 \), that is, \( y_j \) has height 2 in \( L \), \( j = 1, 2, \ldots, n \geq 2 \). If \( S < S_{y_1 \ldots y_n} \), then, taken \( S < S_{y_1 \ldots y_n} \) and the (3) in Remark 1 together, we attain \( S = S_{z_1 \ldots z_m} \) for
holds where 

\[ \exists y \]

\[ \text{ satisfying } \]

\[ \text{ is a contradiction to } \]

\[ \text{ for any } \]

\[ \text{ Then, } \]

\[ \text{ if } p < q \leq m, \]

\[ \text{ then } \]

\[ \text{ and } \]

\[ \text{ Additionally, according to the geometry of } \]

\[ \text{ Since } \]

\[ \text{ Therefore, } \]

\[ \text{ that } \]

\[ \text{ and } \]

\[ \text{ Suppose } \]

\[ \text{ Furthermore, we follow } \]

\[ \text{ Thus, by item (3.1), we arrive } \]

\[ \text{ Suppose } \]

\[ \text{ Since } \]

\[ \text{ If } \]

\[ \text{ Additionally, combining } \]

\[ \text{ We may obtain } \]

\[ \text{ This causes a contradiction to } \]

\[ \text{ Suppose } \]

\[ \text{ If } \]

\[ \text{ If } \]

\[ \text{ and } \]

\[ \text{ for any } \]

\[ \text{ Then, } \]

\[ \text{ Hence, } \]

\[ \text{ So, } m \leq n \text{ is true.} \]

\[ \text{ Therefore, } \]

\[ \text{ when } \]

\[ \text{ where } \]

\[ \text{ We may easily obtain } \]

\[ \text{ in which } \]

\[ \text{ and } \]

\[ \text{ Therefore, } \]

\[ \text{ If } \]

\[ \text{ and } \]

\[ \text{ and } \]

\[ \text{ Then, } \]

\[ \text{ if } \]

\[ \text{ this causes a contradiction to } \]

\[ \text{ Hence, } \]

\[ \text{ Then, } \]

\[ \text{ as true.} \]

\[ \text{ In addition, we may easily explore that if } \]

\[ \text{ for one and only one } \]

\[ \text{ and } \]

\[ \text{ for some } \]

\[ \text{ and } \]

\[ \text{ There is one and only one } \]

\[ \text{ for any } \]

\[ \text{ and } \]

\[ \text{ and } \]

\[ \text{ Thus, } \]

\[ \text{ In addition, if } \]

\[ \text{ then it is easily to obtain } \]

\[ \text{ We may assume } \]

\[ \text{ Some } \]

\[ \text{ Suppose } \]

\[ \text{ Hence, } \]

\[ \text{ if } \]

\[ \text{ and } \]

\[ \text{ and } \]

\[ \text{ Therefore, } \]
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When \( m \leq n \) and \( h(y_n) = 2 \).

Then, we will demonstrate \( m < n \) holds.

Otherwise, \( z_n < y_n \) infers to \( z_n \in A(L) \cup 0 \), a contradiction to \( z_n \in L \setminus (A(L) \cup 0) \).

That is to say, this case will not happen actually.

Combining

\[
p < q \leq m \Rightarrow S_{z_1 \ldots z_p} < S_{z_1 \ldots z_p z_{p+1} \ldots z_q} \leq S_{z_1 \ldots z_m} \leq S_{y_1 \ldots y_m} \leq S_{y_1 \ldots y_m y_{m+1} \ldots y_{n-1}} \leq S_{y_1 \ldots y_n}
\]

and \( z_j < y_j \) \((j = 1, \ldots, m)\) with \( S_{z_1 \ldots z_m} = S < S_{y_1 \ldots y_n} \) and the geometry of \( Cl_\ell(L) \),

we decide \( n - 1 \leq m \).

If \( m = n - 1 \). That is, \(|\{1, \ldots, m\}| = n - 1 \) holds. No matter to suppose \( \{1, \ldots, m\} = \{1, \ldots, n - 1\} \).

Considered \( z_j \leq y_j \) \((j = 1, \ldots, m)\) and \( S = S_{z_1 \ldots z_m} < S_{y_1 \ldots y_n} \), we may confirm \( z_j = y_j \) \((j = 1, \ldots; m; m = n - 1)\).

We will demonstrate \( h(y_n) = 2 \).

Otherwise, \( h(y_n) > 2 \) is true. But, using the definition of height function in a lattice, we affirm the existence of \( S_{y_1 \ldots y_{n-1}} < S_{y_1 \ldots y_n} \).

This follows a contradiction to \( S_{z_1 \ldots z_m} < S_{y_1 \ldots y_n} \).

When \( m = n - 1 \) and \( h(y_n) = 2 \). Since \( S_{z_1 \ldots z_m} \leq S_{y_1 \ldots y_m y_{m+1} \ldots y_{n-1}} < S_{y_1 \ldots y_n} \) holds in light of \( z_j \leq y_j \) \((j = 1, \ldots, m)\) and the given condition \( S_{z_1 \ldots z_m} < S_{y_1 \ldots y_n} \).

We may attain \( S_{y_1 \ldots y_{n-1}} < S \). If \( S_{y_1 \ldots y_{n-1}} \neq S_{y_1 \ldots y_n} \) holds, then there is \( b \in S \setminus (A(L) \cup \{y_1, \ldots, y_{n-1}\}) \) satisfying \( b < y_n \). However, \( h(y_n) = 2 \) and \( b < y_n \) together causes \( b \in A(L) \cup 0 \), a contradiction to \( b \in S \setminus (A(L) \cup \{y_1, \ldots, y_{n-1}\}) \).

Therefore, \( S = S_{y_1 \ldots y_{n-1}} \) holds.

Step 8. To prove item (3.15).

Under the supposition of \( (\alpha) \), \( x \land y_{i_n} = 0 \) will not happen since \( w_n \neq 0, w_n \leq x \)

and \( w_n < y_{i_n} \). That is to say, there is item (3.15.1) and only item (3.15.1) to happen.

According to the given conditions, items (3.5) and (3.14) with (2) in Definition 1, we affirm \( S_{y_1 \ldots y_{n-1} w_n} < S_{y_1 \ldots y_n} \).

Thus, under the supposition of \( (\alpha) \), we may be assured \( S_{y_1 \ldots y_{n-1} w_n} = S_x \land S_{y_1 \ldots y_n} < S_x \land S_{y_1 \ldots y_n} \).

Furthermore, by the geometry of \( Cl_\ell(L) \), we find \( S_x \land S_{y_1 \ldots y_n} < S_x \lor S_{y_1 \ldots y_n} \).

If the supposition of \( (\beta) \) happens. Then we find \( S_{y_{j_1 \ldots j_{n-1}}} = S_z \land S_{y_1 \ldots y_n} \equiv S_z, S_{y_1 \ldots y_n} \) according to item (3.14) and the (2) in Definition 1. Thus, by the geometry of \( Cl_\ell(L) \), we affirm \( S_z, S_{y_1 \ldots y_n} < S_z \lor S_{y_1 \ldots y_n} \).

Step 8.1. To prove item (3.15.1).

Let \( x \land y_{i_n} \neq 0 \). We will prove the needed results as the following two parts.

Part I. We prove \( S_x \lor y_{i_n} = S_x \lor S_{y_1 \ldots y_{n-1}} \).

Let \( S_x \lor S_{y_1 \ldots y_{n-1}} = S_{z_1 \ldots z_m} \) for some \( z_j \in L \setminus 0 \).

The result in item (3.1) and \( x, y_j \in L \setminus (A(L) \cup 0) \) together follows \( z_i \notin A(L) \cup 0 \). \((j = 1, \ldots; n; i = 1, \ldots, m)\).

This implies \( y_{i_j} < x \leq z_1 \), \((j = 1, \ldots, n - 1)\) and \( y_{i_n} < z_2 \). If \( z_1 \neq z_2 \), then \( x \land y_{i_n} \leq z_1 \land z_2 = 0 \). This follows \( x \land y_{i_n} = 0 \), a
contradiction to $x \wedge y_n = 0$. Thus, we produce $z_1 = z_2$. According to item (3.1) and $S_x, S_{y_1 \ldots y_n} \leq S_{z_1} \leq S_{z_1 \ldots z_m} = S_x \lor S_{y_1 \ldots y_n}$, we confirm $S_x \lor S_{y_1 \ldots y_n} = S_{z_1}$.

Therefore, we obtain $S_x, S_{y_1 \ldots y_n} < S_x \lor S_{y_1 \ldots y_n} = S_{z_1}$.

Part II. We will prove the other needed results in item (3.15.1).

In light of the result in Part I, $x \lor (y_1 \lor \ldots \lor y_n) = x \lor y_{in}$ and item (3.1), we find $S_x \lor y_{in} = S_{z_1}$. Combining this result with $S_x < S_x \lor y_{in} = S_x$ and $w_n < y_{in}$, we may easily affirm $x < x \lor y_{in}$. In addition, $S_{y_1 \ldots y_n} < S_x \lor y_{in} = S_{y_{in}}$ illustrates that no element $S \in Cl_L(L)$ satisfies $S_{y_1 \ldots y_n} < S < S_{y_{in}}$. According to item (3.1) and the (3) of Remark 1, we may state that for any $m \leq n$ and $\lor_{j=1}^{m} y_{i_j} \leq q < \lor y_n$, there exists $s \in \{1, \ldots, n\} \setminus \{i_j \mid j = 1, \ldots, m\}$ satisfying $q \wedge y_s \neq 0$ where $\{i_j \mid j = 1, \ldots, m\} \subset \{1, \ldots, n\}$.

Step 9.2. Item (3.15.2) may be verified by similar way to that for item (3.15.1).

Step 9. To prove item (3.16).

Considering the known conditions, item (3.1), the (2) in Definition 1 with Definition 2, we obtain $S_{y_1 \ldots y_{n-1}} < S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n}$. Suppose that there are $S_2, S_3 \in Cl_L(L)$ satisfying $S_{y_1 \ldots y_{n-1}} < S_2 < S_{y_1 \ldots y_{n-1}x_n}$ and $S_{y_1 \ldots y_{n-1}} < S_3 < S_{y_1 \ldots y_{n-1}y_n}$. Then, using items (3.5) and (3.14), we may obtain the following four statements.

if $h(x_n) > 2$, then $S_2 = S_{y_1 \ldots y_{n-1}z_2}$ where $z_2 < x_n$;
if $h(x_n) = 2$, then $S_2 = S_{y_1 \ldots y_{n-1}}$;
if $h(y_n) > 2$, then $S_3 = S_{y_1 \ldots y_{n-1}z_3}$ where $z_3 < y_n$;
if $h(y_n) = 2$, then $S_3 = S_{y_1 \ldots y_{n-1}}$.

Therefore, if there is $S \in Cl_L(L)$ satisfying $S < S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n}$, then there is $S = S_{y_1 \ldots y_{n-1}}$ when $z_n = x_n \wedge y_n \neq A(L) \cup 0$ and $z_n < x_n, y_n$; or there is $S = S_{y_1 \ldots y_{n-1}}$ when $x_n \wedge y_n \in A(L) \cup 0$.

Step 9.1. To verify item (3.16.1).

Since $z_n < x_n$ and $y_n$ taken together follows $z_n = x_n \wedge y_n$. Hence, in view of $h(x_n \wedge y_n) \neq 0$, there is $S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n}$ if $x_n \wedge y_n \in A(L)$, and in addition, there is $S_{y_1 \ldots y_{n-1}z_n} < S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n}$ if $z_n \in L \setminus (A(L) \cup 0)$. No matter which of the above cases happens, using the geometry of $Cl_L(L)$ and Lemma 1, we always obtain $S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n} < S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n}$. Utilizing the (3) in Remark 1 and Definition 2, we may easily gain $S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}x_n} \lor S_{y_1 \ldots y_{n-1}y_n}$. Thus, we arrive at $S_{y_1 \ldots y_{n-1}x_n} < S_{y_1 \ldots y_{n-1}(x_n \lor y_n)}$. Therefore, for any $p, q \in L \setminus (A(L) \cup 0)$, $x_n < p < x_n \lor y_n$ and $y_n < q < x_n \lor y_n$, there are $s, t \in \{1, \ldots, n-1\}$ satisfying $y_s \wedge p \neq 0$ and $y_t \wedge q \neq 0$.

Analogously to the proof above, we may easily obtain the “additionally” part in item (3.16.1).

Step 9.2. To prove item (3.16.2).

Using item (3.16.1) and the induction on $n = (m+1)$, similarly to the proof in Step 9.1, we may easily obtain item (3.16.2).
Remark 2. In the proof of Step 7 for Lemma 6, for the case “when \( m = n - 1 \) and \( h(y_n) = 2 \), we may use item (3.5) to obtain the same result. But, we think that the proof in Lemma 6 for this case is useful to prove the other cases.

Third, we will reveal under what conditions, \( Cl_s(L) \) is geometric for an atomistic complete finite lattice \( L \).

Lemma 7. Let \( L \) be an atomistic complete finite lattice. If \( L \) satisfies items from (3.8) to (3.16), then \( Cl_s(L) \) is a geometric lattice.

Proof of Lemma 7. Applying the information in Subsection 2.1 and Lemma 3, we only need to prove that \( Cl_s(L) \) is atomistic and semimodular.

According to Lemma 4, we may be assured \( A(Cl_s(L)) = \{ S_d \mid d \in F^2 \} \).

Step 1. We prove that every element in \( Cl_s(L) \) is a join of atoms using the following Step 1.1 and Step 1.2.

Step 1.1. To prove: \( S_x = \vee_{d \in F^2(x)} S_d \) for any \( x \in L \setminus (A(L) \cup 0) \).

Let \( x \in F^2 \). The needed result is easily followed.

Let \( x \in L \setminus (A(L) \cup 0) \) and \( h(x) = 3 \). By items (3.8) and (3.11), we find \( 2 \leq |F^2(x)\). Let \( d_1, d_2 \in F^2(x) \) and \( d_1 \land d_2 \neq 0 \). Then in light of item (3.3), we may obtain \( S_{d_1} \lor S_{d_2} = S_{d_1 \lor d_2} = S_x \). Furthermore, we may decide \( S_x = \vee_{d \in F^2(x)} S_d \) since \( S_d \leq S_x \) for any \( d \in F^2(x) \).

Suppose that there is \( S_x = \vee_{d \in F^2(x)} S_d \) for any \( x \in L \setminus (A(L) \cup 0) \) and \( h(x) \leq n - 1 \).

Let \( x \in L \setminus (A(L) \cup 0) \) with \( h(x) = n \). Let \( C(x) = \{ y_i \mid i \in I \} \), that is, \( C(x) \) is the set of elements covered by \( x \) in \( L \).

In view of item (3.8), we produce \( x = \vee_{i \in J} y_i \). Since \( L \) satisfies item (3.12). We obtain \( S_{y_1} \lor S_{y_2} = S_{y_1 \lor y_2} = S_x \) using item (3.3), where \( y_1, y_2 \in C(x) \setminus A(L), y_1 \neq y_2 \) and \( y_1 \land y_2 \neq 0 \). On the other hand, by item (3.1), we confirm \( S_y \leq S_x \) for any \( y \in C(x) \setminus A(L) \).

Moreover, combining the above results, we attain \( S_x = \vee_{y \in C(x)} S_y \).

Considered \( y \in C(x) \) with \( h(x) = n \), we find \( h(y) \leq n - 1 \). Using inductive supposition, we will obtain \( S_y = \vee_{d \in F^2(y)} S_d \). Additionally, we may easily find that if \( d \in F^2(x) \), then \( d \in C(x) \setminus A(L) \) or \( d \leq y_d \) for some \( y_d \in C(x) \). That is to say, \( d \in F^2(y) \) holds for some \( y \in C(x) \setminus A(L) \). Therefore, we arrive at \( S_x = \vee_{d \in F^2(x)} S_d \).

Step 1.2. To prove: \( S_{x_i, t} \in Cl_s(L) \) is a join of atoms where \( |T| \geq 2 \).

Let \( S = S_{x_i, t} \in Cl_s(L) \) with \( 2 \leq |T| \). Using (2) of Definition 1 and item (3.2), we may find \( S = \vee_{t \in T} S_{x_t} \). For every \( S_{x_t} \), applying with Step 1.1, we may obtain \( S_{x_t} = \vee_{d_j \in F^2(x_t)} S_{d_j} \) where \( F^2(x_t) = \{ d_j \mid j \in J_t, t \in T \} \). Therefore, we attain \( S = \vee_{t \in T} \vee_{j \in J_t} S_{d_j} \).

Step 2. We prove that \( Cl_s(L) \) is semimodular.
Let $S_2, S_3 \in Cl_L(L), S_2 || S_3$ and $S_2 \land S_3 \prec S_2, S_3$. According to the (3) of Remark 1, we may point that there are $S_2 = S_x$ or $S_2 = S_{x_i, i \in J}$, and in addition, $S_3 = S_y$ or $S_3 = S_{y_i, i \in J}$, for some $x, y \in L \setminus \emptyset$ and $x_i, y_i \in L \setminus (A(L) \cup 0) (j \in J, i \in I)$ with $|J|, |I| \geq 2$. Based on this statement, we will divide different cases to prove $S_2, S_3 \prec S_2 \lor S_3$ by the following Steps 2.1, 2.2 and 2.3.

Step 2.1. Assume $S_2 = S_x$ and $S_3 = S_y$ for some $x, y \in L \setminus \emptyset$.

Since $S_0 \leq S_2 \land S_3 \prec S_2, S_3$ and $S_2 \land S_3 \in Cl_L(L)$, we may affirm $x, y \in L \setminus (A(L) \cup 0)$. Hence, it follows $S_2, S_3 \prec S_2 \lor S_3$ holds with item (3.10), we may affirm $p < y$.

We will distinguish two cases to fulfill the proof.

Case 1. Suppose $S_2 \land S_3 = S_0$.

Then, $S_x, S_y \in A(Cl_L(L))$ holds. Thus, by item (3.4), we believe $x, y \in F^2$.

When $x \land y = 0$. Using item (3.2), we obtain $S_x, S_y < S_{xy} = S_x \lor S_y$. If $S_x < S < S_{xy}$ holds for some $S \in Cl_L(L)$, then $S = S_{xp}$ holds where $p \in L \setminus (A(L) \cup 0)$ and $p < y$. However, $h(y) = 2$ and $p < y$ taken together follows $p \in A(L) \cup 0$. This is a contradiction to $p \in L \setminus (A(L) \cup 0)$. Thus, $S_x < S_{xy}$ holds. Analogously, $S_y < S_{xy}$ is true.

When $x \land y \neq 0$. Using item (3.3), we affirm $S_x \lor S_y = S_{x \lor y}$. In light of $x, y \in F^2$ and $S_2 \land S_3 = S_0$, we find $x \land y \in A(L)$. So, $x \land y < x, y$ is followed. Using item (3.10), we may affirm $x, y < x \lor y$. Moreover, $h(x \lor y) = h(x) + 1 = 3$ holds.

Hence, it follows $S_x, S_y < S_{x \lor y}$. Therefore, we may obtain $S_x, S_y < S_x \lor S_y$.

Case 2. Suppose $S_x \land S_y \neq S_0$.

Then, $3 \leq h(x), h(y)$ hold since $S_x \land S_y < S_x, S_y$.

By item (3.1), it is easily to find $S_x \land S_y \leq S_x \land S_y$.

We will demonstrate $S_x \land S_y = S_x \land S_y$.

Otherwise, $S_x \land S_y < S_x \land S_y$ holds. If $S_x \land S_y = S_z$ for some $z \in L$, then $x \land y < z < x, y$. According to item (3.1), we may achieve $S_x \land S_y < S_z < S_x, S_y$, a contradiction to $S_x \land S_y < S_x, S_y$. That is to say, we attain $S_x \land S_y = S_{z_i, i \in J} \in Cl_L(L)$ and $2 \leq |J|$. By the (2) of Definition 1, the (2) of Lemma 2, and Definition 2, we may obtain $A(x \land y) \subseteq A(z_{i_0})$ for some $i_0 \in J$. Considering this result with the atomistic property of $L$ and $2 \leq |J|$, we provide $x \land y \leq z_{i_0} < \land_{i \in J} z_i \leq x, y$, and further, $S_x \land S_y \leq S_{z_{i_0}} \land S_{\land_{i \in J} z_i} < S_x, S_y$ or $S_x \land S_y \leq S_{z_{i_0}} \land S_{\lor_{i \in J} z_i} \leq S_y$ since $S_x \neq S_y$. No matter which of the above cases to happen, it causes a contradiction to $S_x \land S_y < S_x$ or $S_x \land S_y < S_y$.

In other words, $S_x \land S_y = S_x \land S_y$ is real.

In fact, $S_x \land S_y \neq S_0$ and $S_x \land S_y = S_x \land S_y$ taken together infers to $h(x \land y) \geq 2$.

Moreover, $x \land y < x, y$ holds according to $S_x \land S_y = S_x \land S_y < S_x, S_y$. Combining with item (3.10), we may get $x, y \prec x \lor y$. Hence, since $L$ satisfies item (3.13) and there are $S_x \land S_y < S_x, S_y$, we may determine $S_x, S_y \prec S_x \lor S_y$. Additionally, $S_0 \neq S_x \land S_y$ and $S_x \land S_y = S_x \land S_y$ follow $x \land y \neq 0$. According to item (3.3), we find $S_{x \lor y} = S_x \lor S_y$.

Therefore, we decide $S_x, S_y < S_x \lor S_y$. 

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Step 2.2. Assume $S_2 = S_X$ for some $x \in L \setminus 0$ and $S_3 = \{y_j \mid j \in \mathcal{Y}\} \cup \{a \in A(L) \mid a \not\leq y_j, \forall j \in \mathcal{Y}\} \subset \mathcal{Y}$.

Combining items (3.5) and (3.10) with $S_2 \land S_3 \prec S_2$, we may state that there is $\mathcal{Y}_2 \prec \mathcal{Y}$ satisfying $|\mathcal{Y}_2| = 2$, $S_2 \land S_3 \supset S_2$, we may indicate $x \geq \forall j \in \mathcal{Y}_2$. No matter to assume $\mathcal{Y} = \{1, 2, \ldots, n\}$ and $\mathcal{Y}_2 = \{1, 2, \ldots, n - 1\}$. In view of item (3.14) and $S_2 \land S_3 \prec S_{y_1 \ldots y_n} = S_X$, we may obtain that

- if $h(y_n) = 2$, then $S_2 \land S_3 = S_{y_1 \ldots y_{n-1}}$;
- if $h(y_n) \geq 3$, then $S_2 \land S_3 = S_{y_1 \ldots y_{n-1}}z_n$ where $z_n < y_n$ and $z_n \in L \setminus (A(L) \cup 0)$.

Suppose $h(y_n) = 2$. Then $S_{y_1 \ldots y_{n-1}} = S_X \land S_{y_1 \ldots y_n}$ produces that $x$, which $x = \vee_{j=1}^n y_j$, satisfies: for any $p \in L \setminus (A(L) \cup 0)$ with $p \not\leq x$, there is $y_p \land p \not= 0$ for some $s \in \{1, \ldots, n - 1\} \setminus \{i \mid j \leq 1, \ldots, m\}$ where $i \leq 1, \ldots, m \leq n$. We will use the following two statuses to fulfill the proof.

Status 1. When $x \land y_n = 0$.

Then, $S_{xy_n} \subset C\mathcal{L}(L)$ is obtained from the (2) of Definition 1. Hence, it follows $S_X \land S_{y_1 \ldots y_n} \preceq S_{xy_n}$. Moreover, we may be assured $S_X \land S_{y_1 \ldots y_n} \preceq S_{xy_n}$.

Considering the (3) of Remark 1 and $S_{y_1} \prec S_X \land S_{y_1 \ldots y_n}$, we may suppose $S_X \land S_{y_1 \ldots y_n} = S_b$ for some $b \in L \setminus (A(L) \cup 0)$. Evidently, this supposition will cause $S_{x_{y_n}} \preceq S_b$. Thus, we decide $S_{xy_n} = S_X \land S_{y_1 \ldots y_n} = S_b$. This is a contradiction to $x \not= y_n$ and $|\{b\}| = 1$.

On the other hand, we may also suppose $S_X \land S_{y_1 \ldots y_n} = S_z_1 \ldots z_m$ where $2 \leq m$ according to the (3) of Remark 1. This supposition will cause $x \leq z_1$ and $y_n \leq z_m$ in which $z_1, z_m \in \{z_1, \ldots, z_m\}$. Considered both of the (1) and (2) in Lemma 2 with Definition 2, we may determine $S_{xy_n} \leq S_{y_1 \ldots y_m} \leq S_{z_1 \ldots z_m}$. Thus, it follows $S_{xy_n} = S_{z_1 \ldots z_m} = S_X \land S_{y_1 \ldots y_n}$. If $z_1 = z_m$, then $S_{xy_n} = S_{z_1 \ldots z_m}$. This expression transfers to the above case. But the above case shows that this expression is wrong. In other words, $z_1 \neq z_m$ is true.

Therefore, we obtain $S_X \land S_{y_1 \ldots y_n} = S_{xy_n}$.

If $S_X < S < S_{xy_n}$ for some $S \in C\mathcal{L}(L)$. Then $x \in S$ and $b_n \in S$, where $b_n \in L \setminus (A(L) \cup 0)$ and $b < y_n$. However, $h(y_n) = 2$ follows the non-existence of $b_n$. Thus, there does not exist $S \in C\mathcal{L}(L)$ satisfying $S_X < S < S_{xy_n}$. That is to say, $S_X < S_{xy_n}$ holds.

If $S_{y_1 \ldots y_n} < S < S_{xy_n}$ for some $S \in C\mathcal{L}(L)$. Then $y_n \in S, \vee_{t=1}^u y_t \leq q \in S, \{j_t \mid t \leq 1, \ldots, u\} \leq \{1, \ldots, u\}$, $2 \leq |\{j(1) \mid t = 1, \ldots, u\}|$ and $A(y_j) \leq A(z_j)$ for $z_j \in S, \{j(1) \mid t = 1, \ldots, u\} \setminus \{j_t \mid t = 1, \ldots, u\}$ and $|\{j(1) \mid p = 1, \ldots, v\}| \leq |\{1, \ldots, n\} \setminus \{j_t \mid t = 1, \ldots, u\}|$. In addition, we easily find $(\vee_{t=1}^u y_t) \land z_{j_T} = 0$ for every $z_{j_T} \in S, (p = 1, \ldots, v)$. However, we already know that there exists $x \in \{1, \ldots, n\} \setminus \{j_t \mid t = 1, \ldots, u\}$ satisfying $(\vee_{t=1}^u y_t) \land y_x \not= 0$. That is to say, it does not exist $S \in C\mathcal{L}(L)$ satisfying $S_{y_1 \ldots y_n} < S < S_{xy_n}$. Thus, there is $S_{y_1 \ldots y_n} < S_{xy_n} = S_X \land S_{y_1 \ldots y_n}$.
Status 2. When \( x \land y_n \neq 0 \).

Actually, using item (3.15.2), we will attain \( S_{x \lor y_n} = S_x \lor S_{y_1 \ldots y_n} \) and \( S_x, S_{y_1 \ldots y_n} < S_{x \land y_n} \).

Suppose \( h(y_n) \neq 2 \). Then, \( S_2 \land S_3 = S_{y_1 \ldots y_{n-1} \land z_n} \) is true where \( z_n < y_n \). Additionally, \( S_2 \land S_3 < S_x \) follows \( x = \lor_{j=1}^{n-1} y_j \lor z_n \). So, \( z_n = x \land y_n \in L \setminus (A(L) \cup 0) \) holds. This implies \( x \land y_n \neq 0 \).

We will prove \( S_x \lor S_{y_1 \ldots y_n} = S_{x \lor y_n} \).

It is easily found \( S_x, S_{y_1 \ldots y_n} < S_{x \lor y_n} \) since item (3.1). Thus, we obtain \( S_x \lor S_{y_1 \ldots y_n} = S_{x \lor y_n} \) for \( b_1 \in L \setminus (A(L) \cup 0) \) \((i = 1, \ldots, m)\). By item (3.1), it follows \( y_j \leq x < b_1 \) \((j = 1, \ldots, n - 1)\) and \( y_n \leq b_t \) for some \( t \in \{1, \ldots, m\} \). If \( t \neq 1 \), then \( 0 \leq x \land y_n \leq b_1 \land b_t = 0 \). So, \( x \land y_n = 0 \) is followed. This causes a contradiction to \( x \land y_n \neq 0 \). Moreover, \( y_j \leq x \leq b_1 \) \((j = 1, \ldots, n - 1)\). Thus, \( m = 1 \) holds. In addition, according to item (3.1) and \( x \land y_n \leq b_1 \), we may present \( S_x \lor S_{y_1 \ldots y_n} = S_{x \lor y_n} \leq S_{b_1} \). Hence, \( S_{x \land y_n} = S_x \lor S_{y_1 \ldots y_n} \) holds.

Next, according to \( S_{y_1 \ldots y_{n-1} \land z_n} < S_x, S_{y_1 \ldots y_n} \), we may indicate that for any \( \lor_{j=1}^{n-1} y_j \leq p < x \lor \lor_{j=1}^{n-1} y_j \lor z_n \leq q < x \) where \( \{i_j \mid j = 1, \ldots, t\} \subseteq \{1, \ldots, n - 1\} \), there exists \( s \in \{1, \ldots, n - 1\} \setminus \{i_j \mid j = 1, \ldots, t\} \) satisfying \( y_s \land p \neq 0 \) or \( z_n \land p \neq 0 \) or \( y_3 \land q \neq 0 \). Using items (3.15.1) and (3.1), it follows \( S_2, S_3 < S_2 \lor S_3 \).

Step 2.3. Let \( S_2 = S_{x_1 x_2 \ldots x_m} \) and \( S_3 = S_{y_1 y_2 \ldots y_n} \). By item (3.5) and \( S_2 \land S_3 < S_2, S_3 \), we follow \( S_2 \land S_3 \leq S_{x_1 \ldots x_m \land y_1 \ldots y_n} = S_{x_1 \ldots x_m} \land S_{y_1 \ldots y_n} \) \((i = 1, \ldots, n)\) and \( \{|x_1, \ldots, x_m\} \cap \{|y_1, \ldots, y_n\}| = n - 1 \) if \( |\{|x_1, \ldots, x_m\} \cap \{|y_1, \ldots, y_n\}| = n - 1 \). No matter to suppose \( \{|x_1, \ldots, x_m\} \cap \{|y_1, \ldots, y_n\}| = n - 1 \). Thus, we may reveal \( n = m, x_m \neq y_n \), and \( S_2 = S_{y_1 y_2 \ldots y_{n-1} x_m} \), \( S_3 = S_{y_1 y_2 \ldots y_{n-1} y_n} \).

We will demonstrate \( h(x_n) = 2 \) if and only if \( h(y_n) = 2 \).

In fact, if \( h(y_n) = 2 \), then by item (3.14) and \( S_2 \land S_3 < S_{y_1 \ldots y_n} \), there is \( S_2 \land S_3 = S_{y_1 \ldots y_{n-1}} \). Meanwhile, \( S_{y_1 \ldots y_{n-1}} = S_2 \land S_3 \leq S_3 \) \( \leq S_{y_1 \ldots y_{n-1} x_m} \), \( S_{y_1 \ldots y_{n-1} y_n} \) follows \( h(x_n) = 2 \) by item (3.14).

Similarly, if \( h(x_n) = 2 \), then \( h(y_n) = 2 \).

Additionally, if \( h(y_n) = 2 \), then according to items (3.5) and (3.14) and \( h(x_n) = 2 \), we obtain \( S_2 \land S_3 = S_{y_1 \ldots y_{n-1}} < S_{y_1 \ldots y_{n-1} x_m}, S_{y_1 \ldots y_{n-1} y_n} \) holds.

When \( h(y_n) = 2 \) and \( x_n \land y_n = 0 \). We will prove \( S_{y_1 \ldots y_n x_m} = S_{y_1 \ldots y_n} \lor S_{y_1 \ldots y_{n-1} x_m} = S_2 \lor S_3 \).

Suppose \( S \in Cl_s(L) \) and \( S_{y_1 \ldots y_{n-1} x_m} \land S_{y_1 \ldots y_{n-1} x_m} < S_{y_1 \ldots y_{n-1} x_m} \). Then, by \( y_p \land y_q = 0, x_n \land y_p = 0, (p \neq q; p, q = 1, \ldots, n) \), it follows \( y_j \in S. (j = 1, \ldots, n - 1) \) and \( x_n \in S \). Additionally, by the supposition, we decide that there is \( z \in L \setminus (A(L) \cup 0) \) with \( z < y_n \) satisfying \( y_j = z \land x_n = 0 \). This implies \( \leq y_j < h(y_n), a contradiction to h(y_n) = 2 \). Moreover, \( S_{y_1 \ldots y_{n-1} x_m} < S_{y_1 \ldots y_{n-1} x_m} \) is real.

Analogously, \( S_{y_1 \ldots y_n} \land S_{y_1 \ldots y_n x_m} \) is true. Hence, \( S_{y_1 \ldots y_n} \lor S_{y_1 \ldots y_{n-1} x_m} = S_{y_1 \ldots y_n x_m} \) is true and \( S_{y_1 \ldots y_n} \land S_{y_1 \ldots y_{n-1} x_m} = S_{y_1 \ldots y_{n-1} x_m} \) holds.
When \( h(y_n) = 2 \) and \( x_n \land y_n \neq 0 \).

Since \( h(x_n) = h(y_n) = 2 \) and \( x_j \land y_n \neq 0 \) taken together follows \( x_n \land y_n < x_n \land y_n \) and \( x_n \land y_n \in A(L) \).

We prove \( S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_n} = S_{y_1 \ldots y_{n-1} (x_n \lor y_n)} \) if \( y_j \land (x_n \lor y_n) = 0, (j = 1, \ldots, n-1) \).

Using the (3) in Remark 1, item (3.1) and Definition 1, it is easily to obtain \( S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_n} = S_{z_1 \ldots z_m} \), \((z_i \in L \setminus (A(L) \cup 0); i = 1, \ldots, m)\). According to \( z_t \land z_s = 0, (t, s = 1, \ldots, m; t \neq s) \), we follow \( y_j = z_j, (j = 1, \ldots, n - 1) \). In view of \( x_n \land y_n \neq 0 \), we attain \( x_n \land y_n \leq z_n \). And further, we gain \( x_n \land y_n \leq z_n \).

Moreover, we decide \( S_{y_1 \ldots y_{n-1} (x_n \lor y_n)} \leq S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_{n-1} x_n} \). On the other hand, combining \( x_n \land y_n \leq x_n \land y_n \) with items (3.1) and (3.2), we may easily arrive at \( S_{y_1 \ldots y_n} \lor S_{y_1 \ldots y_{n-1} x_n} \leq S_{y_1 \ldots y_{n-1} (x_n \lor y_n)} \).

So,

\[
S_{y_1 \ldots y_n} \lor S_{y_1 \ldots y_{n-1} x_n} \leq S_{y_1 \ldots y_{n-1} (x_n \lor y_n)}
\]

is followed.

Combining the above, we affirm \( S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_n} = S_{y_1 \ldots y_{n-1} (x_n \lor y_n)} \).

Using the “Additionally” part in item (3.16.1), we obtain \( S_{y_1 \ldots y_{n-1} x_n} \land S_{y_1 \ldots y_n} < S_{y_1 \ldots y_{n-1} (x_n \lor y_n)} \).

If \( y_j \land (x_n \lor \lor_{i=m+1}^n y_i) = 0 \), \((j = 1, \ldots, n)\) and the above proof for the case of \( y_j \land (x_n \lor y_n) = 0 \), \((j = 1, \ldots, n - 1)\), we may gain \( S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_n} = S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_{n-1} x_n} \lor S_{y_1 \ldots y_n} \lor S_{y_1 \ldots y_{n-1} x_n} \). Applying the “Additionally” part in item (3.16.2), we gain \( S_{y_1 \ldots y_n} \lor S_{y_1 \ldots y_{n-1} x_n} \). 

When \( h(y_n) > 2 \). Applying with the result above, we find \( h(x_n) > 2 \). In virtue of \( S_2 \land S_3 < S_{y_1 \ldots y_n} = S_3 \) and item (3.14), we obtain \( S_2 \land S_3 = S_{y_1 \ldots y_{n-1} z_n} \) in which \( z_n < y_n \) and \( z_n \in L \setminus (A(L) \cup 0) \). At the same time, we may attain \( z_n < x_n \) since \( S_2 \land S_3 < S_{y_1 \ldots y_{n-1} x_n} \). This follows \( z_n = x_n \land y_n \in L \setminus (A(L) \cup 0) \).

Furthermore, we reveal \( S_{y_1 \ldots y_{n-1} z_n} < S_{y_1 \ldots y_{n-1} x_n} \), \( S_{y_1 \ldots y_n} \). Thus, for any \( \lor_{i=1}^m y_i \leq p < x_n \land \lor_{i=m+1}^n y_i \leq q < y_n \), and \( \{j_i \mid i = 1, \ldots, m_2\}, \{k_i \mid i = 1, \ldots, m_2\} \subseteq \{1, \ldots, n - 1\} \) there is \( x_{x_i} \in \{1, \ldots, n - 1\} \setminus \{j_i \mid i = 1, \ldots, m_2\} \), \( y_{x_j} \in \{1, \ldots, n - 1\} \setminus \{k_i \mid i = 1, \ldots, m_2\} \) satisfying \( y_{x_k} \land p \neq 0, y_{x_j} \land q \neq 0 \). Owing to items (3.16.2) and (3.1), we confirm \( S_2 \land S_3 < S_2 \land S_3 = S_{y_1 \ldots y_{n-1} (x_n \lor y_n)} \).

Combining Lemma 6 with Lemma 7, we may express the following theorem.

**Theorem 2.** Let \( L \) be an atomistic complete finite lattice with \( |F^2| \geq 2 \). Then \( C_{L_q}(L) \) is a geometric lattice if and only if \( L \) satisfies items from (3.8) to (3.16).

4. Conclusion

To sum up our results, we make the following remarks.
(1) Though some of conditions in items from (3.8) to (3.16) seem to be complex, they are actually expressed in a detailed and applicable way. In addition they complete the check process, since items from (3.8) to (3.16) are suitable for finite cases. Additionally, from Lemma 7, or from the results of Section 2, we can confirm that items from (3.8) to (3.16) are necessary and essential when we decide the geometry of $C_{ik}(L)$ for an atomistic complete finite lattice $L$.

(2) It is well known that an atomistic complete lattice is finite or infinite. Theorem 1 and Theorem 2 together answer the open problem of S. Radeleczki for finite cases. In fact, Theorem 1 is also true for infinite atomistic complete lattice. However, many preparatory works for Theorem 2 of this paper are proved with inductive method. This illustrates that Theorem 2 cannot be directly generalized to the infinite case. Even though, we may hope that the results of this paper will assist the solution of S. Radeleczki’s open problem for infinite cases. We intend to pursue this line of research in the future.

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