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APPROXIMATION BY A GENERALIZED SZÁSZ TYPE OPERATOR FOR FUNCTIONS OF TWO VARIABLES

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Abstract. In the present paper, we define a new Szász-Mirakjan type operator in exponential weighted spaces for functions of two variables having exponential growth at infinity using a method given by Jakimovski-Leviatan. This operator is a generalization of two variables of an operator defined by A. Ciupa [1]. In this study, we investigate approximation properties and also estimate the rate of convergence for this new operator.

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Keywords: linear positive operator, Jakimovski-Leviatan operator, weighted space, modulus of continuity, rate of convergence

1. INTRODUCTION

For a real function of real variable $f : [0,\infty) \to \mathbb{R}$, the Szász-Mirakjan operators are defined in [2] as

$$S_n(f;x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f(\frac{j}{n}) \ , \ x \in [0,\infty),$$

where the convergence of $S_n(f;x)$ to $f(x)$ under the exponential growth condition on f that is $|f(x)| \le Ce^{Bx}$, for all $x \in [0,\infty)$, with $C, B > 0$ was proved. Then, various modifications and further properties of the Szász-Mirakjan operators have been studied intensively by many authors (e.g. [1, 3–9]).

In [4], A. Jakimovski and D. Leviatan investigated approximation properties of a generalization of the Szász-Mirakjan operators which are stated as follows:

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the disk $|z| < R, R > 1$ and suppose $g(1) \neq 0$. Define the Appell polynomials $p_k(x) = p_k(x,g)$ $(k \ge 0)$ by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k.$$

For each function f defined in $[0,\infty)$, they considered the operators L_n defined by

$$L_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right),n>0$$

and also the authors obtained several approximation properties of these operators. A. Ciupa [1] introduced a Szász-Mirakjan type operator that is a generalization of the operator defined by M. Lesniewicz and L. Rempulska [5] using the method given by Jakimovski-Leviatan. A. Ciupa studied the properties of approximation for functions of one variable in the space of continuous functions having an exponential growth at infinity.

In this paper, inspired by [1], for each function f defined in $[0,\infty)\times[0,\infty)$, we define the operators $L_{n,m}$ by

$$L_{n,m}(f;x,y)=\frac{e^{-nx}e^{-my}}{(g(1))^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_k(nx)\,p_j(my)\,f\left(\frac{k}{n},\frac{j}{m}\right)$$

where

$$g(u_1)e^{u_1x}g(u_2)e^{u_2y}=\sum_{k=0}^{\infty}p_k(x)\,u_1^k\sum_{j=0}^{\infty}p_j(y)\,u_2^j.$$

Now, we consider the function $g(x)=\sum\limits_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}=\sinh x$ where $\sinh x$ is the hyperbolic function of x and let p_k be the polynomials generated by relation

$$\sinh u_1\sinh(u_1x)\sinh u_2\sinh(u_2y)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(x)\,p_{2j}(y)\,u_1^{2k}u_2^{2j}.$$

Using the following equalities

$$\sinh u_1\sinh(u_1x)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(1+x)^{2k}-(1-x)^{2k}}{(2k)!}u_1^{2k}$$

$$\sinh u_2\sinh(u_2y)=\frac{1}{2}\sum_{j=0}^{\infty}\frac{(1+y)^{2j}-(1-y)^{2j}}{(2j)!}u_2^{2j},$$

we have

$$p_{2k}(x)=\frac{(1+x)^{2k}-(1-x)^{2k}}{2(2k)!}\quad,\quad p_{2j}(y)=\frac{(1+y)^{2j}-(1-y)^{2j}}{2(2j)!}.$$

Let $C\left(R_1^2\right)$ be the set of all real-valued continuous functions of two variables on $R_1^2:=\{(x,y):x\geq 1,y\geq 1\}$.

For $p,q>0$ and $(x,y)\in R_1^2$, we define

$$w_{p,q}(x,y)=w_p(x)\,w_q(y)=e^{-px}e^{-qy}$$
$$C_{p,q}=\left\{f\in C\left(R_1^2\right):w_{p,q}f \text{ is uniformly continuous and bounded on } R_1^2\right\}$$
$$\|f\|_{p,q}=\sup_{(x,y)\in R_1^2} w_p(x)\,w_q(y)\,|f(x,y)|$$

and also for $h,k\geq 0$, $\delta\geq 0$, $f\in C_{p,q}$, the first order modulus of continuity given by

$$\omega\left(f,C_{p,q};\delta\right)=\sup_{0\leq h,k\leq\delta}\left\|\Delta_{h,k}f\right\|_{p,q}$$

where

$$\Delta_{h,k}f(x,y)=f(x+h,y+k)-f(x,y).$$

In this study, in the space $C_{p,q}$, $p,q>0$, we introduce the following positive linear operators

$$P_{n,m}(f;x,y)=\frac{1}{(\sinh 1)^2\sinh(nx)\sinh(my)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(nx)\,p_{2j}(my)\,f\left(\frac{2k}{n},\frac{2j}{m}\right)\tag{1}$$

$n,m\in\mathbb{N}$, $(x,y)\in R_1^2$ and investigate the theorems on convergence of $P_{n,m}(f;x,y)$ operators to functions of two variables. We also estimate the rate of convergence for this new operator by using the modulus of continuity.

2. Auxiliary Results

In this section, we will give some useful results in order to study the convergence of the sequence $\left(P_{n,m}f\right)$ to the function $f\in C_{p,q}$.

Lemma 1. *If $(x,y)\in R_1^2$ and $n,m\in N$, we have*

$$P_{n,m}\left(e_{0,0};x,y\right)=1$$
$$P_{n,m}\left(e_{1,0};x,y\right)=\frac{1}{n}\coth 1+x\coth(nx)$$
$$P_{n,m}\left(e_{0,1};x,y\right)=\frac{1}{m}\coth 1+y\coth(my)$$
$$P_{n,m}\left(e_{1,0}^2+e_{0,1}^2;x,y\right)=\left(x^2+y^2\right)+\left(\frac{1}{n^2}+\frac{1}{m^2}\right)(1+\coth 1)+(1+2\coth 1)\left(\frac{x}{n}\coth(nx)+\frac{y}{m}\coth(my)\right)$$

where $e_{i,j}(t_1,t_2)=t_1^i t_2^j$; $i,j\in\{0,1\}$ and $\coth u$ *is the hyperbolic function of u.*

Lemma 2. *If* $(x,y) \in R_1^2$, $p,q>0$ *and* $n,m \in N$, *then we have*

$$P_{n,m}\left(e^{pt_1};x,y\right)=\frac{1}{\sinh 1 \sinh(nx)}\sinh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)$$

$$P_{n,m}\left(e^{qt_2};x,y\right)=\frac{1}{\sinh 1 \sinh(my)}\sinh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)$$

$$P_{n,m}\left(t_1e^{pt_1};x,y\right)=\frac{e^{p/n}}{n}\frac{1}{\sinh 1 \sinh(nx)}\left\{\cosh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)+nx\sinh\left(e^{p/n}\right)\cosh\left(nxe^{p/n}\right)\right\}$$

$$P_{n,m}\left(t_2e^{qt_2};x,y\right)=\frac{e^{q/m}}{m}\frac{1}{\sinh 1 \sinh(my)}\left\{\cosh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)+my\sinh\left(e^{q/m}\right)\cosh\left(mye^{q/m}\right)\right\}$$

$$P_{n,m}\left(t_1^2e^{pt_1};x,y\right)=\frac{1}{\sinh 1 \sinh(nx)}\left\{\frac{e^{2p/n}}{n^2}\sinh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)+\frac{2x}{n}e^{2p/n}\cosh\left(e^{p/n}\right)\cosh\left(nxe^{p/n}\right)+x^2e^{2p/n}\sinh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)+\frac{1}{n^2}e^{p/n}\cosh\left(e^{p/n}\right)\sinh(nxe^{p/n})+\frac{x}{n}e^{p/n}\sinh\left(e^{p/n}\right)\cosh\left(nxe^{p/n}\right)\right\}$$

$$P_{n,m}\left(t_2^2e^{qt_2};x,y\right)=\frac{1}{\sinh 1 \sinh(my)}\left\{\frac{e^{2q/m}}{m^2}\sinh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)+\frac{2y}{m}e^{2q/m}\cosh\left(e^{q/m}\right)\cosh\left(mye^{q/m}\right)+y^2e^{2q/m}\sinh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)+\frac{1}{m^2}e^{q/m}\cosh\left(e^{q/m}\right)\sinh(mye^{q/m})+\frac{y}{m}e^{q/m}\sinh\left(e^{q/m}\right)\cosh\left(mye^{q/m}\right)\right\}.$$

Lemma 3. *For all* $(x,y)\in R_1^2$ *and* $n,m\in N$, *we have*

$$P_{n,m}\left((t_1-x)^2e^{pt_1};x,y\right)=\frac{1}{\sinh 1 \sinh nx}\left\{x^2\sinh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)\left[e^{p/n}-1\right]^2+\sinh\left(nxe^{p/n}\right)\left[\frac{e^{2p/n}}{n^2}\sinh\left(e^{p/n}\right)+\frac{e^{p/n}}{n^2}\cosh\left(e^{p/n}\right)-\frac{2x}{n}e^{p/n}\cosh\left(e^{p/n}\right)\right]+\cosh\left(nxe^{p/n}\right)\left[\frac{2x}{n}e^{2p/n}\cosh\left(e^{p/n}\right)\right.\right.$$

$$+ \frac{x}{n} e^{p/n} \sinh\left(e^{p/n}\right)\Big] - 2x^2 e^{p/n} \sinh\left(e^{p/n}\right) e^{-nxe^{p/n}}\Big\}$$

$$P_{n,m}\left((t_2-y)^2 e^{qt_2}; x, y\right)$$
$$= \frac{1}{\sinh 1 \sinh my}\Big\{ y^2 \sinh\left(e^{q/m}\right) \sinh\left(mye^{q/m}\right)\left[e^{q/m}-1\right]^2$$
$$+ \sinh\left(mye^{q/m}\right)\Big[\frac{e^{2q/m}}{m^2} \sinh\left(e^{q/m}\right) + \frac{e^{q/m}}{m^2} \cosh\left(e^{q/m}\right)$$
$$- \frac{2y}{m} e^{q/m} \cosh\left(e^{q/m}\right)\Big] + \cosh\left(mye^{q/m}\right)\Big[\frac{2y}{m} e^{2q/m} \cosh\left(e^{q/m}\right)$$
$$+ \frac{y}{m} e^{q/m} \sinh\left(e^{q/m}\right)\Big] - 2y^2 e^{q/m} \sinh\left(e^{q/m}\right) e^{-mye^{q/m}}\Big\}.$$

Lemma 4. *For all $(x,y) \in R_1^2$ and $n, m \in N$, we have*

$$P_{n,m}\left((t_1-x)^2; x, y\right) \le \frac{3(x+1)}{n}$$
$$P_{n,m}\left((t_2-y)^2; x, y\right) \le \frac{3(y+1)}{m}.$$

Proof. By Lemma 1, we get

$$P_{n,m}\left((t_1-x)^2; x, y\right) = (\coth(nx)-1)\, 2x\left(\frac{1}{n}\coth 1 - x\right) + \frac{x}{n}\coth nx + \frac{1}{n^2}(1+\coth 1).$$

Thus for $(x,y) \in R_1^2$, we can write

$$P_{n,m}\left((t_1-x)^2; x, y\right) \le \frac{x-1}{n} + \frac{2x}{n} + \frac{3}{n^2} \le \frac{3(x+1)}{n}.$$

Similarly, we can easily obtain

$$P_{n,m}\left((t_2-y)^2; x, y\right) \le \frac{3(y+1)}{m}.$$

□

Lemma 5. *Let $p,q > 0$, $r > p$, $s > q$ and let $n_0 = n_0(p,r), m_0 = m_0(q,s)$ be fixed natural numbers such that $n_0 > p/(\ln r - \ln p)$ and $m_0 > q/(\ln s - \ln q)$. Then there exist positive constants $C_{p,r}$ and $C_{q,s}$ depending only on p,r and q,s such that*

$$w_r(x)\, P_{n,m}\left((t_1-x)^2 e^{pt_1}; x, y\right) \le C_{p,r} \frac{\sinh\left(e^{p/n}\right)}{\sinh 1} \frac{x+2}{n}$$

$$w_s(y)\,P_{n,m}\left((t_2-y)^2 e^{qt_2};x,y\right) \le C_{q,s}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}\frac{y+2}{m}$$

for all $(x,y) \in R_1^2$ *and* $n \ge n_0, m \ge m_0$.

Proof. Firstly, for $m,n \in \mathbb{N}$, we consider the sequence of real numbers (p_n) and (q_m),

$$p_n = n\left(e^{p/n}+1\right) \tag{2.1}$$

$$q_m = m\left(e^{q/m}+1\right) \tag{2.2}$$

which are decreasing and $\lim\limits_{n\to\infty} p_n = p$, $\lim\limits_{m\to\infty} q_m = q$. Thus

$$p < p_n < pe^{p/n} \le pe^p \tag{2.3}$$

$$q < q_m < qe^{q/m} \le qe^q \tag{2.4}$$

Since $n_0 > p/(\ln r - \ln p)$, we have $e^{p/n_0} < e^{\ln(r/p)} = r/p$ and $r > pe^{p/n_0} > p_{n_0} > p_n$ for $n \ge n_0$. Also, because $m_0 > q/(\ln s - \ln q)$, we get $e^{q/m_0} < e^{\ln(s/q)} = s/q$ and $s > qe^{q/m_0} > q_{m_0} > q_m$ for $m \ge m_0$.

Applying 2.1, we obtain

$$\begin{aligned} \sinh\left(nxe^{p/n}\right)(\sinh nx)^{-1} &\le 2e^{p_n x} \\ \cosh\left(nxe^{p/n}\right)(\sinh nx)^{-1} &\le e^{p_n x} \\ x^2(\sinh nx)^{-1} &\le \frac{x}{n}. \end{aligned} \tag{2.5}$$

Also using 2.2, we get

$$\begin{aligned} \sinh\left(mye^{q/m}\right)(\sinh my)^{-1} &\le 2e^{q_m y} \\ \cosh\left(mye^{q/m}\right)(\sinh my)^{-1} &\le e^{q_m y} \\ y^2(\sinh my)^{-1} &\le \frac{y}{m}. \end{aligned} \tag{2.6}$$

By writing the last inequalities in Lemma 2, we get respectively

$$P_{n,m}\left(e^{pt_1};x,y\right) \le \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}2e^{p_n x}$$

$$P_{n,m}\left(e^{qt_2};x,y\right) \le \frac{\sinh\left(e^{q/m}\right)}{\sinh 1}2e^{q_m y}$$

and

$$\left\| P_{n,m}\left(e^{pt_1};x,y\right)\right\|_r \leq \sup \frac{\sinh\left(e^{p/n}\right)}{\sinh 1} 2e^{(p_n-r)x}$$
$$\left\| P_{n,m}\left(e^{qt_2};x,y\right)\right\|_s \leq \sup \frac{\sinh\left(e^{q/m}\right)}{\sinh 1} 2e^{(q_m-s)y}.$$

Taking into account Lemma 3 and 2.5, we obtain

$$\begin{aligned}
&P_{n,m}\left(\left(t_1-x\right)^2 e^{pt_1};x,y\right)\\
&\leq \frac{1}{\sinh 1}\left\{2x^2 e^{p_n x}\frac{p_n^2}{n^2}\sinh\left(e^{p/n}\right)+\frac{2x}{n}\sinh\left(e^{p/n}\right)e^{\frac{p}{n}-nxe^{p/n}}\right.\\
&\quad+\frac{2}{n^2}e^{p_n x}e^{2p/n}\sinh\left(e^{p/n}\right)+\frac{2}{n^2}e^{p_n x}e^{p/n}\cosh\left(e^{p/n}\right)\\
&\quad-\frac{4x}{n}e^{p_n x}e^{p/n}\cosh\left(e^{p/n}\right)+\frac{2x}{n}e^{p_n x}e^{2p/n}\cosh\left(e^{p/n}\right)\\
&\quad\left.+\frac{x}{n}e^{p_n x}e^{p/n}\sinh\left(e^{p/n}\right)\right\}\\
&\leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{2x^2\frac{p_n^2}{n^2}e^{p_n x}+\frac{2x}{n}e^{p/n}+\frac{2}{n^2}e^{p_n x}e^{2p/n}\right.\\
&\quad+\frac{2}{n^2}e^{p_n x}e^{p/n}\coth\left(e^{p/n}\right)+\frac{2x}{n}e^{p_n x}e^{2p/n}\coth\left(e^{p/n}\right)\\
&\quad\left.+\frac{x}{n}e^{p_n x}e^{p/n}\right\}
\end{aligned}$$

Since $\coth\left(e^{p/n}\right) \leq \coth 1 < 2$ and $e^{p/n} < e^p$, we can write

$$\begin{aligned}
P_{n,m}\left(\left(t_1-x\right)^2 e^{pt_1};x,y\right) \leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}&\left\{\frac{2x^2}{n^2}p_n^2 e^{p_n x}+\frac{2x}{n}e^p+\frac{2}{n^2}e^{p_n x}e^{2p}\right.\\
&\left.+\frac{4}{n^2}e^{p_n x}e^p+\frac{4x}{n}e^{p_n x}e^{2p}+\frac{x}{n}e^{p_n x}e^p\right\}.
\end{aligned}$$

Say $w_r(x) = e^{-rx}$. Thus, we get

$$\begin{aligned}
w_r(x)P_{n,m}\left(\left(t_1-x\right)^2 e^{pt_1};x,y\right) \leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}&\left\{\frac{2x}{n}e^{(p_n-r)x}\left(\frac{x}{n}p_n^2+\frac{e^p}{2}+2e^{2p}\right)\right.\\
&\left.+\frac{2x}{n}e^{p-rx}+\frac{2}{n^2}e^{(p_n-r)x}\left(e^{2p}+2e^p\right)\right\}.
\end{aligned}$$

Now, by using 2.1 and inequalities

$$\frac{x}{n}p_n^2 < \frac{x}{n}p^2 e^{2p/n} < xp^2 e^{2p},$$

it follows

$$\begin{aligned}
& w_r(x) P_{n,m}\left((t_1-x)^2 e^{pt_1}; x, y\right) \\
& \leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{\frac{2x}{n}e^{(p_n-r)x}\left(xp^2e^{2p}+\frac{e^p}{2}+2e^{2p}\right)\right. \\
& \left.+\frac{2x}{n}e^p+\frac{2}{n^2}e^{(p_n-r)x}\left(e^{2p}+2e^p\right)\right\}.
\end{aligned}$$

Also, we have $r-p_n \geq r-p_{n_0} > 0$ and $xe^{-(r-p_n)x} \leq xe^{-(r-p_{n_0})x} \leq 1/(r-p_{n_0})$ for $n \geq n_0$. Applying $e^{(p_n-r)x} < 1$, we obtain

$$\begin{aligned}
& w_r(x) P_{n,m}\left((t_1-x)^2 e^{pt_1}; x, y\right) \\
& \leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{\frac{2}{n}\frac{1}{r-p_{n_0}}\left(xp^2e^{2p}+2e^{2p}+\frac{e^p}{2}\right)\right. \\
& \left.+\frac{2x}{n}e^p+\frac{2}{n^2}\left(e^{2p}+2e^p\right)\right\} \\
& \leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{\frac{2x}{n}\frac{e^{2p}}{r-p_{n_0}}p^2+\frac{2}{n}\frac{1}{r-p_{n_0}}\left(2e^{2p}+\frac{e^p}{2}\right)\right. \\
& \left.+\frac{2x}{n}e^p+\frac{2}{n^2}\left(e^{2p}+2e^p\right)\right\} \\
& \leq C_{p,r}\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\frac{x+2}{n}.
\end{aligned}$$

Similarly as above, applying 2.6 to Lemma 3, by simple calculations we easily obtain the required inequality

$$w_s(y) P_{n,m}\left((t_2-y)^2 e^{qt_2}; x, y\right) \leq C_{q,s}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}\frac{y+2}{m}.$$

□

Lemma 6. *If $p,q>0$, $r>p$, $s>q$ and $n_0=n_0(p,r)$, $m_0=m_0(q,s)$ be fixed natural numbers such that $n_0>p/(\ln r-\ln p)$, $m_0>q/(\ln s-\ln q)$ and if $f\in C_{p,q}$, then we have*

$$\|P_{n,m}(f;x,y)\|_{r,s} \leq 2\|f\|_{p,q}\frac{\sinh(e^p)\sinh(e^q)}{(\sinh 1)^2}.$$

Proof. By (1), we can write

$$e^{-rx}e^{-sy}\left|P_{n,m}(f;x,y)\right|$$
$$=\frac{e^{-rx}e^{-sy}}{(\sinh 1)^2\sinh(nx)\sinh(my)}\left|\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(nx)\,p_{2j}(my)\,e^{-\frac{2kp}{n}}e^{-\frac{2jq}{m}}\right.$$
$$\left.\cdot f\left(\frac{2k}{n},\frac{2j}{m}\right)e^{\frac{2kp}{n}}e^{\frac{2jq}{m}}\right|.$$

Since $\|f\|_{p,q}=\sup_{(x,y)\in R_1^2}e^{-px}e^{-qy}\left|f(x,y)\right|$, it follows that

$$e^{-rx}e^{-sy}\left|P_{n,m}(f;x,y)\right|$$
$$\le\frac{e^{-rx}e^{-sy}}{(\sinh 1)^2\sinh(nx)\sinh(my)}\|f\|_{p,q}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(nx)\,p_{2j}(my)\,e^{\frac{2kp}{n}}e^{\frac{2jq}{m}}$$
$$=e^{-rx}e^{-sy}\|f\|_{p,q}\,P_{n,m}\left(e^{pt_1};x,y\right)P_{n,m}\left(e^{qt_2};x,y\right)$$
$$=e^{-rx}e^{-sy}\|f\|_{p,q}\frac{\sinh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)}{\sinh 1\sinh(nx)}\frac{\sinh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)}{\sinh 1\sinh(my)}.$$

Using the notations in Lemma 5, the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}e^{-rx}e^{-sy}\left|P_{n,m}(f;x,y)\right|&\le\|f\|_{p,q}e^{-rx}e^{-sy}\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}2e^{p_nx}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}2e^{q_my}\\&=4\|f\|_{p,q}e^{-x(r-p_n)}e^{-y(s-q_m)}\frac{\sinh\left(e^{p/n}\right)\sinh\left(e^{q/m}\right)}{(\sinh 1)^2}\\&\le 4\|f\|_{p,q}\frac{\sinh\left(e^{p}\right)\sinh\left(e^{q}\right)}{(\sinh 1)^2}.\end{aligned}$$

□

3. Approximation by $\mathbf{P}_{n,m}$ Operators

In this section, we give theorems on the degree of approximation of functions of two variables by these operators.

Theorem 1. *Let* $p,q>0$ *,* $r>p$*,* $s>q$ *and* $n_0=n_0(p,r)$*,* $m_0=m_0(q,s)$ *be fixed natural numbers such that* $n_0>p/(\ln r-\ln p)$ *and* $m_0>q/(\ln s-\ln q)$*. If* $f\in C_{p,q}^1$*, where* $C_{p,q}^1=\left\{f\in C_{p,q}:f_x,f_y\in C_{p,q}\right\}$*, then there exists a positive constant* $M_{p,q,r,s}$ *depending only on* p,q,r,s *such that*

$$w_{r,s}(x,y)\left|P_{n,m}(f;x,y)-f(x,y)\right|$$
$$\leq M_{p,q,r,s}\left\{\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\sqrt{\frac{x+2}{n}}+\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\sqrt{\frac{y+2}{m}}\right\}$$

Proof. Let (x,y) be a fixed point in R_1^2. For $f\in C_{p,q}^1$ and $(t_1,t_2)\in R_1^2$ we have

$$f(t_1,t_2)-f(x,y)=\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du+\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv.$$

Using $P_{n,m}(1;x,y)=1$, it results that

$$P_{n,m}(f(t_1,t_2);x,y)-f(x,y)$$
$$=P_{n,m}\left(\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du;x,y\right)+P_{n,m}\left(\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv;x,y\right).$$

For $r>p, s>q$ and $m,n\in\mathbb{N}$, we have

$$w_{r,s}(x,y)\left|P_{n,m}(f(t_1,t_2);x,y)-f(x,y)\right|$$
$$\leq w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du\right|;x,y\right)$$
$$+\,w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv\right|;x,y\right).$$

By using the following inequalities

$$\left|\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du\right|\leq\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\left|\int_x^{t_1}\frac{1}{w_{p,q}(u,t_2)}du\right|$$
$$\leq\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\frac{1}{w_q(t_2)}\left(\frac{1}{w_p(t_1)}+\frac{1}{w_p(x)}\right)|t_1-x|,$$

$$\left|\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv\right|\leq\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\left|\int_y^{t_2}\frac{1}{w_{p,q}(x,v)}dv\right|$$
$$\leq\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\frac{1}{w_p(x)}\left(\frac{1}{w_q(t_2)}+\frac{1}{w_q(y)}\right)|t_2-y|$$

and Hölder inequality, we can write

$$w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du\right|;x,y\right)$$
$$\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q} w_{r,s}(x,y)\,P_{n,m}\left(\frac{1}{w_q(t_2)};x,y\right)\left\{P_{n,m}\left(\frac{|t_1-x|}{w_p(t_1)};x,y\right)\right.$$
$$\left.+\frac{1}{w_p(x)}P_{n,m}(|t_1-x|;x,y)\right\}$$
$$\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q}\left\{\left[w_r(x)P_{n,m}\left((t_1-x)^2e^{pt_1};x,y\right)\right]^{1/2}\left[w_r(x)P_{n,m}\left(e^{pt_1};x,y\right)\right]^{1/2}\right.$$
$$\left.+\,e^{(p-r)x}\left[P_{n,m}\left((t_1-x)^2;x,y\right)\right]^{1/2}\right\}w_s(y)P_{n,m}\left(e^{qt_2};x,y\right),$$

$$w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv\right|;x,y\right)$$
$$\leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q}\frac{w_r(x)}{w_p(x)}P_{n,m}(1;x,y)\,w_s(y)\left\{P_{n,m}\left(\frac{|t_2-y|}{w_q(t_2)};x,y\right)\right.$$
$$\left.+\frac{1}{w_q(y)}P_{n,m}(|t_2-y|;x,y)\right\}$$
$$\leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q}\left\{\left[w_s(y)\,P_{n,m}\left((t_2-y)^2e^{qt_2};x,y\right)\right]^{1/2}\right.$$
$$\left.\cdot\left[w_s(y)\,P_{n,m}\left(e^{qt_2};x,y\right)\right]^{1/2}+\left[P_{n,m}\left((t_2-y)^2;x,y\right)\right]^{1/2}\right\}.$$

Applying the inequalities 2.5 and 2.6 to Lemma 2, we obtain

$$\begin{aligned}w_r(x)P_{n,m}\left(e^{pt_1};x,y\right) &\leq e^{-rx}\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}2e^{p_nx}\\ &= 2\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}e^{-x(r-p_n)}\\ &\leq 2\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\end{aligned}$$

and

$$w_s(y)P_{n,m}\left(e^{qt_2};x,y\right)\leq e^{-sy}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}2e^{q_my}$$

$$
\begin{aligned}
&= 2\frac{\sinh\left(e^{q/m}\right)}{\sinh 1} e^{-y(s-q_m)} \\
&\leq 2\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}.
\end{aligned}
$$

By these inequalities, Lemma 5 and Lemma 4, we get to

$$
\begin{aligned}
& w_{r,s}(x,y)\, P_{n,m}\left(\left|\int_x^{t_1} \frac{\partial f}{\partial u}(u,t_2)\,du\right|; x,y\right) \\
&\leq 2\left\|\frac{\partial f}{\partial x}\right\|_{p,q} \frac{\sinh\left(e^{p/n}\right)\sinh\left(e^{q/m}\right)}{(\sinh 1)^2}\sqrt{2C_{p,r}\frac{x+2}{n}} \\
&+2\left\|\frac{\partial f}{\partial x}\right\|_{p,q} \frac{\sinh\left(e^{q/m}\right)}{\sinh 1} e^{-x(r-p)}\sqrt{\frac{3(x+1)}{n}} \\
&\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q} M_{p,r}\sqrt{\frac{x+2}{n}},
\end{aligned}
$$

and

$$
\begin{aligned}
& w_{r,s}(x,y)\, P_{n,m}\left(\left|\int_y^{t_2} \frac{\partial f}{\partial v}(x,v)\,dv\right|; x,y\right) \\
&\leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q} \frac{\sinh\left(e^{q/m}\right)}{\sinh 1}\sqrt{2C_{q,s}\frac{y+2}{m}} + \left\|\frac{\partial f}{\partial y}\right\|_{p,q}\sqrt{\frac{3(y+1)}{m}} \\
&\leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q} M_{q,s}\sqrt{\frac{y+2}{m}}
\end{aligned}
$$

for all $m \geq m_0$ and $n \geq n_0$. This proves the theorem. □

Theorem 2. *Suppose that $f \in C_{p,q}$ and p,r,q,s,n_0,m_0 satisfy the conditions of Theorem 1. Then there exists positive constant $M^* = M_{p,q,r,s}$ depending only on p,q,r,s such that*

$$
w_{r,s}(x,y)\left|P_{n,m}(f;x,y) - f(x,y)\right| \leq M^*\omega\left(f, C_{p,q}; \left(\frac{x+2}{n}\right)^{1/2}, \left(\frac{y+2}{m}\right)^{1/2}\right)
$$

for all $(x,y) \in R_1^2$ and $m \geq m_0$, $n \geq n_0$.

Proof. Similarly as in [5], we consider the Steklov means for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^h\int_0^\delta f(x+u,y+v)\,du\,dv \quad , \quad h,\delta>0,(x,y)\in R_1^2.$$

We have

$$f_{h,\delta}(x,y) - f(x,y) = \frac{1}{h\delta}\int_0^h\int_0^\delta \Delta_{u,v} f(x,y)\,du\,dv,$$

$$\frac{\partial f_{h,\delta}}{\partial x}(x,y) = \frac{1}{h\delta}\int_0^\delta (f(x+h,y+v)-f(x,y+v))\,dv,$$

$$\frac{\partial f_{h,\delta}}{\partial y}(x,y) = \frac{1}{h\delta}\int_0^h (f(x+u,y+\delta)-f(x+u,y))\,du,$$

which implies $f_{h,\delta} \in C^1_{p,q}$ $(h,\delta>0)$ and

$$\| f_{h,\delta} - f \|_{p,q} \le w\left(f, C_{p,q}; h,\delta\right),$$

$$\begin{aligned}\left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{p,q} &\le \sup_{(x,y)\in R_1^2} w_{p,q}(x,y)\frac{1}{h\delta}\int_0^\delta \left(\left|\Delta_{h,v} f(x,y)\right| + \left|\Delta_{0,v} f(x,y)\right|\right) dv \\ &\le \frac{2}{h} w\left(f, C_{p,q}; h,\delta\right)\end{aligned}$$

and

$$\left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{p,q} \le \frac{2}{\delta} w\left(f, C_{p,q}; h,\delta\right)$$

for $h,\delta>0$.

For every fixed $(x,y)\in R_1^2$, $r>p$, $s>q$ and $n,m\in\mathbb{N}$, $h,\delta>0$ we have

$$\begin{aligned}&w_{r,s}(x,y)\left|P_{n,m}(f;x,y)-f(x,y)\right| \\ &\le w_{r,s}(x,y)\left\{\left|P_{n,m}\left(f-f_{h,\delta};x,y\right)\right|\right. \\ &\quad \left. + \left|P_{n,m}\left(f_{h,\delta};x,y\right)-f_{h,\delta}(x,y)\right| + \left|f_{h,\delta}(x,y)-f(x,y)\right|\right\}.\end{aligned}$$

By Lemma 5, one obtains

$$w_{r,s}(x,y)\left|P_{n,m}\left(f-f_{h,\delta};x,y\right)\right| \le 4w\left(f, C_{p,q}; h,\delta\right)$$

for all $m \ge m_0$ and $n \ge n_0$. Since Theorem 1, we can write

$$w_{r,s}(x,y)\left|P_{n,m}\left(f_{h,\delta};x,y\right) - f_{h,\delta}(x,y)\right|$$

$$\leq M_{p,q,r,s} w\left(f, C_{p,q}; h, \delta\right) \left\{ \frac{1}{h}\sqrt{\frac{x+2}{n}} + \frac{1}{\delta}\sqrt{\frac{y+2}{m}} \right\}$$

for $m \geq m_0$, $n \geq n_0$. Therefore

$$\begin{aligned} & w_{r,s}(x,y) \left| P_{n,m}(f; x, y) - f(x,y) \right| \\ & \leq 4w\left(f, C_{p,q}; h, \delta\right) + M_{p,q,r,s} w\left(f, C_{p,q}; h, \delta\right) \left\{ \frac{1}{h}\sqrt{\frac{x+2}{n}} + \frac{1}{\delta}\sqrt{\frac{y+2}{m}} \right\} \\ & \quad + w_{r,s}(x,y) \left| f_{h,\delta}(x,y) - f(x,y) \right| \\ & \leq w\left(f, C_{p,q}; h, \delta\right) \left(5 + M_{p,q,r,s} \left\{ \frac{1}{h}\sqrt{\frac{x+2}{n}} + \frac{1}{\delta}\sqrt{\frac{y+2}{m}} \right\} \right) \end{aligned}$$

for all $h, \delta > 0$ and $m \geq m_0$, $n \geq n_0$. Setting $h = \sqrt{\frac{x+2}{n}}$, $\delta = \sqrt{\frac{y+2}{m}}$ we obtain the desired result. □

Corollary 1. *If $f \in C_{p,q}$, then for all $(x,y) \in R_1^2$*

$$\lim_{m,n \to \infty} P_{n,m}(f; x, y) = f(x,y).$$

Also, the convergence is uniform on every rectangle $1 \leq x \leq a$, $1 \leq y \leq b$.

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