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# New integral inequalities involving beta function via $P$-convexity

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# NEW INTEGRAL INEQUALITIES INVOLVING BETA FUNCTION VIA $P$-CONVEXITY

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*Abstract.* In this note we establish some estimates, involving the Euler Beta function, of the integral $\int_a^b (x-a)^p (b-x)^q f(x)dx$ for functions when a power of the absolute value is $P-$convex. An extension to functions of several variables is also obtained.

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## 1. INTRODUCTION

Let $I$ be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\}$$

holds for any $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [11]).

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \mathcal{R}_m[f] \tag{1.1}$$

for certain $B_{m,k}, \gamma_k$ and rest term $\mathcal{R}_m[f]$ (see [22]).

In [17], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.1) via some kinds of convexity. Especially, they discussed the following result connecting with quasi-convex function:

**Theorem 1.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$. If $f$ is quasi-convex on $[a,b]$, then for some fixed $p,q>0$, we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \le (b-a)^{p+q+1}\beta(p+1,q+1)\max\{f(a),f(b)\},$$

*where $\beta(x,y)$ is the Euler Beta function.*

Recently, Liu [12] established some new integral inequalities for quasi-convex functions as follows:

**Theorem 2.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$ and let $k>1$. If $|f|^{\frac{k}{k-1}}$ is quasi-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)dx\\ \le&(b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left(\max\left\{|f(a)|^{\frac{k}{k-1}},|f(b)|^{\frac{k}{k-1}}\right\}\right)^{\frac{k-1}{k}}.\end{aligned}$$

**Theorem 3.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$ and let $l\ge 1$. If $|f|^l$ is quasi-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)dx\\ \le&(b-a)^{p+q+1}\beta(p+1,q+1)\left(\max\left\{|f(a)|^l,|f(b)|^l\right\}\right)^{\frac{1}{l}}.\end{aligned}$$

On the other hand, Dragomir et al. in [6] defined the following class of functions of $P$-convex.

**Definition 1.** Let $I\subseteq\mathbb{R}$ be an interval. The function $f:I\to\mathbb{R}$ is said to belong to the class $P(I)$ (or to be $P$-convex) if it is nonnegative and, for all $x,y\in I$ and $t\in[0,1]$, satisfies the inequality

$$f(tx+(1-t)y)\le f(x)+f(y).$$

Note that $P(I)$ contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category; see [1, 2, 4, 5, 7–10, 13–16, 18–21, 23–26] and references therein.

The main purpose of this note is to establish some new estimates of the integral $\int_a^b (x-a)^p(b-x)^q f(x)dx$ for functions when a power of the absolute value is $P-$convex. An extension to functions of several variables is also obtained. That is, this study is a continuation and further generalization of [12, 17] via $P$-convexity.

## 2. New integral inequalities via $P$-convexity

In this section we generalize Theorems 1-3 with a $P$-convex function setting. For this purpose, we need the following lemma (see [17, Lemma 2.1]):

**Lemma 1.** *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $a < b$. Then the equality*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \tag{2.1}$$

*holds for some fixed $p,q > 0$.*

The next theorem gives a new result for $P$-convex functions.

**Theorem 4.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $0 \le a < b < \infty$. If $|f|$ is $P$-convex on $[a,b]$, for some fixed $p,q > 0$, then*

$$\begin{aligned} &\int_a^b (x-a)^p (b-x)^q f(x)dx \\ \le&(b-a)^{p+q+1} \beta(p+1,q+1)(|f(a)|+|f(b)|), \end{aligned} \tag{2.2}$$

*where $\beta(x,y)$ is the Euler Beta function.*

*Proof.* By Lemma 1, the Beta function which is defined for $x,y > 0$ as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

and the fact that $f$ is $P$-convex on $[a,b]$, we have

$$\begin{aligned} \int_a^b (x-a)^p (b-x)^q f(x)dx \le&(b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|dt \\ \le&(b-a)^{p+q+1} \int_0^1 (1-t)^p t^q (|f(a)|+|f(b)|)dt \\ =&(b-a)^{p+q+1} \beta(q+1,p+1)(|f(a)|+|f(b)|), \end{aligned}$$

which completes the proof. □

The corresponding version for powers of the absolute value is incorporated in the following result.

**Theorem 5.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $0 \le a < b < \infty$ and let $k > 1$. If $|f|^{\frac{k}{k-1}}$ is $P$-convex on $[a,b]$, for some fixed $p,q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$

$$\leq (b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}. \tag{2.3}$$

*Proof.* By Lemma 1, Hölder's inequality, the definition of Beta function and the fact that $|f|^{\frac{k}{k-1}}$ is $P$-convex on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx \\
\leq &(b-a)^{p+q+1}\left[\int_0^1 (1-t)^{kp}t^{kq}dt\right]^{\frac{1}{k}}\left[\int_0^1 |f(ta+(1-t)b)|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}} \\
\leq &(b-a)^{p+q+1}\left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}}\left[\int_0^1 \left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}} \\
= &(b-a)^{p+q+1}\left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}},
\end{aligned}$$

which completes the proof. □

A more general inequality using Lemma 1 is as follows:

**Theorem 6.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\leq a<b<\infty$ and let $l>1$. If $|f|^l$ is $P$-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx \\
\leq &(b-a)^{p+q+1}\beta(p+1,q+1)\left(|f(a)|^l+|f(b)|^l\right)^{\frac{1}{l}}.
\end{aligned} \tag{2.4}$$

*Proof.* By Lemma 1, Hölder's inequality, the definition of Beta function and the fact that $|f|^l$ is $P$-convex on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx \\
= &(b-a)^{p+q+1}\int_0^1 \left[(1-t)^p t^q\right]^{\frac{l-1}{l}}\left[(1-t)^p t^q\right]^{\frac{1}{l}} f(ta+(1-t)b)dt \\
\leq &(b-a)^{p+q+1}\left[\int_0^1 (1-t)^p t^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|^l dt\right]^{\frac{1}{l}} \\
\leq &(b-a)^{p+q+1}\left[\beta(q+1,p+1)\right]^{\frac{l-1}{l}}\left[\left(|f(a)|^l+|f(b)|^l\right)\beta(q+1,p+1)\right]^{\frac{1}{l}} \\
= &(b-a)^{p+q+1}\beta(p+1,q+1)\left(|f(a)|^l+|f(b)|^l\right)^{\frac{1}{l}},
\end{aligned}$$

which completes the proof. □

## 3. AN EXTENSION TO FUNCTIONS OF SEVERAL VARIABLES

In this section some new integral inequalities for functions of several variables on convex subsets of $\mathbb{R}^n$ will be given. First we recall the notion of $P$-convexity for functions on a convex subset $U$ of $\mathbb{R}^n$.

**Definition 2** ([3, Definition 3.1]). The function $f : U \to \mathbb{R}$ is said to be $P$-convex on $U$ if it is nonnegative and, for all $x, y \in U$ and $\lambda \in [0,1]$, satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y).$$

The following proposition will be used throughout this section.

**Proposition 1** ([3, Proposition 3.2]). *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and $f : U \to \mathbb{R}$ be a function. Then $f$ is $P$-convex on $U$ if and only if, for every $x, y \in U$, the function $\varphi : [0,1] \to \mathbb{R}$, defined by*

$$\varphi(t) := f((1-t)x + ty),$$

*is $P$-convex on $I$ with $I = [0,1]$.*

We have the following inequalities for functions of several variables on convex subsets of $\mathbb{R}^n$.

**Theorem 7.** *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$. Assume that $f : U \to \mathbb{R}^+$ is a $P$-convex function on $U$. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\begin{aligned} &\int_a^b (t-a)^p (b-t)^q f((1-t)x+ty)dt \\ \leq &(b-a)^{p+q+1}\beta(p+1,q+1)[f((1-a)x+ay)+f((1-b)x+by)]. \end{aligned} \tag{3.1}$$

*Proof.* Let $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$. Since $f : U \to \mathbb{R}^+$ is a $P$-convex function, by Proposition 1 the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f((1-t)x + ty),$$

is $P$-convex on $I$ with $I = [0,1]$. Applying Theorem 4 to the function $\varphi$ implies that

$$\begin{aligned} &\int_a^b (t-a)^p (b-t)^q \varphi(t)dt \\ \leq &(b-a)^{p+q+1}\beta(p+1,q+1)(|\varphi(a)|+|\varphi(b)|), \end{aligned}$$

and we deduce that (3.1) holds. □

Similarly, we have

**Theorem 8.** *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and let $k > 1$. Assume that $f^{\frac{k}{k-1}} : U \to \mathbb{R}^+$ is a $P$-convex function on $U$. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\int_a^b (t-a)^p (b-t)^q f((1-t)x+ty)dt$$
$$\leq (b-a)^{p+q+1} [\beta(kp+1,kq+1)]^{\frac{1}{k}}$$
$$\left[ f^{\frac{k}{k-1}}((1-a)x+ay) + f^{\frac{k}{k-1}}((1-b)x+by)\right]^{\frac{k-1}{k}}.$$

**Theorem 9.** *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and let $l > 1$. Assume that $f^l : U \to \mathbb{R}^+$ is a $P$-convex function on $U$. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\int_a^b (t-a)^p (b-t)^q f((1-t)x+ty)dt$$
$$\leq (b-a)^{p+q+1}\beta(p+1,q+1)[f^l((1-a)x+ay)+f^l((1-b)x+by)]^{\frac{1}{l}}.$$


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