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ON GENERALIZED DOUBLE STATISTICAL CONVERGENCE IN LOCALLY SOLID RIESZ SPACES

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Abstract. The concept of statistical convergence is one of the most active area of research in the field of summability. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. In this paper, we introduce the idea of double $\mathcal{I}_{\bar{\lambda}}$-statistical convergence in a locally solid Riesz space and study some of its properties by using the mathematical tools of the theory of topological vector spaces.

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1. Introduction

The notion of Riesz space was first introduced by F. Riesz [17] in 1928 and since then it has found several applications in measure theory, operator theory, optimization and also in economics (see [3]). It is well known that a topology on a vector space that makes the operations of addition and scalar multiplication continuous is called a linear topology and a vector space endowed with a linear topology is called a topological vector space. A Riesz space is an ordered vector space which is also a lattice, endowed with a linear topology. Further if it has a base consisting of solid sets at zero then it is known as a locally solid Riesz space. More investigations and applications of Riesz space can be found in [11, 25].

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [8] and Steinhaus [24] independently in the same year 1951 and its topological consequences were studied first by Fridy [9] and Šalát [18]. The notion has also been defined and studied in different steps, for example, in the locally convex space [12]; in topological groups ([4]); in probabilistic normed spaces[19]; in intuitionistic fuzzy normed spaces [15]; in random two normed spaces [21]. Recently, in [1, 2], Albayrak and Pehlivan studied this notion in locally solid Riesz spaces. Quite recently, Mohiuddine and et.al [13] studied statistically convergent, statistically bounded and statistically Cauchy for double sequences in locally solid Riesz spaces. Also in [5], for single sequences, the ideas of I_λ-statistical-τ-convergence, and I_λ-statistical-τ- Cauchy condition of sequences in a locally solid Riesz space were investigated.

The more general idea of λ-statistical convergence was introduced by Mursaleen in [16]. Subsequently a lot of interesting investigations have been done on this convergence (see, for example [20, 21], where more references can be found).

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [10] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [5, 7, 10, 22, 23], where many important references can be found.

Recently in [6, 23] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}_\lambda$-statistical convergence and investigated their properties.

It is quite natural to expect that the idea of $\mathcal{I}_\lambda$-statistical convergence in a locally solid Riesz space can be extend for double sequences. As a natural consequence, in this paper, we introduce the idea of double $\mathcal{I}_{\bar{\lambda}}$-statistical convergence in a locally solid Riesz space and study some of its properties by using the mathematical tools of the theory of topological vector spaces.

It should be noted that our paper contains the results of [13] as special cases. However one can see that the methods of proofs are not at all analogous to those of [13] and are more complicated.

2. Preliminaries

In this section we recall some of the basic concepts of Riesz spaces .

Definition 1. Let L be a real vector space and let $\leq$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties:

(i) If $x, y \in L$ and $y \leq x$ then $y + z \leq x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \leq x$ then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

If in addition L is a lattice with respect to the partial ordering, then L is said to be a Riesz space (or a vector lattice).

For an element x of a Riesz space L the positive part of x is defined by $x^+ = x \bigvee \theta$, the negative part of x by $x^- = (-x) \bigvee \theta$ and the absolute value of x by $|x| = x \bigvee (-x)$, where θ is the element zero of L.

A subset S of a Riesz space L is said to be solid if $y \in S$ and $|x| \leq |y|$ imply $x \in S$.

A topology τ on a real vector space L that makes the addition and scalar multiplication continuous is said to be a linear topology, that is when the mappings

$$(x, y) \to x + y \quad (\text{from } (L \times L, \tau \times \tau) \to (L, \tau))$$
$$(\lambda, x) \to \lambda x \quad (\text{from } (\mathbb{R} \times L, \sigma \times \tau) \to (L, \tau))$$

are continuous where σ is the usual topology on $\mathbb{R}$. In this case the pair (L, τ) is called a topological vector space.

Every linear topology τ on a vector space L has a base $\mathcal{N}$ for the neighborhoods of θ satisfying the following properties:

a) Each $V \in \mathcal{N}$ is a balanced set, that is $\lambda x \in V$ holds for all $x \in V$ and every $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

b) Each $V \in \mathcal{N}$ is an absorbing set, that is for every $x \in L$, there exists a $\lambda > 0$ such that $\lambda x \in V$.

c) For each $V \in \mathcal{N}$ there exists some $W \in \mathcal{N}$ with $W + W \subset V$.

Definition 2. A linear topology τ on a Riesz space L is said to be locally solid if τ has a base at zero consisting of solid sets. A locally solid Riesz space (L, τ) is a Riesz space L equipped with a locally solid topology τ.

$\mathcal{N}_{sol}$ will stand for a base at zero consisting of solid sets and satisfying the properties (a),(b) and (c) in a locally solid topology.

We now recall the following basic facts from [10].

A family $\mathcal{I}$ of subset of a non-empty set X is said to be an ideal if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$. $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$. $\mathcal{I}$ is admissible if it contains all singletons. If $\mathcal{I}$ is a proper non-trivial ideal then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset X : M^c \in \mathcal{I}\}$ is a filter on X (where c stands for the complement.) It is called the filter associated with the ideal $\mathcal{I}$.

If we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal of $\mathbf{N}$.

Definition 3 ([10]). Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.

(i) The sequence $\{x_k\}$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $x_0 \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\} \in \mathcal{I}$.

(ii) The sequence $\{x_k\}$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}^*$-convergent to $x_0 \in \mathbb{R}$ if there exists $M \in F(\mathcal{I})$ such that $\{x_k\}_{k \in M}$ converges to x_0.

3. Double $I_{\bar{\lambda}}$- statistical topological convergence in locally solid Riesz spaces

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$, the set of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \ exists.$$

Note that if $K \subset \mathbb{N}$ is finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbf{N}, \delta(K^C) = 1 - \delta(K)$.

Definition 4. A sequence $x = (x_k)$ is said to be *statistically convergent* to x_0 if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}) = 0.$$

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$\liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $(\frac{K_{m,n}}{mn})_{m,n=1,1}^{\infty}$ has a limit then we say that K has a natural density and is defined as

$$\lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : (i,j) \in \mathbb{N} \times \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{m,n} \frac{K_{m,n}}{mn} \leq \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero).

Recently Mursaleen and Edely [14] presented the notion statistical convergence for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be statistically convergent to x_0, provided that for each $\epsilon > 0$

$$\lim_{m,n} \frac{1}{mn} |\{(k,l) : k \leq m \text{ and } l \leq n, |x_{k,l} - x_0| \geq \epsilon\}| = 0.$$

Let $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1.$$

Definition 5 ([23])**.** A sequence $x = (x_k)$ is said to be $\mathcal{I}_\lambda-$statistically convergent or $\mathcal{I}_\lambda - st-$ convergent to $x_0 \in \mathbb{R}$, if for every $\epsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : |x_k - x_0| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I},$$

where $I_n = [n - \lambda_n + 1, n]$. In this case we write $\mathcal{I}_\lambda - st - \lim x = L$.

Remark 1. For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}_\lambda - st$-convergence coincides with $\lambda-$statistical convergence [16]. Again taking $\lambda_n = n$ it is easy to observe that $\mathcal{I}_\lambda - st$-convergence becomes only $\mathcal{I}$-statistical convergence [6, 23] which again coincides with statistical convergence for $\mathcal{I} = \mathcal{I}_{fin}$.

Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers both of which tends to ∞ as m and n approach ∞, respectively. Also let $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 0$ and $\mu_{n+1} \leq \mu_n + 1, \mu_1 = 0$. The collection of such sequence (λ, μ) will be denoted by Δ.

Throughout this paper we shall denote $\lambda_m \mu_n$ by $\bar{\lambda}_{m,n}$ and $(k \in I_m, l \in I_n)$ by $(k,l) \in I_{m,n}$. We now introduce the definition of $\mathcal{I}_{\bar{\lambda}}$- double statistical convergence in a locally solid Riesz space.

Definition 6. Let $x = (x_{kl})$ be a double sequence in a locally solid Riesz space (L, τ). We say that x is $\mathcal{I}_{\bar{\lambda}} - st_\tau$-convergent to x_0 if for every $\tau-$ neighborhood U of zero, and for $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{m,n}} \left|\{(k,l) \in \mathcal{I}_{mn} : x_{kl} - x_0 \notin U\}\right| \geq \delta\right\} \in \mathcal{I}$$

In this case we write $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = x_0$ (or $x_{kl} \xrightarrow{\mathcal{I}_{\bar{\lambda}} - st_\tau} x_0$ briefly).

Remark 2. For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}_{\bar{\lambda}} - st_\tau$-convergence becomes $\bar{\lambda}$ -statistical τ- convergence in a locally solid Riesz space which becomes double statistical τ-convergence in a locally sold Riesz space taking $\lambda_{n,m} = n, m$.

Definition 7. Let $x = (x_{kl})$ be a double sequence in locally solid Riesz space(L, τ). We say that $\mathcal{I}_{\bar{\lambda}} - st_\tau$ bounded if for every $\tau-$ neighborhood U of zero and $\delta > 0$, there exists some $\alpha > 0$ such that

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{m,n}} \left|\{(k,l) \in I_{mn} : \alpha x_{kl} \notin U\}\right| \geq \delta\right\} \in \mathcal{I}$$

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Definition 8. Let $x = (x_{kl})$ be a double sequence in a locally solid Riesz space (L, τ). We say that x is $I_{\bar{\lambda}} - st_\tau$-Cauchy if for every $\tau-$ neighborhood U of zero and $\delta > 0$, there exist $p, q \in \mathbb{N} \times \mathbb{N}$ such that

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{m,n}} \left|\left\{(k,l) \in I_{mn} : x_{kl} - x_{pq} \notin U\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

We are now ready to present some basic properties of this new convergence in a locally solid Riesz space.

Theorem 1. *Let (L, τ) be a Hausdorff locally solid Riesz space, $x = (x_{kl})$ and $y = (y_{kl})$ be two sequences in L. Then the following hold.*

a) *If $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = y_0$ and $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = z_0$ then $y_0 = z_0$.*
b) *If $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = x_0$, then $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim \alpha x_{kl} = \alpha x_0$ for each $\alpha \in \mathbb{R}$.*
c) *If $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = x_0$ and $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim y_{kl} = y_0$ then $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim (x_{kl} + y_{kl}) = x_0 + y_0$.*

Proof. a) Let U be any $\tau-$ neighborhood of zero. Then there exists a $V \in \mathcal{N}_{sol}$ such that $V \subset U$. Choose a $W \in \mathcal{N}_{sol}$ such that $W + W \subset V$. Let $\delta = \frac{1}{5}$. Since $I_{\bar{\lambda}} - st_\tau - \lim x_{kl} = y_0$ and $I_{\bar{\lambda}} - st_\tau - \lim x_{kl} = z_0$ so

$$K_1 = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} \left|\{(k,l) \in I_{nm} : x_{kl} - y_0 \notin W\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$$

and

$$K_2 = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - z_0 \notin W\}| < \delta \right\} \in \mathcal{F}(\mathcal{I}).$$

Then $K = K_1 \cap K_2 \in \mathcal{F}(\mathcal{I})$ and for $(m,n) \in K$,

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - y_0 \notin W\}| < \delta$$

i.e.

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - y_0 \in W\}| > 1 - \delta = \frac{4}{5}. \tag{3.1}$$

Similarly

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - z_0 \in W\}| > \frac{4}{5}. \tag{3.2}$$

Now note that $\{(k,l) \in I_{nm} : x_{kl} - y_0 \in W\}$ and $\{(k,l) \in I_{nm} : x_{kl} - z_0 \in W\}$ can not be disjoint for then we will have $\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm}\}| > \frac{8}{5}$ which is impossible. So there is a $(k_n, l_m) \in I_{nm}$ for which

$$x_{k_n,l_m} - y_0 \in W \text{ and } x_{k_n,l_m} - z_0 \in W.$$

Then

$$x_0 - z_0 = y_0 - x_{k_{k_n,l_m}} + x_{k_n,l_m} - z_0 \in W + W \subset V \subset U.$$

Thus $y_0 - z_0 \in U$ for every $\tau-$ neighborhood U of zero. Since (L,τ) is Hausdorff, the intersection of all $\tau-$ neighborhoods of zero is the singleton $\{\theta\}$ and so $y_0 - z_0 = \theta$ i.e. $y_0 = z_0$.

b) Let $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = x_0$ and let U be an arbitrary $\tau-$ neighborhood of zero. Choose $V \in \mathcal{N}_{sol}$ such that $V \subset U$. For any $1 > \delta > 0$,

$$K = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - x_0 \notin V\}| < \delta \right\} \in \mathcal{F}(\mathcal{I})$$

i.e. $\forall (n,m) \in K$,

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - x_0 \in V\}| > 1 - \delta.$$

First let $|\alpha| \leq 1$. Since V is balanced, $x_{kl} - x_0 \in V$ implies that $\alpha(x_{kl} - x_0) \in V$. Hence

$$\{(k,l) \in I_{nm} : \alpha x_{kl} - \alpha x_0 \in V\} \supset \{(k,l) \in I_{nm} : x_{kl} - x_0 \in V\}$$

and so $\forall (n,m) \in K$,

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : \alpha x_{kl} - \alpha x_0 \in V\}|$$

$$\geq \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - x_0 \in V\}| > 1 - \delta$$

which implies that

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : \alpha x_{kl} - \alpha x_0 \notin V\}| < \delta\right\} \supset K$$

and consequently

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : \alpha x_{kl} - \alpha x_0 \notin V\}| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

Now if $|\alpha| > 1$ and $[|\alpha|]$ is the smallest integer greater or equal to $|\alpha|$, choose $W \in \mathcal{N}_{sol}$ such that $[|\alpha|]\, W \subset V$. Again for $1 > \delta > 0$, taking

$$K = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{k \in I_{nm} : x_{kl} - x_0 \notin W\}| < \delta\right\} \in \mathcal{F}(\mathcal{I})$$

and in view of the fact that

$$|\alpha x_0 - \alpha x_{kl}| = |\alpha|\, |x_0 - x_{kl}| \leq [|\alpha|]\, |x_{kl} - x_0| \in [|\alpha|]\, W \subset V \subset U$$

which consequently implies that $\alpha x_0 - \alpha x_{kl} \in V \subset U$ (since V is solid), proceeding as before, we can conclude that

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : \alpha x_{kl} - \alpha x_0 \notin U\}| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

This prove that $I_{\bar{\lambda}} - st_\tau - \lim \alpha x_{kl} = \alpha x_0$.

c) Let U be an arbitrary $\tau-$ neighborhood of zero. Then there are $V, W \in \mathcal{N}_{sol}$ such that $W + W \subset V \subset U$. Since $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = x_0$ and $\mathcal{I}_\lambda - st_\tau - \lim y_{kl} = y_0$ we have for $0 < \delta < 1$,

$$K_1 = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{m,n}} |\{(k,l) \in I_{nm} : x_{kl} - x_0 \notin W\}| < \frac{\delta}{3}\right\} \in \mathcal{F}(\mathcal{I})$$

and

$$K_2 = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : y_{kl} - y_0 \notin W\}| < \frac{\delta}{3}\right\} \in \mathcal{F}(\mathcal{I}).$$

If $K = K_1 \cap K_2$, then $\forall (n,m) \in K$,

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - x_0 \notin W\}| < \frac{\delta}{3}$$

i.e.

$$\frac{1}{\bar{\lambda}_{n,m}} |\{(k,l) \in I_{nm} : x_{kl} - x_0 \in W\}| > 1 - \frac{\delta}{3}$$

and also

$$\frac{1}{\bar{\lambda}_{n,m}}\left|\{(k,l)\in I_{nm}:y_{kl}-y_0\notin W\}\right|<\frac{\delta}{3}.$$

But

$$(x_{kl}+y_{kl})-(x_0+y_0)=(x_{kl}-x_0)+(y_{kl}-y_0)\in W+W\subset V\subset U$$

$\forall(k,l)\in I_{nm}$ such that $(k,l)\in A\cap B$ when $\{(k,l)\in I_{nm}:x_{kl}-x_0\in W\}=A$ and $\{(k,l)\in I_{nm}:y_{kl}-y_0\in W\}=B$. Note that

$$|A|=|A\cap B|+|A\backslash B|\le|A\cap B|+|B^c|$$

i.e.

$$\begin{aligned}\frac{1}{\bar{\lambda}_{m,n}}|A| &\le \frac{1}{\bar{\lambda}_{m,n}}|A\cap B|+\frac{1}{\bar{\lambda}_{m,n}}|B^c|\\ &< \frac{1}{\bar{\lambda}_{m,n}}|A\cap B|+\frac{\delta}{3}\end{aligned}$$

i.e.

$$\begin{aligned}\frac{1}{\bar{\lambda}_{m,n}}|A\cap B| &= \frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-x_0\in W\wedge y_{kl}-y_0\in W\}\right|\\ &> \frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-x_0\in W\}\right|-\frac{\delta}{3}\\ &> 1-\frac{\delta}{3}-\frac{\delta}{3}\\ &> 1-\delta.\end{aligned}$$

Since

$$\{(k,l)\in I_{nm}:(x_{kl}+y_{kl})-(x_0+y_0)\in U\}\supset A\cap B$$

so for all $(n,m)\in K$,

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:(x_{kl}+y_{kl})-(x_0+y_0)\in U\}\right|\ge\frac{1}{\bar{\lambda}_{n,m}}|A\cap B|>1-\delta$$

i.e.

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:(x_{kl}+y_{kl})-(x_0+y_0)\notin U\}\right|<\delta.$$

Therefore

$$K\subset\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:(x_{kl}+y_{kl})-(x_0+y_0)\notin U\}\right|<\delta\right\}$$

and so

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:(x_{kl}+y_{kl})-(x_0+y_0)\in U\}\right|<\delta\right\}\in\mathcal{F}(\mathcal{I})$$

which completes the proof of the result. □

Theorem 2. *Let (L,τ) be a locally solid Riesz space. Let $x=(x_{kl})$, $y=(y_{kl})$ and $z=(z_{kl})$ be three sequences in L such that $x_{kl}\le y_{kl}\le z_{kl}$ for each $(k,l)\in\mathbb{N}\times\mathbb{N}$. If $\mathcal{I}_{\bar{\lambda}}-st_\tau-\lim x_{kl}=a=\mathcal{I}_{\bar{\lambda}}-st_\tau-\lim z_{kl}$ then $\mathcal{I}_{\bar{\lambda}}-st_\tau-\lim y_{kl}=a$.*

Proof. Let U be an arbitrary $\tau-$ neighborhood of zero. Choose $V,W\in\mathcal{N}_{sol}$ such that $W+W\subset V\subset U$. Since $\mathcal{I}_{\bar{\lambda}}-st_\tau-\lim x_{kl}=a=\mathcal{I}_{\bar{\lambda}}-st_\tau-\lim z_{kl}$, so for $0<\delta<1$,

$$K_1=\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-a\notin W\}\right|<\frac{\delta}{3}\right\}\in\mathcal{F}(\mathcal{I})$$

and

$$K_2=\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:z_{kl}-a\notin W\}\right|<\frac{\delta}{3}\right\}\in\mathcal{F}(\mathcal{I}).$$

Then we see that $\forall(n,m)\in K$,

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-a\notin W\}\right|<\frac{\delta}{3}$$

i.e.

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-a\in W\}\right|>1-\frac{\delta}{3}$$

and

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:z_{kl}-a\notin W\}\right|<\frac{\delta}{3}.$$

Writing $A=\{(k,l)\in I_{nm}:x_{kl}-a\in W\}$ and $B=\{(k,l)\in I_{nm}:z_{kl}-a\in W\}$ we see that $\forall(k,l)\in A\cap B$,

$$x_{kl}\le y_{kl}\le z_{kl},$$

$$x_{kl}-a\le y_{kl}-a\le z_{kl}-a,$$

$$|y_{kl}-a|\le|x_{kl}-a|+|z_{kl}-a|\in W+W\subset V$$

and as V is solid so

$$y_{kl}-a\in V\subset U.$$

Clearly $\{(k,l)\in I_{nm}:y_{kl}-a\in U\}\supset A\cap B$ and as in the previous theorem we can show that $\forall(n,m)\in K$,

$$\frac{1}{\bar{\lambda}_{n,m}}\left|\{(k,l)\in I_{nm}:y_{kl}-a\in U\}\right|\ge\frac{1}{\bar{\lambda}_{m,n}}|A\cap B|>1-\delta$$

i.e.

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:y_{kl}-a\notin U\}\right|<\delta.$$

Therefore

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:y_{kl}-a\notin U\}\right|<\delta\right\}\supset K$$

where $K\in\mathcal{F}(\mathcal{I})$ and so

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{nm}}\left|\{(k,l)\in I_{nm}:y_{kl}-a\notin U\}\right|\geq\delta\right\}\in\mathcal{I}$$

This proves that $\mathcal{I}_{\bar{\lambda}}-st_{\tau}-\lim y_{kl}=a$. □

Theorem 3. *An $\mathcal{I}_{\bar{\lambda}}-$statistically $\tau-$convergent sequence $x=(x_{kl})$ in a locally solid Riesz space (L,τ) is $\mathcal{I}_{\bar{\lambda}}-$ statistically $\tau-$bounded.*

Proof. Let $x=(x_{kl})$ be $\mathcal{I}_{\bar{\lambda}}$- statistically $\tau-$convergent to $x_0\in L$. Let U be an arbitrary $\tau-$neigbourhood of zero. Choose $V,W\in\mathcal{N}_{sol}$ such that $W+W\subset V\subset U$. Since W is absorbing there is a $\mu>0$ such that $\mu x_0\in W$. Choose $\alpha\leq 1$ so that $\alpha\leq\mu$. Since W is solid and $|\alpha x_0|\leq|\mu x_0|$, we have $\alpha x_0\in W$. Again as W is balanced, $x_{nm}-x_0\in W$ implies that $\alpha(x_{kl}-x_0)\in W$. Now for any $0<\delta<1$,

$$K=\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-x_0\notin W\}\right|<\delta\right\}\in\mathcal{F}(\mathcal{I}).$$

Thus for all $(n,m)\in K$,

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-x_0\notin W\}\right|<\delta$$

i.e.

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-x_0\in W\}\right|>1-\delta.$$

If $B_{nm}=\{(k,l)\in I_{nm}:x_{kl}-x_0\in W\}$ then $\forall(k,l)\in B_{nm}$

$$\alpha x_{kl}=\alpha(x_{kl}-x_0)+\alpha x_0\in W+W\subset V\subset U$$

and so for all $(n,m)\in K$,

$$\begin{aligned}\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:\alpha x_{kl}\in W\}\right|&\geq\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:x_{kl}-x_0\in W\}\right|\\&>1-\delta\end{aligned}$$

i.e.

$$\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:\alpha x_{kl}\notin W\}\right|<\delta.$$

Hence

$$K\subset\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{\lambda}_{m,n}}\left|\{(k,l)\in I_{nm}:\alpha x_{kl}\notin W\}\right|<\delta.\right\}$$

Since $K \in \mathcal{F}(\mathcal{I})$ so the set on the right hand side also belongs of $\mathcal{F}(\mathcal{I})$ and it is proven that (x_{kl}) is $\mathcal{I}_{\bar{\lambda}}$- statistically τ -bounded. □

Theorem 4. *If a sequence $x = (x_{kl})$ in a locally solid Riesz space (L,τ) is $\mathcal{I}_{\bar{\lambda}}-$ statistically $\tau-$convergent then it is $\mathcal{I}_{\bar{\lambda}}-$ statistically $\tau-$ Cauchy.*

Proof. Let $x = (x_{kl})$ be $\mathcal{I}_{\bar{\lambda}}$- statistically τ -convergent to $x_0 \in L$. Let $0 < \delta < 1$. Then

$$K = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{m,n}} \left|\{(k,l) \in I_{nm} : x_{kl} - x_0 \notin W\}\right| < \delta \right\} \in \mathcal{F}(\mathcal{I}).$$

For all $(n,m) \in K$,

$$\frac{1}{\bar{\lambda}_{m,n}} \left|\{(k,l) \in I_{nm} : x_{kl} - x_0 \notin W\}\right| < \delta$$

i.e.

$$\frac{1}{\bar{\lambda}_{m,n}} \left|\{(k,l) \in I_{nm} : x_{kl} - x_0 \in W\}\right| > 1 - \delta.$$

Choose $(n,m) \in K$ and in view of above we can choose $p,q \in \{(k,l) \in I_{nm} : x_{kl} - x_0 \in W\}$ (since this set can not be empty). Then $x_{pq} - x_0 \in W$. Now observe that if for $(k,l) \in I_{nm}, x_{kl} - x_0 \in W$ then

$$x_{kl} - x_{pq} = x_{kl} - x_0 + x_0 - x_{pq} \in W + W \subset V \subset U.$$

Hence as in the earlier proofs we can prove that

$$K \subset \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{m,n}} \left|\left\{(k,l) \in I_{nm} : x_{kl} - x_{pq} \notin W\right\}\right| < \delta \right\}$$

which consequently implies that (x_{kl}) is $\mathcal{I}_{\bar{\lambda}}$- statistically τ- Cauchy. □

Finally we conclude this paper by presenting the following theorem. Before presenting the next theorem, let us consider the following definition.

Definition 9. Let (L_1,τ_1) and (L_2,τ_2) be locally solid Riesz spaces and $A \subset L_1$. A mapping $f : A \to L_2$ is said to be $\mathcal{I}_{\bar{\lambda}}-$ statistically $\tau-$ continuous at a point $x_0 \in A$ if $\mathcal{I}_{\bar{\lambda}} - st_\tau - \lim x_{kl} = x_0$ implies that $I_\lambda - st_\tau - \lim f(x_{kl}) = f(x_0)$.

Theorem 2.17 in [3] states that the basic lattice operations are uniformly continuous. So we can state the following theorem.

Theorem 5. *If a function $f : L_1 \to L_2$ is uniformly continuous then f is $\mathcal{I}_{\bar{\lambda}}-$ statistically $\tau-$ continuous.*

The proof of this theorem is omitted since it can be proved by using the techniques present by Das and Savas in [5].

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