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N. K. Prykarpatska, D. L. Blackmore, Anatoly K. Prykarpatsky, and M. Pytel-Kudela

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ON THE INF-TYPE EXTREMALITY SOLUTIONS TO HAMILTON–JACOBI EQUATIONS, THEIR REGULARITY PROPERTIES, AND SOME GENERALIZATIONS

N. K. PRYKARPATSKA, D. L. BLACKMORE, A. K. PRYKARPATSKY, AND M. PYTEL-KUDELA

This paper is dedicated to the memory of our friend Prof. Jerzy Zagrodzinski (†2001)

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ABSTRACT. The extremality problem representation of the Lax type [1] is studied in detail for some class of Hamilton-Jacobi equations in the many-dimensional case. The regularity properties of solutions of the Cauchy problem in the class of convex lower semicontinuous functions are established. A generalisation to a wider class of functions is obtained.

The Hamilton-Jacobi equation on the sphere is considered, and its exact solutions are found in terms of a Lax type extremality problem. Some generalisation of the results for the general case of many-dimensional Hamilton-Jacobi equations is obtained by using the Fan-Brouwder fixed point techniques in a Banach space.

Mathematics Subject Classification: Primary 34A30, 34B05; Secondary 34B15

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1. THE EXTREMALITY PROBLEM FUNCTIONAL ANALYSIS: CONVEX BSC-CLASS SOLUTIONS

1.1. Introduction. It is well-known that equations like

$$u_t + f(t, x; u, \nabla u) = 0 \tag{0.1}$$

for $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ with $f : \mathbb{R}_+ \times J^{(1)}(\mathbb{R}^n) \to \mathbb{R}$ being some fixed mapping on the jet-manifold $J^{(1)}(\mathbb{R}^n; \mathbb{R})$ are called *Hamilton–Jacobi equations*, and are related to the motion of certain mechanical systems. As was shown earlier in [1–3], these equations possess the stabilisation property as $t \to \infty$, namely, a solution to (0.1) for any Cachy data $u|_{t=0^+} = v$ from some appropriate class of functions tends to these Cauchy data.

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In case the mapping $f : J^{(1)}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is smooth and does not depend on variables $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, S. Kruzhkov [8–9] obtained the following analytical representation for the solution to (0.1) with the smooth Cauchy data $v : \mathbb{R}^n \to \mathbb{R}$:

$$u(x,t) = v(x - t\nabla f(p_0)) + t\left[\langle p_0, \nabla f(p_0)\rangle - f(p_0)\right], \tag{0.2}$$

where $u|_{t=0^+} = v$ and the function $p_0 = \nabla v\,(x - t\nabla f(p_0))$.

In this work, we develop the theory of equations like (0.1) for some special kinds of mappings $f : J^{(1)}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$, namely,

$$u_t + f(x;u,\nabla u) = 0 \tag{0.3}$$

and demonstrate the inf-type extremality structure of their solutions which was first observed by P. Lax (see [1]). In particular, we prove that this inf-type extremality structure really gives rise to solutions of (0.3) for the Cauchy data from the class $\mathrm{BSC}(\mathbb{R}^n)$ of convex lower semicontinuous functions on $\mathbb{R}^n$. In tparticular, for the case where

$$f(x;u,\nabla u) := \frac{1}{2}\langle \nabla u, \nabla u\rangle, \qquad u \in \mathrm{BSC}(\mathbb{R}^n),$$

we prove in Section 1 the aforementioned Lax result and give its generalisation to a wider class of Cauchy data.

In Section 2, we study the extremality structure of solutions to equations (0.3) which reduce to a fixed point problem and show its well-posedness. Another generalisation considered in this work is related to a Hamilton-Jacobi equation (0.3) on the n-dimensional sphere $\mathbb{S}^n$. The corresponding inf-type extremality solution to this Hamilton-Jacobi equation is proved to exist also for the BSC-class of functions on $\mathbb{S}^n$.

1.2. Problem setting. The review article [1] devoted to viscosity solutions of first and second order partial differential equations contains the following exact formula, suggested by P. Lax,

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left\{v(y) + \frac{1}{2t}\|x-y\|^2\right\}, \tag{1.1}$$

for the solutions to the Hamilton–Jacobi nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\|\nabla u\|^2 = 0, \quad u|_{t=0^+} = v, \tag{1.2}$$

with Cauchy data $v \in \mathrm{BSC}(\mathbb{R}^n)$ being chosen as a properly convex and lower semicontinuous function. Here, $\|\cdot\| = \langle\cdot,\cdot\rangle$ is the usual norm in $\mathbb{R}^n, n \in \mathbb{Z}$, and $t \in \mathbb{R}_+$ is a positive evolution parameter. It is also stated in [1] that there is no exact proof of the Lax formula (1.1) based on general properties of the Hamilton-Jacobi equation (1.2).

The present section is devoted to such an exact proof of the Lax formula (1.1) and to the study of some of its properties.

1.3. Analysis of the Hamilton-Jacobi dynamics. Consider the following canonical Hamiltonian system associated naturally [2] with (1.2):

$$\frac{dx}{dt} = \frac{\partial H_0}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_0}{\partial x}, \tag{2.1}$$

where the Hamiltonian function $H_0 \in C^2(T(\mathbb{R}^n); \mathbb{R})$ is

$$H_0(x, p) = \frac{1}{2}\|p\|^2 \tag{2.2}$$

for $(x, p) \in T^*(\mathbb{R}^n)$, $T^*(\mathbb{R}^n)$ being the canonical phase space of coordinates. The solution to (2.1) with Cauchy data at $(x_0, p_0) \in T^*(\mathbb{R}^n)$ is given for all $t \in \mathbb{R}_+$ as follows:

$$x = x_0 + p_0 t, \quad p = p_0. \tag{2.3}$$

Introduce now the so-called "action function" $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ which can be defined [2] locally as

$$du = -H_0(x, p)dt + \langle p, dx \rangle, \tag{2.4}$$

where, by (2.3), $p = (x - x_0)/t$, and let $u|_{t=0^+} = v \in \mathrm{BSC}(\mathbb{R}^n)$. From (2.4) one obtains immediately that

$$\frac{\partial u}{\partial t} = -H_0(x, p), \quad \frac{\partial u}{\partial x} = p \tag{2.5}$$

for all points $(x, p) \in T^*(\mathbb{R}^n)$. Substituting (2.2) into (2.5), one gets the following.

Lemma 1.1. *The action function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ satisfies exactly the Hamilton-Jacobi equation (1.2), that is*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\|\nabla u\|^2 = 0, \quad u|_{t=0^+} = v. \tag{2.6}$$

Now we shall proceed to computing an expression for the action function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, defined by (2.4). From (2.4) one finds that

$$\begin{aligned} u(x,t) = & \int_0^t d\tau \left(\frac{du}{d\tau}\right)\Bigg|_{\substack{x=x_0+p_0\tau \\ p=p_0}} + v(x_0)|_{x=x_0+p_0 t} = \\ & = \int_0^t d\tau \left(\left\langle p, \frac{dx}{d\tau}\right\rangle - H_0(x,p)\right)\Bigg|_{\substack{x=x_0+p_0\tau \\ p=p_0}} + v(x_0)|_{x=x_0+p_0 t} = \\ & = \left(\frac{1}{2}\|p_0\|^2 t + v(x_0)\right)\Bigg|_{x=x_0+p_0 t}. \end{aligned} \tag{2.7}$$

Since, due to (2.5), at a fixed $x \in \mathbb{R}^n$, the function

$$\frac{\partial u(x,t)}{\partial x}\Big|_{t=0^+} = p_0(x), \tag{2.8}$$

is defined, where $p_0 : \mathbb{R}^n \to \mathbb{R}^n$ is some mapping naturally defined from (2.3), (2.8) and (2.7), one arrives at the formula

$$u(x,t) = v\,(x - p_0(x_0)t) + \frac{t}{2}\|p_0(x_0)\|^2. \tag{2.9}$$

Here, for some $x_0 \in C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}^n)$, the equation

$$x_0 = x - tp(x_0)$$

holds, giving an unwieldy solution to the Hamilton-Jacobi equation (2.6). The expression (2.9) can be easily transformed into the following useful form:

$$u(x,t) = v(\xi) + \frac{1}{2t}\|x - \xi\|^2, \tag{2.10}$$

where the mapping $\xi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is defined as a solution to the functional equations

$$\xi(x,t) := x - tp_0(\xi(x,t)), \quad p_0(x) := \left.\frac{\partial u(x,t)}{\partial x}\right|_{t=0^+} \tag{2.11}$$

for any $x \in \mathbb{R}^n$, and $t \in \mathbb{R}_+$.

For the expression (2.10) to be interpreted more exactly, it is useful to recall that Hamiltonian equations (2.1) are completely equivalent to the following shortened extremal Lagrange action principle:

$$\delta\tilde{u}[x_0;\tilde{x},t]\Big|_{\substack{\tilde{x}|_{\tau=0^+}=x_0\in\mathbb{R}^n\\ \tilde{x}|_{\tau=t}=x\in\mathbb{R}^n\\ \tilde{x}\in C^1(\mathbb{R}_+;\mathbb{R}^n)}} = 0, \qquad \tilde{u}[x_0;\tilde{x},t] := \int_0^t d\tau L_0(\tilde{x},\dot{\tilde{x}}) + v(x_0), \tag{2.12}$$

where, by definition, the Lagrangian function is

$$L_0(\tilde{x},\dot{\tilde{x}}) := \langle p, \dot{\tilde{x}}\rangle - H_0(\tilde{x},p)|_{\dot{\tilde{x}}=\partial H_0(\tilde{x},p)/\partial p}. \tag{2.13}$$

Based on (2.4) and (2.13), one infers easily that the extremum expression

$$\begin{aligned}\tilde{u}(x,t) &:= \inf_{x_0\in\mathbb{R}^n}\left\{v(x_0) + \frac{1}{2t}\|x - x_0\|^2\right\}\\ &= v(\tilde{\xi}) + \frac{1}{2t}\left\|x - \tilde{\xi}\right\|^2\end{aligned} \tag{2.14}$$

holds if it is assumed that the infimum in the parenthesis exists and is attained at a unique point $x_0 = \tilde{\xi}(x,t)$ for fixed $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. For the above motivation to be validated, we shall study in detail properties of the solution $\tilde{\xi} = \tilde{\xi}(x,t)$ to the extremal problem (2.14) aiming to prove that $\tilde{\xi}(x,t) = \xi(x,t)$ for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, where

$$\xi(x,t) := x - tp_0(\xi(x,t)), \qquad p_0(x) := \frac{\partial u(x,t)}{\partial x}|_{t=0^+},$$

as it was found in (2.11).

1.4. Analysis of the extremality problem. Let us consider the problem (2.14) in the case when a function $v : \mathbb{R}^n \to \mathbb{R}$ is properly convex and semicontinuous from below, that is $v \in \mathrm{BSC}(\mathbb{R}^n)$. Then the following lemma (similar to lemma A5 in [1]) is true.

Lemma 1.2. *There exists a unique solution* $x_0 = \tilde{\xi}(x,t) \in \mathbb{R}^n$ *to the extremum problem* (2.14) *characterised by the inequality*

$$\frac{1}{t}\left\langle \tilde{\xi} - x, \tilde{\xi} - y \right\rangle \le v(y) - v(\tilde{\xi}) \tag{3.1}$$

for all $y \in \mathbb{R}^n$, $t \in \mathbb{R}_+$.

Proof. We first prove the inequality (3.1) assuming the existence of a solution to (2.14). Let us take any

$$z = \tilde{\xi} + \vartheta(y - \tilde{\xi}) = \vartheta y + (1-\vartheta)\tilde{\xi},$$

where $\vartheta \in (0,1)$. Then one easily obtains the inequality

$$\begin{aligned} v(\tilde{\xi}) + \frac{1}{2t}\|x - \tilde{\xi}\|^2 \le{}& v(\tilde{\xi} + \vartheta(y - \tilde{\xi})) \\ &+ \frac{1}{2t}\left\langle \tilde{\xi} + \vartheta(y-\tilde{\xi}) - x, \tilde{\xi} + \vartheta(y - \tilde{\xi}) - x \right\rangle \\ &\le (1-\vartheta)v(\tilde{\xi}) + \vartheta v(y) + \frac{1}{2t}\left\langle \tilde{\xi} - x, \tilde{\xi} - x \right\rangle \\ &\quad + \frac{1}{t}\vartheta\left\langle \tilde{\xi} - x, y - \tilde{\xi} \right\rangle + \frac{\vartheta^2}{2t}\left\langle y - \tilde{\xi}, y - \tilde{\xi} \right\rangle, \end{aligned} \tag{3.2}$$

whence we find that

$$v(\tilde{\xi}) - v(y) + \frac{1}{t}\left\langle \tilde{\xi} - x, \tilde{\xi} - y \right\rangle \le \frac{\vartheta}{2t}\|y - \tilde{\xi}\|^2, \tag{3.3}$$

for any $\vartheta \in (0,1)$. Passing to the limit in (3.3) as $\vartheta \to 0$, one gets exactly the inequality (3.1).

Now we shall proceed to the proof of the existence of a solution to (2.4). We use a standard minimizing sequence $\tilde{\xi}_j \in \mathbb{R}^n$, $j \in \mathbb{Z}_+$, satisfying the inequality

$$v(\tilde{\xi}_j) + \frac{1}{2t}\left\langle \tilde{\xi}_j - x, \tilde{\xi}_j - x \right\rangle \le \tilde{u}(x,t) + \frac{1}{j} \tag{3.4}$$

for any $j \in \mathbb{Z}_+$ and prove first that it is a Cauchy sequence.

One deduces now that

$$\|\tilde{\xi}_j - \tilde{\xi}_i\|^2 = 2\|\tilde{\xi}_j - x\|^2 + 2\|\tilde{\xi}_i - x\|^2 - 4\left\|\frac{\tilde{\xi}_j + \tilde{\xi}_i}{2} - x\right\|^2 \leq$$
$$\leq 4t\left[\frac{1}{i} + \frac{1}{j} + 2\tilde{u}(x,t) - v(\tilde{\xi}_j) - v(\tilde{\xi}_j)\right] + 8t\left(v\left(\frac{\tilde{\xi}_j + \tilde{\xi}_i}{2}\right) - \tilde{u}(x,t)\right) \leq$$
$$\leq 4t\left(\frac{1}{i} + \frac{1}{j}\right), \quad (3.5)$$

using inequalities (3.4) and the convexity of $v \in \mathrm{BSC}(\mathbb{R}^n)$. Inequality (3.5) means that for any fixed $t \in \mathbb{R}_+$, there exists the limit element $\lim_{j\to\infty} \tilde{\xi}_j = \tilde{\xi} \in \mathbb{R}^n$, by virtue of the completeness of the Euclidean space $\mathbb{R}^n$. On the other hand, the lower semicontinuity of $v : \mathbb{R}^n \to \mathbb{R}$ yields

$$\tilde{u}(x,t) \leq v(\tilde{\xi}) + \frac{1}{2t}\|\tilde{\xi} - x\|^2$$
$$\leq \liminf_{j\to\infty}\left(v(\tilde{\xi}_j) + \frac{1}{2}\|\tilde{\xi}_j - x\|^2\right) \leq \tilde{u}(x,t), \quad (3.6)$$

which implies the following identity related to (2.14):

$$\tilde{u}(x,t) \equiv v(\tilde{\xi}) + \frac{1}{2t}\|\tilde{\xi} - x\|^2. \quad (3.7)$$

The uniqueness of the solution $\tilde{\xi} \in \mathbb{R}^n$ of problem (2.14) for fixed $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ is proved as follows. If $\tilde{\xi}_1$ and $\tilde{\xi}_2 \in \mathbb{R}^n$ are two different solutions of (2.14), then it follows from (3.1) that

$$\begin{aligned} v(\tilde{\xi}_1) - v(\tilde{\xi}_2) + \frac{1}{t}\left\langle \tilde{\xi}_1 - x, \tilde{\xi}_1 - \tilde{\xi}_2\right\rangle \leq 0, \\ v(\tilde{\xi}_2) - v(\tilde{\xi}_1) + \frac{1}{t}\left\langle \tilde{\xi}_2 - x, \tilde{\xi}_2 - \tilde{\xi}_1\right\rangle \leq 0, \end{aligned} \quad (3.8)$$

for any $t \in \mathbb{R}_+$. Summing inequalities (3.8), one gets readily that

$$\frac{1}{t}\|\tilde{\xi}_1 - \tilde{\xi}_2\| \leq 0 \quad (3.9)$$

for any $t \in \mathbb{R}_+$ and, hence, $\tilde{\xi}_1 \equiv \tilde{\xi}_2$. Thus, the solution $\tilde{\xi} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ of problem (2.14) is unique. □

Now we shall investigate differential properties of the solution $\tilde{\xi} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, which will be used essentially in the main required equality $\tilde{\xi} = \xi$, thus ensuring that the extremum function $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ belongs to $\mathrm{BSC}(\mathbb{R}^n)$ and coincides with the solution $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ of problem (1.2).

The following lemma is almost obvious from inequality (3.1).

Lemma 1.3. *The mappings $\tilde{P}_t : \mathbb{R}^n \to \mathbb{R}^n$ and $(1 - \tilde{P}_t) : \mathbb{R}^n \to \mathbb{R}^n$, where, by definition, $\tilde{P}_t x := \tilde{\xi}(x,t)$ for any $t \in \mathbb{R}_+$, are Lipschitzian, that is, for any $x, y \in \mathbb{R}^n$*

$$||\tilde{P}_t x - \tilde{P}_t y|| \leq ||x - y||, \qquad ||(1 - \tilde{P}_t)x - (1 - \tilde{P}_t)y|| \leq ||x - y||. \tag{3.10}$$

Proof. From (3.1) one easily obtains

$$||\tilde{P}_t x - \tilde{P}_t y||^2 + ||(1 - \tilde{P}_t)x - (1 - \tilde{P}_t)y||^2 \leq ||x - y||^2, \tag{3.11}$$

which yields inequalities (3.10). □

Consider now the minimum function $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ for problem (2.14), which is realized at a unique element $\tilde{\xi} = \tilde{P}_t \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. We can formulate the following useful lemma.

Lemma 1.4. *The mapping $\tilde{u}_t : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\tilde{u}_t(x) := \tilde{u}(x,t)$ for any $x \in \mathbb{R}^n$, with $t \in \mathbb{R}_+$ fixed, is convex and differentiable with respect to $x \in \mathbb{R}^n$, and satisfies the equality*

$$\nabla \tilde{u}_t(x) = \frac{1}{t}\left(x - \tilde{P}_t(x)\right). \tag{3.12}$$

Moreover, as $t \to 0^+$, for any $x \in \mathbb{R}^n$ the following limits exist:

$$\lim_{t \to 0^+} \tilde{u}_t(x) = v(x), \qquad \lim_{t \to 0^+} \tilde{P}_t(x) = x. \tag{3.13}$$

Proof. We first verify equalities (3.13) making use of the following Fenkhel characterisation [3] of convex lower semicontinuous functions: *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex and lower semicontinuous if it coincides with its second conjugate via a Fenkhel function*, that is $f^{**}(x) = f(x)$, $x \in \mathbb{R}^N$, where, by definition,

$$f^*(p) := \sup_{x \in \mathbb{R}^n} \{\langle x, p \rangle - f(x)\}, \tag{3.14}$$

for any $f : \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$. Since (2.14) yields

$$\tilde{u}(x,t) \leq \left\{ v(y) + \frac{1}{2t}||y - x||^2 \right\}\Bigg|_{y=x} = v(x)$$

for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, we get the following chain of inequalities using the semicontinuity of the function $v : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} \frac{1}{2t}\left\|\tilde{\xi}(x,t) - x\right\|^2 &\leq v(x) + v^*(p) - \langle p, x \rangle + \langle p, x - \xi(x,t) \rangle \leq \\ &\leq \frac{1}{4t}\left\|\tilde{\xi}(x,t) - x\right\|^2 + v(x) + v^*(p) - \langle p, x \rangle + t||p||^2. \end{aligned} \tag{3.15}$$

Whence one arrives at the inequality

$$\left\|\tilde{\xi}(x,t) - x\right\|^2 \leq 4t\left(v(x) + v^*(p) - \langle p, x \rangle + t||p||^2\right), \tag{3.16}$$

which means evidently that $\lim_{t \to 0^+} \tilde{\xi}(x,t) = x$ for all $x \in \mathbb{R}^n$.

Note that this result can also be obtained by making use of inequality (3.1) and the convexity of the function $v : \mathbb{R}^n \to \mathbb{R}$.

Since the function $v : \mathbb{R}^n \to \mathbb{R}$ is convex and lower semi-continuous, one can write the following chain of inequalities:

$$\tilde{u}(x,t) \leq \left\{ v(y) + \|x-y\|^2 \frac{1}{2t} \right\} \Big|_{y=x} = v(x) \leq \liminf_{t \to 0^+} v(\tilde{\xi}(x,t)) =$$
$$= \liminf_{t \to 0^+} \left\{ \tilde{u}(x,t) - \frac{1}{2t}\|x - \tilde{\xi}\|^2 \right\} \leq \liminf_{t \to 0^+} \tilde{u}(x,t). \quad (3.17)$$

Thus, applying the $\limsup_{t \to 0^+}$ operation to the left hand side of (3.17), one gets easily that

$$\limsup_{t \to 0^+} \tilde{u}(x,t) \leq v(x) \leq \liminf_{t \to 0^+} \tilde{u}(x,t),$$

hence the limit

$$\lim_{t \to 0^+} \tilde{u}(x,t) = v(x)$$

exists. To prove now equality (3.12), we note that the inequality

$$\tilde{u}(x,t) - \tilde{u}(y,t) \leq \left\langle \frac{1}{t}(x - \tilde{\xi}(x,t)), x - y \right\rangle \quad (3.18)$$

holds for any $x, y \in \mathbb{R}^n$ and fixed $t \in \mathbb{R}_+$. From (3.18) and (3.10), applying the change of variables $x \rightleftarrows y$, one arrives at the following inequality:

$$\tilde{u}(x,t) - \tilde{u}(y,t) \geq \left\langle \frac{1}{t}(y - \tilde{\xi}(y,t)), x - y \right\rangle = \left\langle \frac{1}{t}(x - \tilde{\xi}(x,t)), x - y \right\rangle +$$
$$+ \left\langle \frac{1}{t}(y - x + \tilde{\xi}(x,t) - \tilde{\xi}(y,t)), x - y \right\rangle = \left\langle \frac{1}{t}(x - \tilde{\xi}(x,t)), x - y \right\rangle +$$
$$+ \left\langle \frac{1}{t}(1 - \tilde{P}_t)y - \frac{1}{t}(1 - \tilde{P}_t)x, x - y \right\rangle \geq \left\langle \frac{1}{t}(x - \tilde{P}_i x), x - y \right\rangle - \frac{1}{t}\|x-y\|^2. \quad (3.19)$$

It follows from (3.19) that, for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

$$\left| \tilde{u}(x,t) - \tilde{u}(y,t) - \left\langle \tfrac{1}{t}(1 - \tilde{P}_i)x, x - y \right\rangle \right| \leq \frac{1}{t}\|x-y\|^2, \quad (3.20)$$

that is,

$$\frac{1}{t}(1 - \tilde{P}_t)x = \nabla \tilde{u}(x,t),$$

which is the result desired. As a consequence of expressions (3.12) and (2.14), one obtains the equality

$$\lim_{t \to 0^+} \nabla \tilde{u}(x,t) = \tilde{p}_0(x) \equiv \frac{x - \tilde{\xi}(x,t)}{t} \quad (3.21)$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, defining a certain function $\tilde{p}_0 : \mathbb{R}^n \to \mathbb{R}^n$. The latter proves exactly that $\xi(x,t) \equiv \tilde{\xi}(x,t)$ and $u(x,t) \equiv \tilde{u}(x,t)$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ if only one identifies $p_0(x) = \tilde{p}_0(x)$, $x \in \mathbb{R}^n$. □

Summing up the results above, we obtain the following characterisation theorem.

Theorem 1.5. *The solution of the extremum problem* (2.14) *realized at the point* $\tilde{\xi} = \tilde{\xi}(x,t) \in \mathbb{R}^n$ *is a convex, lower semicontinuous function* $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *coinciding exactly with expression* (2.10) *containing the vector* $\xi = \xi(x,t) \in \mathbb{R}^n$ *equal to the vector* $\tilde{\xi} = \tilde{\xi}(x,t) \in \mathbb{R}^n$ *constructed above.*

Having stated in Theorem 1.5 the identity between two expressions (2.10) and (2.14), we obtain our ultimate theorem.

Theorem 1.6. *The extremum Lax expression* (1.1) *solves the Hamilton–Jacobi equation* (1.2) *for the Cauchy data from the class of convex lower semicontinuous functions* $\mathrm{BSC}(\mathbb{R}^n)$.

2. The extremality problem functional analysis revisited: the BSC-class solutions

2.1. A general description of results. This subsection deals with the study of the validity of the Lax formula (1.1) of Section 1 for the solution to Hamilton-Jacobi nonlinear partial differential equation (1.2), Section 1, with the Cauchy data $v : \mathbb{R}^n \to \mathbb{R}$ being a lower semicontinuous function which is not necessarily convex. We shall prove that the Lax formula solves the Cauchy problem (1.2), Section 1, at any point $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ fixed, save for an exceptional set of points Q of the F_δ type having Lebesgue measure zero.

2.2. Problem setting and formulation of results. Suppose that $n \in \mathbb{Z}_+$ and define $\mathrm{BSC}(\mathbb{R}^n)$ as the set of all lower semicontinuous functions $v : \mathbb{R}^n \to \mathbb{R}$ for which

$$v^-(y) = o(\|y\|^2) \tag{1.1}$$

as $\|y\| \to \infty$, where $v^- := \frac{1}{2}(|v| - v)$ and $y \in \mathbb{R}^n$. Let us put $\Lambda := \mathbb{R}^n \times \mathbb{R}_+$ and let $\mathcal{B}_{loc}(\Lambda)$ be the set of all locally bounded functions $u : \Lambda \to \mathbb{R}$. Consider now the following essentially nonlinear operator

$$u(x) := L(v)(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ v(y) + \frac{1}{2t}\|x - y\|^2 \right\} \tag{1.2}$$

from $\mathrm{BSC}(\mathbb{R}^n)$ to $\mathcal{B}_{loc}(\Lambda)$, where $z := (x,t) \in \Lambda$ is an arbitrary point.

Remark 2.1. It follows from (1.1) that

$$L(v)(z) \le v(0) + \frac{\|x\|^2}{2t}, \tag{1.3}$$

and therefore, $L : \mathrm{BSC}(\mathbb{R}^n) \to \mathcal{B}_{loc}(\Lambda)$, that is operator (1.2) is well-defined.

Below we prove the Lax formula (1.1), Section 1, for the solution to the Hamilton-Jacobi equation (1.2), Section 1, that is, the formula $u(z) = L(v)(z)$ for almost all $z \in \Lambda$ does solve the Cauchy problem (1.2), Section 1. Namely, we develop a background for stating the Lax formula (1.1), Section 1, and prove the validity of Theorem 2.2 below.

Theorem 2.2. *Let* $v \in \mathrm{BSC}(\mathbb{R}^n)$ *and* $u = L(v)$*. Then:*

(i) *The function* $u \in \mathcal{B}_{loc}(\Lambda) \cap \mathrm{Lip}_{1,loc}(\Lambda)$ *and, for almost all* $(x, t) \in \Lambda$*, is differentiable in the usual sense;*
(ii) *The set of points where the function* $u : \Lambda \to \mathbb{R}$ *is not differentiable is contained in some exceptional set* Q *of type* F_δ *having Lebesgue measure zero;*
(iii) *The derivatives* $\partial u/\partial t$ *and* ∇u *are continuous on* $\Lambda \setminus Q$*;*
(iv) *For all* $(x, t) \in \Lambda \setminus Q$*, the function* $u : \Lambda \to \mathbb{R}$ *solves the Cauchy problem (1.2), Section 1;*
(v) *For all* $x \in \mathbb{R}^n$*, we have* $\lim_{t\to 0^+} u(x, t) = v(x)$*.*

2.3. Regularity properties of the operator *L*. We divide our *proof* of Theorem 2.2 into several steps and formulate some regularity lemmas.

Definition 2.3. For any $z \in \Lambda$, we put

$$A_z := \left\{ y_0 \in \mathbb{R}^n : \psi(z; y_0) = \inf_{y\in\mathbb{R}^n} \psi(z; y) \right\}, \tag{2.1}$$

where

$$\psi(z; y) := v(y) + \frac{\|x - y\|^2}{2t}, \tag{2.2}$$

for $z = (x, t) \in \Lambda$, and, for arbitrary $K \subset \Lambda$, we set

$$A(K) := \bigcup\nolimits_{z\in K} A_z. \tag{2.3}$$

Remark 2.4. Since $v \in \mathrm{BSC}(\mathbb{R}^n)$, we see that the mapping $\psi_z : \mathbb{R}^n \ni y \mapsto \psi(z; y) \in \mathbb{R}$ is also lower semicontinuous and, hence, the sets A_z, $z \in \Delta$, are nonempty. In the sequel, for the sake of convenience, we put

$$r(z; y) = \frac{\|x - y\|^2}{2t}, \tag{2.4}$$

where $y \in \mathbb{R}^n$ and $z = (x, t) \in \Lambda$.

Lemma 2.5. *Suppose* K *is a nonempty compact subset of* Λ*. Then* $A(K)$ *is nonempty and compact in* $\mathbb{R}^n$*. In particular, for arbitrary* $z \in \Lambda$*, the set* A_z *is compact.*

Proof. Let us first prove that the set $A(K)$ is bounded. In accordance with the notation above, we have that, for all $z \in \Lambda$,

$$u(z) = \inf_{y\in\mathbb{R}^n} \psi(z; y). \tag{2.5}$$

Thus, having taken into account (2.1) and (2.2), one gets that

$$A(K) = \{y \in \mathbb{R}^n : \exists\, z \in K \quad \psi(z; y) = u(z)\}. \tag{2.6}$$

From (2.6) one easily deduces that for any $y \in A(K)$

$$\inf_{z\in K} \psi(z; y) \le \sup_{z\in K} u(z),$$

whence, by virtue of the local boundedness of $u : \Lambda \to \mathbb{R}$ (see Remark 1.1), it follows that there exists a constant $c_0 \in \mathbb{R}_+$ such that, for any $y \in A(K)$,

$$\inf_{z\in K} \psi(z; y) \le c_0. \tag{2.7}$$

It is also evident that there exist constants $c_1, c_2 \in \mathbb{R}_+$ such that

$$\inf_{z\in K} r(z; y) \ge c_1\|y\|^2 - c_2. \tag{2.8}$$

From (2.7) and (2.8) it follows that for any $y \in A(K)$

$$v(y) + c_1 \|y\|^2 - c_2 \le \inf_{z\in K} \psi(z; y) \le c_0,$$

and therefore, for any $y \in A(K)$

$$c_1\|y\|^2 - v^-(y) \le c_0 + c_2. \tag{2.9}$$

The last inequality, in view of condition (1.1), implies the boundedness of the set $A(K)$.

We shall now prove that the set $A(K)$ is closed. Let a sequence $\{y_m \in \mathbb{R}^n : m \in \mathbb{Z}_+\} \subset A(K)$ be convergent and $\tilde{y} := \lim_{m\to\infty} y_m$. From (2.5) and (2.6) we get that there exists a sequence $\{z_m \in K : m \in \mathbb{Z}_+\}$ such that, for any $y \in \mathbb{R}^n$ and $m \in \mathbb{Z}_+$,

$$\psi(z_m; y_m) \le \psi(z_m; y). \tag{2.10}$$

Taking, if necessary, a suitable subsequence, we may assume that the sequence $\{z_m \in K : m \in \mathbb{Z}_+\}$ is also convergent. Putting $z := \lim_{m\to\infty} z_m$ and taking into account that $v \in \mathrm{BSC}(\mathbb{R}^n)$, one concludes immediately that, for any $y \in \mathbb{R}^n$,

$$\begin{aligned}\psi(z; \overline{y}) = v(\overline{y}) + r(z; \overline{y}) \le \lim_{k\to\infty} \psi(z_k, y_k) \le \lim_{k\to\infty} \psi(z_k, y) = \\ = v(y) + \lim_{k\to\infty} r(z_k; y) = v(y) + r(z; y) + \psi(z; y).\end{aligned}$$

Thus, for any $y \in \mathbb{R}^n$, we have $\psi(z; \overline{y}) \le \psi(z; y)$, that is, $\overline{y} \in A_z \subset A(K)$. Since $z \in K$, this proves the lemma. □

Lemma 2.6. *The function* $u : \Lambda \to \mathbb{R}$ (2.5) *belongs to the space* $\mathrm{Lip}_{1,loc}(\Lambda)$ *of locally Lipschitzian functions.*

Proof. Let us fix an arbitrary closed ball $K \subset \Lambda$ and take $z_1, z_2 \in \overline{K}, y_1 \in A_{z_1}, y_2 \in A_{z_2}$. Then

$$\begin{aligned} u(z_1) &= v(y_1) + r(z_1; y_1),\\ u(z_2) &= v(y_2) + r(z_2; y_2),\end{aligned} \tag{2.11}$$

and, by the definition of infimum, we get

$$\begin{aligned} u(z_1) &\leq v(y_2) + r(z_1; y_2), \\ u(z_2) &\leq v(y_1) + r(z_2; y_1). \end{aligned} \tag{2.12}$$

From (2.11) and (2.12) it follows immediately that

$$u(z_1) - u(z_2) \leq r(z_1; y_2) - r(z_2; y_2),$$

$$u(z_2) - u(z_1) \leq r(z_2; y_1) - r(z_1; y_1),$$

whence the inequality

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq \max_{j=\overline{1,2}} |r(z_1; y_1) - r(z_2; y_j)| \\ &\leq \sup_{y \in A(K)} |r(z_1; y) - r(z_2; y)|. \end{aligned} \tag{2.13}$$

follows.

Since $r \in C^\infty(\Lambda \times \mathbb{R}^n; \mathbb{R})$ and $A(\overline{K})$ is compact by virtue of Lemma 1.1, we deduce that there exists a constant $\bar{c} \in \mathbb{R}_+$, depending only on the ball $\overline{K}$, such that for any $z_1, z_2 \in \overline{K},\ y \in A(\overline{K})$ one has

$$|r(z_1; y) - r(z_2; y) \leq \bar{c}||z_1 - z_2||.$$

Thus, our lemma is proved. □

Lemma 2.7. *For all $x \in \mathbb{R}^n$,*

$$\lim_{t \to 0^+} u(x, t) = v(x).$$

Proof. Take any $x \in \mathbb{R}^n$, $t_1 \leq t_2 \in \mathbb{R}_+$ and $z_j := (x, t_j) \in \Lambda,\ j = \overline{1, 2}$. Then, for any $y \in \mathbb{R}^n$, we have $\psi(z_1; y) \geq \psi(z_2; y)$ and, hence,

$$u(x, t_1) \geq u(x, t_2) \tag{2.14}$$

for any $x \in \mathbb{R}^n$ and $t_1 \leq t_2 \in \mathbb{R}_+$. It is also evident that, for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

$$u(x, t) \leq v(x). \tag{2.15}$$

Now, from (2.14) and (2.11), we easily get that for all $x \in \mathbb{R}^n$ the limit $\bar{v}(x) = \lim_{t \to 0^+} u(x, t)$ exists and satisfies inequality

$$\bar{v}(x) \leq v(x). \tag{2.16}$$

Let now $x \in \mathbb{R}^n$ and a sequence $\{t_j \in \mathbb{R}_+ : j \in \mathbb{Z}_+\}$ be convergent to zero. Put $z_k := (x, t_k) \in \Lambda$, $k \in \mathbb{Z}_+$ and choose a sequence $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ with $y_k \in A_{z_k}$, $k \in \mathbb{Z}_+$. Then evidently for all $k \in \mathbb{Z}_+$, $x \in \mathbb{R}^n$

$$u(x, t_k) = v(y_k) + r(z_k; y_k). \tag{2.17}$$

Using the same considerations as in Lemma 1.1 when proving the boundedness of the set $A(K)$, it easy to see that $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ is also bounded. Passing to a subsequence if necessary, we may assume that the sequence $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ is convergent.

Put $\overline{y} := \lim_{k\to\infty} y_k$. Then, by virtue of the lower semicontinuity of the function $v \in \mathrm{BSC}(\mathbb{R}^n)$, we obtain

$$\lim_{k\to\infty} v(y_k) \geq v(\overline{y}). \tag{2.18}$$

Hence the sequence $\{v(y_k) : k \in \mathbb{Z}_+\}$ is bounded from below. As a result, using (2.17) and (2.18), one arrives at the boundedness of the sequence $\{r(z_k; y_k) : k \in \mathbb{Z}_+\}$, which leads to the following limit relations:

$$\|x - \overline{y}\|^2 = \lim_{k\to\infty} \|x - y_k\|^2 = \lim_{k\to\infty} 2t_k r(z_k; y_k) = 0.$$

Thus it is evident that $\overline{y} \equiv x \in \mathbb{R}^n$. Due to (2.17), we see that $u(x, t_k) \geq v(y_k)$ for any $x \in \mathbb{R}^n$, $t_k \in \mathbb{R}_+$ and $y_k \in A_{z_k}$, $k \in \mathbb{Z}_+$, and consequently, taking into account (2.14) and the equality $\overline{y} \equiv x \in \mathbb{R}^n$, one gets the following limit relations:

$$\overline{v}(x) = \lim_{k\to\infty} u(x, t_k) \geq \lim_{k\to\infty} v(y_k) \geq v(\overline{y}) = v(x).$$

This obviously implies that $\overline{v}(x) \geq v(x)$ for any $x \in \mathbb{R}^n$. Thus, on account of (2.16), the required assertion is proved. □

Lemma 2.8. *Take* $\{K_m \subset \Lambda : m \in \mathbb{Z}_+\}$ *as a sequence of embedded compact subsets of* Λ*, where, for any* $m \in \mathbb{Z}_+$, $K_m \subset K_{m+1}$*. Then*

$$\bigcap_{j=0}^{\infty} A(K_j) = A\left(\bigcap_{j=0}^{\infty} K_j\right).$$

Proof. Let us set $K := \bigcap_{j=0}^{\infty} K_j$ and $B := \bigcap_{j=0}^{\infty} A(K_j)$. It is evident that it is sufficient to prove the inclusion $B \subset A(K)$. By virtue of Lemma 1.1, the set B is compact. Suppose also here that $B \neq \varnothing$, and choose $y \in B$. Evidently there exists a sequence $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ such that $y := \lim_{k\to\infty} y_k$ and $y_m \in A(K_m)$, $m \in \mathbb{Z}_+$. It follows from the consideration above that there exists a sequence $\{z_m \in K_m\ m \in \mathbb{Z}_+\}$ such that $y_m \in A_{z_m}$ for any $m \in \mathbb{Z}_+$. Clearly, the sequence may be assumed to be convergent.

Let $\lim_{t\to\infty} z_m = z$; it is evident that $z \in K$. Using Definition (2.2) and taking into account the continuity of the mapping $u : \Lambda \to \mathbb{R}$, one obtains

$$\begin{aligned}\psi(z; y) = v(y) + r(z; y) \leq \lim_{m\to\infty} v(y_m) + \lim_{m\to\infty} r(z_m; y_m) = \\ = \lim_{m\to\infty} \psi(z_m; y_m) = \lim_{m\to\infty} u(z_m) = u(z),\end{aligned}$$

which yields the equality $\psi(z; y) = u(z)$ for $z \in K$, or $y \in A_z \subset A(K)$, thus proving the lemma. □

2.4. Analysis of the exceptional set Q. Denote now by $\operatorname{diam} M$ the diameter of a set M in a metrics space. Let us set $a(z) := \operatorname{diam} A_z$ for any $z \in \Lambda$ and put

$$Q := \{z \in \Lambda : a(z) > 0\}. \tag{3.1}$$

One of the crucial points in the proof of Theorem 2.2 is based on the following statement.

Proposition 2.9. *The set Q is a set of type F_δ, and it has Lebesgue measure zero.*

The proof of Proposition 2.9 follows from the lemmas formulated and proved below.

Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ and for $z \in \Lambda$, $e \in \mathbb{S}^{n-1}$, put

$$A_z^e := \{\lambda = \langle w, e\rangle : w \in A_z\}, \quad G_e := \{x, t) \in \Lambda : \langle x, e\rangle = 0\}, \tag{3.2}$$

and define, for $e \in \mathbb{S}^{n-1}$, the mapping

$$\Lambda \ni z \longmapsto a_e(z) := \operatorname{diam} A_z^e \in \overline{\mathbb{R}}_+. \tag{3.3}$$

In addition, for any fixed $e \in \mathbb{S}^{n-1}$ and $z \in G_e$, we define the mappings

$$\begin{aligned} \mathbb{R} \ni \xi &\longmapsto c_{e,z}(\xi) = \operatorname{diam} A^e_{z+(\xi e,0)} \in \overline{\mathbb{R}}_+, \\ \mathbb{R} \ni \xi &\longmapsto c^+_{e,z}(\xi) = \sup A^e_{z+(\xi e,0)} \in \overline{\mathbb{R}}, \\ \mathbb{R} \ni \xi &\longmapsto c^-_{e,z}(\xi) = \inf A^e_{z+(\xi e,0)} \in \overline{\mathbb{R}}. \end{aligned} \tag{3.4}$$

Lemma 2.10. *The mappings a, a_e, $c_{e,z}$, $c^+_{e,z}$, and $-c^-_{e,z}$ are upper semicontinuous for any $e \in \mathbb{S}^{n-1}$ and $z \in G_e$.*

Proof. Take, for instance, the mapping a: $\Lambda \to \overline{\mathbb{R}}_+$. For the other mappings, the proof is completely similar.

Let $\{z_m \in \Lambda : m \in \mathbb{Z}_+\}$ and $\lim_{m\to\infty} z_m = z \in \Lambda$. Let us put $K := \{z\} \cup \{z_m : m \in \mathbb{Z}_+\}$. Since K is compact, we see that, due to Lemma 1.1, $A(K)$ is also compact. Thus, for any $m \in \mathbb{Z}_+$, $a(z_m) = \operatorname{diam} A_{z_m} \le \operatorname{diam} A(K) < \infty$, whence it follows that the sequence $\{a(z_m) \in \mathbb{R} : m \in \mathbb{Z}_+\}$ is bounded.

Put

$$\alpha := \overline{\lim_{m\to\infty}} a(z_m); \tag{3.5}$$

then there exists a subsequence $\{z_{m_p} \in \Lambda : p \in \mathbb{Z}_+\}$ for which

$$\alpha := \lim_{p\to\infty} a(z_{m_p}) \tag{3.6}$$

together with sequences $\{y_p \in A(K_p) : p \in \mathbb{Z}_+\}$ and $\{\tilde{y}_p \in A(K_p) : p \in \mathbb{Z}_+$, $K_p := \{z\} \cup \{z_{m_j} : j = \overline{p, \infty}\}$ satisfying for all $p \in \mathbb{Z}_+$ the condition

$$a(z_{m_p}) = \|y_p - \tilde{y}_p\|. \tag{3.7}$$

It is evident that, without loss of generality, we may assume that the sequences $\{y_p \in A(K_p) : p \in \mathbb{Z}_+$ and $\{\tilde{y}_p \in A(K_p) : p \in \mathbb{Z}_+\}$ are convergent.

Put $y = \lim_{p\to\infty} y_p$ and $\tilde{y} = \lim_{p\to\infty} \tilde{y}_p$. Then, taking into account Lemma 2.8, one obtains that $y, \tilde{y} \in \bigcap_{p=0}^{\infty} A(K_p) = A(\bigcap_{p=0}^{\infty} K_p) = A_z$. Having used relations (3.5)–(3.7), one arrives at the relation

$$\overline{\lim_{m\to\infty}}\, a(z_m) = \lim_{p\to\infty} \|y_p - \tilde{y}_p\| = \|y - \tilde{y}\| \le \operatorname{diam} A_z = a(z),$$

which evidently proves the lemma. □

Let us now define, for an arbitrary $e \in \mathbb{S}^{n-1}$ (see (3.3)), the set

$$Q^e := \{z \in \Lambda : a_e(z) > 0\}. \tag{3.8}$$

Lemma 2.11. *The sets Q and Q^e are sets of F_δ type, that is, they are at most countable unions of closed sets in $\mathbb{R}^{n+1}$. In particular, the sets Q and Q^e are Lebesgue measurable.*

Proof. We confine ourselves to the case of the set Q. The proof for the set Q^e is completely similar.

The mapping $a : \Lambda \to \overline{\mathbb{R}}_+$ is upper semicontinuous, hence, the following sets are closed for all $m \in \mathbb{Z}_+$:

$$Q_m = \left\{(x,t) \in \Lambda : a(z) \ge \frac{1}{m+1},\ t \ge \frac{1}{m+1}\right\}.$$

Now the obvious equality $Q = \bigcup_{m=0}^{\infty} Q_m$ proves the lemma. □

Lemma 2.12. *For arbitrary $e \in \mathbb{S}^{n-1}$, $z \in G_e$, the set*

$$P_{e,z} := \{\xi \in \mathbb{R} : c_{e,z}(\xi) \ne 0\} \tag{3.9}$$

is countable.

Proof. Fix $e \in \mathbb{S}^{n-1}$ and $z = (x,t) \in G_e$ and put, for the sake of convenience, $c(\xi) := c_{e,z}(\xi)$ and $c^\pm(\xi) := c^\pm_{e,z}(\xi)$ for $\xi \in \mathbb{R}$, and $P := P_{e,z}$. We also set, for any $\xi \in \mathbb{R}$, $\Delta_\xi := (c^-(\xi), c^+(\xi))$.

It is evident that $|\Delta_\xi| = c^+(\xi) - c^-(\xi) = c(\xi)$ for any $\xi \in \mathbb{R}$, where $|\Delta_\xi| = \operatorname{diam} \Delta_\xi$, and $P = \{\xi \in \mathbb{R} : \Delta_\xi \ne \varnothing\}$. This means that the lemma follows if one proves that, for any $\xi \ne \eta \in \mathbb{R}$, we have $\Delta_\xi \cap \Delta_\eta = \varnothing$.

Put $z_\xi := z + (\xi e, 0)$, $z_\eta = z + (\eta e, 0)$ and $\xi < \eta$. Then for any $y \in \mathbb{R}^n$

$$\begin{aligned}\psi(z_\eta; y) - \psi(z_\xi; y) &= \frac{1}{2t}\left(\|x - y + \eta e\|^2 - \|x - y + \xi e\|^2\right)\\ &= \frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle y, e\rangle).\end{aligned} \tag{3.10}$$

Since the set A_{z_ξ} is compact and

$$c^+(\xi) = \sup\{\langle y, e\rangle : y \in A_{z_\xi}\},$$

it follows that there exists an element $\overline{y} \in A_{z_\xi}$ such that $\langle \overline{y}, e\rangle = c^+(\xi)$.

Put now

$$E_\xi := \{y \in \mathbb{R}^n : \langle y, e\rangle < c^+(\xi)\}.$$

Since $\xi \in \mathbb{R}^n$ is fixed, it is easy to see that for all $y \in E_\xi$

$$\frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle \overline{y}, e\rangle) < \frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle y, e\rangle), \tag{3.11}$$

whence, by virtue of (3.10), (3.11) and the inequality $\psi(z_\xi; y) \geq \psi(z_\xi; \overline{y})$ valid for any $y \in \mathbb{R}^n$, one obtains

$$\psi(z_\eta; y) > \psi(z_\xi; \overline{y}) + \frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle \overline{y}, e\rangle) = \psi(z_\eta; \overline{y}).$$

Thus, for any $\xi < \eta$, we have $A_{z_\eta} \subset \mathbb{R}^n \setminus E_\xi$, which results in the inequality $c^-(\eta) \geq c^+(\xi)$. The latter is equivalent to the equality

$$\Delta_\xi \cap \Delta_\eta = \varnothing$$

for any $\xi \neq \eta \in \mathbb{R}$, so the lemma is proved. □

Lemma 2.13. *The set Q has zero Lebesgue measure.*

Proof. Let $e \in \mathbb{S}^{n-1}$ and χ_{Q^e} be the characteristic function of the corresponding set Q^e. It follows from Lemma 2.11 that the function χ_{Q^e} is locally integrable in Λ. Let K_e^0 and K be compact sets in G_e and $\mathbb{R}$, respectively, and

$$K_e := \left\{z + (\xi e, 0) \in \mathbb{R}^{n+1} : z \in K_e^0,\ \xi \in K\right\}.$$

The Fubini theorem [5, 6] implies that

$$\int_{K_e} \chi_{Q^e} d\mu_{n+1} = \int_{K_e^0} d\mu_n(x) \int_K \chi_{Q^e}(z + (\xi e, 0)) d\mu_1(\xi) \tag{3.12}$$

for any $e \in \mathbb{S}^{n-1}$, where μ_k is the Lebesgue measure in $\mathbb{R}^k$, $k \in \mathbb{Z}_+$. It is now easy to see that for arbitrary $e \in \mathbb{S}^{n-1}$, $z \in G_e$,

$$\left\{\xi \in \mathbb{R} : \chi_{Q^e}(z + (\xi e, 0) \neq 0\right\} = \{\xi \in \mathbb{R} : c_{e,z}(\xi) \neq 0)\} = P_{e,z}.$$

By virtue of Lemma 2.12, the set $P_{e,z}$, at fixed $e \in \mathbb{S}^{n-1}$, $z \in G_e$, is countable, and therefore, for $z \in G_e$, we have

$$\int_K \chi_{Q^e}(z + (\xi e, 0)) d\mu_1(\xi) = 0.$$

Hence, using (3.12) and the condition $\chi_{Q^e} \geq 0$, one concludes that the function χ_{Q^e} is equal to zero almost everywhere on K. Taking into account the arbitrariness of the compact sets K_e^0 and K, one easily gets that χ_{Q^e} is equal to zero almost everywhere on Λ. Fix now an arbitrary orthonormal basis $\{e_j \in \mathbb{S}^{n-1} : j = \overline{1, n}\}$ in the space $\mathbb{R}^n$. It is easy to see, by (3.1) and (3.8), that $Q = \bigcup_{j \in \overline{1,n}} Q^{e_j}$. By virtue of the results proved above, each set Q^{e_j}, $j = \overline{1, n}$, is of zero Lebesgue measure and, hence, the set Q has zero Lebesgue measure as well. This proves the lemma. □

By using Lemmas 2.11 and 2.13, one readily obtains the proof of Proposition 2.9.

2.5. Analysis of the differential operator *L*. For Theorem 1.3 to be proved completely, we introduce a function $\rho : \Lambda \to \mathbb{R}^n$ such that for any $z \in \Lambda$

$$\Lambda \ni z :\to \rho(z) \in A_z. \tag{4.1}$$

The existence of such a function (4.1) follows from the axiom of choice [3].

Define the differential operator

$$Hu = \frac{\partial u}{\partial t} + \frac{1}{2}\|\nabla u\|^2, \tag{4.2}$$

where mapping $u : \Lambda \to \mathbb{R}$ is differentiable almost everywhere on Λ.

Lemma 2.14. *A function $\rho : \Lambda \to \mathbb{R}^n$ satisfying condition* (4.1) *is continuous at all points $z \in \Lambda \setminus Q$.*

Proof. Take $z \in \Lambda \setminus Q$ and choose a convergent sequence $\{z_m \in \Lambda : m \in \mathbb{Z}_+\}$ such that $\lim_{m\to\infty} z_m = z$. Owing to Lemma 2.7, the sequence

$$\{\rho(z_m) \in \mathbb{R}^n : m \in \mathbb{Z}_+\}$$

is bounded.

Fix now an arbitrary convergent subsequence $\{\rho(z_{m_k}) \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ of the sequence $\{\rho(z_m) \in \mathbb{R}^n : m \in \mathbb{Z}_+\}$. The lemma will be proved if we show that

$$\lim_{k\to\infty} \rho(z_{m_k}) = \rho(z). \tag{4.3}$$

For any $y \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$,

$$v(\rho(z_{m_k})) + r(z_{m_k}; \rho(z_{m_k})) \le v(y) + r(z_{m_k}; y), \tag{4.4}$$

whence, passing to the limit as $k \to \infty$, one gets

$$v(\bar{y}) + r(z; \bar{y}) \le v(y) + r(z; y), \tag{4.5}$$

where, by definition, $\lim_{k\to\infty} \rho(z_{m_k}) = \bar{y}$. From (4.5) it follows that $\bar{y} \in A_z$. Since $z \in \Lambda \setminus Q$ and the set A_z is a singleton, one concludes immediately that $\bar{y} = \rho(z)$. Thus the lemma is proved. □

Lemma 2.15. *The function $u := L(v)$ is differentiable at all points $z \in \Lambda \setminus Q$ and*

$$(Hu)(z) = (Hr)(z; y)|_{y=\rho(z)} = 0. \tag{4.6}$$

Proof. Choose $z \in \Lambda \setminus Q$, $e \in \mathbb{S}^n$, and a sequence $\{\tau_k \in \mathbb{R}_+ : k \in \mathbb{Z}_+\}$ convergent to zero. Put also $z_k := (z + \tau_k e) \in \Lambda, k \in \mathbb{Z}_+$. From the definition of the function $u = L(v)$ one obtains the inequalities

$$\begin{aligned} u(z) &\le v(\rho(z_k) + r(z; \rho(z_k)), \\ u(z_k) &\le v(\rho(z) + r(z_k; \rho(z)) \end{aligned} \tag{4.7}$$

for all $k \in \mathbb{Z}_+$. As a result of (4.7), having taken into account that, for all $k \in \mathbb{Z}_+$,

$$u(z) = v(\rho(z) + r(z;\rho(z)),$$
$$u(z_k) = v(\rho(z_k) + r(z_k;\rho(z_k)),$$

one gets, also for all $k \in \mathbb{Z}_+$, the following inequalities:

$$\begin{aligned} \frac{u(z+\tau_k e) - u(z)}{\tau_k} &\leq \frac{r(z+\tau_k e;\rho(z)) - r(z;\rho(z))}{\tau_k}, \\ \frac{u(z+\tau_k e) - u(z)}{\tau_k} &\geq \frac{r(z+\tau_k e;\rho(z_k)) - r(z;\rho(z_k))}{\tau_k}. \end{aligned} \tag{4.8}$$

Lemma 2.14 now implies that $\lim_{k\to\infty}\rho(z_k) = \rho(z)$. Taking this into account and passing in (4.8) to the limit when $k \to \infty$, one arrives at the relation

$$\lim_{k\to\infty} \frac{u(z+\tau_k e) - u(z)}{\tau_k} = \frac{\partial}{\partial\tau} r(z+\tau e;\rho(z)|_{\tau=0}. \tag{4.9}$$

Since the right hand side of equality (4.9) is independent of the choice of the sequence $\{\tau_k \in \mathbb{R}_+ : k \in \mathbb{Z}_+\}$, the mapping $\tau \mapsto u(z+\tau e)$ is differentiable at zero, and the equality

$$\frac{\partial}{\partial\tau} u(z+\tau e)|_{\tau=0} = \frac{\partial}{\partial\tau} r(z+\tau e;\rho(z)|_{\tau=0}$$

is true for all $z \in \Lambda \setminus Q$. Thus, we obtain the proof of the lemma. □

The final steps of proof of Theorem 2.2. It is now easy to complete the proof of Theorem 1.2. The validity of (i) follows from Lemmas 2.6 and 2.15; (ii) follows from Proposition 2.9 and Lemma 2.15; (iv) follows from Lemma 2.15; and (v) follows from Lemma 2.15.

To prove statement (iii) of Theorem 2.2, we note that Lemma 2.15 implies that the relation

$$\nabla u(z) = \nabla r(z;y)|_{y=\rho(z)}, \tag{4.10}$$

it true for all $z \in \Lambda \setminus Q$. From Lemma 2.14 (by the continuity of the mapping $\rho : \Lambda \setminus Q \to \mathbb{R}^n$) and (4.10) we now immediately obtain assertion (iii). Thus, the theorem is proved completely. □

2.6. Generalizations of the inf-type Lax formula. There are many applications in modern mathematical physics of the following nonlinear Hamilton-Jacobi type first order partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}||\nabla u||^2 + \frac{\beta u}{2}||x||^2 + \frac{1}{2} < Jx, x >= 0, \tag{5.1}$$

where $t \in \mathbb{R}_+, \beta \in \mathbb{R}_+$, and $J : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal positive definite matrix, with the Cauchy data

$$u|_{t=0^+} = v \in \mathrm{BSC}(\mathbb{R}^n). \tag{5.2}$$

It is natural to seek for a generalisation of the Lax formula (0.1) to represent the solution of the Cauchy problem (5.1), (5.2) by using the dynamical systems approach developed in Section 1 (see also [4, 7]). As a result of such an analysis, we obtain an inf-type representation for problem (5.1), (5.2), which is described below. Some related results concerning generalised solutions to Hamilton-Jacobi equations can be also found in [8, 9].

Proposition 2.16. *The Cauchy problem (5.1), (5.2) admits the following inf-type fixed point problem representation:*

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\{v(y) + P_u(x,t;y)\}, \tag{5.3}$$

where, by definition, for any $(z;y) \in \Lambda\times\mathbb{R}^n$ *the nonlocal functional*

$$P(z;y) : \mathrm{BSC}(\mathbb{R}^n;\mathbb{R}) \ni u \to P_u(z;y) \in \mathbb{R}$$

is given by

$$P_u(x,t;y) := \frac{1}{4}\frac{d}{d\tau}\|\alpha(\tau|x,t;y)\|^2\Big|_{0^+}^{t} - \frac{\beta}{16}(\|x\|^4 - \|y\|^4), \tag{5.4}$$

and the mapping $\alpha : \mathbb{R}_+\times(\Lambda\times\mathbb{R}^n) \to \mathbb{R}^n$ *is the functional solution to the following nonlinear second order ordinary differential equation (with respect to the first variable* $\tau\in\mathbb{R}_+$*):*

$$\ddot{\alpha} + \beta u(\alpha,\tau)\alpha + \beta\|\alpha\|^2\dot{\alpha} = -J\alpha, \quad \alpha|_{\tau=0^+} = y, \quad \alpha|_{\tau=t} = x, \tag{5.5}$$

for any $y, x \in \mathbb{R}^n$ *and fixed* $u : \Lambda \to \mathbb{R}$.

Letting $\beta \to 0^+$ in the exact expressions (5.3)–(5.5), we get that the fixed point problem (5.3) becomes that of the Lax type exact form. Consider the fixed point problem (5.3) and discuss conditions under which it possesses a unique solution $u \in \mathrm{BSC}(\mathbb{R}^n;\mathbb{R})$ satisfying the Cauchy data

$$\lim_{t\to 0^+} u(x,t) = v(x)$$

for all $x\in\mathbb{R}^n$.

For the infimum expression (5.3) to exist at almost all points $z := (x;t) \subset \Lambda$, one needs to find conditions at which the function $P_u(z;\cdot) : \mathbb{R}^n \to \mathbb{R}$ will be convex and lower semicontinuous at each $u \in \mathrm{BSC}(\mathbb{R}^n;\mathbb{R})$. Since

$$P_u(z;y) \;=\; \frac{1}{2}\left\langle\dot{\alpha}(t|z;y),\ \alpha(t|z;y)\right\rangle - \frac{1}{2}\left\langle\dot{\alpha}(0|z;y), \alpha(0|z;y)\right\rangle - \frac{\beta}{16}\left(\|x\|^4 - \|y\|^4\right),$$

where $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)^T \in \mathbb{R}^n$ and for every $j = \overline{1,n}$ the set of equations

$$\ddot{\alpha}_j + \beta u(\alpha,\tau)\alpha_j + \beta\|\alpha\|^2\dot{\alpha}_j = -\omega_j^2\alpha_j,\ \alpha_j|_{\tau=0^+} = y_j\ ,\ \alpha_j|_{\tau=t} = x_j, \tag{5.6}$$

holds, where $\omega_j^2 \in \mathbb{R}_+,\ j = \overline{1,n}$, are some fixed eigenvalues of the matrix $J : \mathbb{R}^n \to \mathbb{R}^n$ in (5.1), one gets easily from (5.6) that

$$P_u(z;y) = \frac{1}{2}\int_0^t \|\dot{\alpha}\|^2 ds - \frac{\beta}{2}\int_0^t \|\alpha\|^2 (u(\alpha;s) + \langle \dot{\alpha}, \alpha \rangle) ds - \\ - \frac{1}{2}\int_0^t \langle J\alpha, \alpha \rangle \, ds - \frac{\beta}{16}(\|x\|^4 - \|y\|^4) \quad (5.7)$$

for all $(z;y) \in \Lambda \times \mathbb{R}^n$. Thereby, making use of expression (5.7) (for $\beta \in \mathbb{R}_+$), we can claim that $P_u(z;\cdot) \in \mathrm{BSC}(\mathbb{R}^n)$, and that the mapping $\mathcal{P}_v : \mathrm{BSC}(\mathbb{R}^n) \to \mathrm{BSC}(\mathbb{R}^n)$ is well-defined, where, for any $u \in \mathrm{BSC}(\mathbb{R}^n)$,

$$\mathcal{P}_v(u) := \inf_{y \in \mathbb{R}^n} \{v(y) + P_u(z;y)\}. \quad (5.8)$$

In view of (5.8) and (5.3), we arrive at the following fixed point problem:

$$\mathcal{P}_v(u) = u \quad (5.9)$$

for $u \in \mathrm{BSC}(\mathbb{R}^n)$ with an arbitrary, but fixed, function $v \in \mathrm{BSC}(\mathbb{R}^n)$.

In relation to the nonlinear mapping (5.8) one can define the topological space $\mathcal{B}_{loc}(\mathbb{R}^n)$ of all locally bounded functions $u : \Lambda \times \mathbb{R}^n \to \mathbb{R}$ with the topology of uniform consequence over all compact subsets of $\Lambda \times \mathbb{R}^n$. For the further analysis of problem (5.9), we need the Fan-Browder fixed point theorem [11], which reads as follows.

Theorem 2.17. *In a topological vector space $\mathcal{B}$, let $\mathcal{P}_v$ be a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ such that there exists a non-empty subset $\overline{A} \subset \mathcal{B}$ satisfying the following condition:*

> *The set $\mathcal{B} \setminus \left(\bigcup_{u \in \overline{A}} \mathcal{P}_v u\right)$ is either compact or empty, and the subset* $\operatorname{conv} \overline{A}$ *is compact.*

Then there exists a point $\bar{u} \in \mathcal{B}$ such that $\bar{u} = \mathcal{P}_v \bar{u}$.

Keeping in mind the conditions above, we define, for any $v \in \mathrm{BSC}(\mathbb{R}^n)$, the subset

$$A_v := \bigcup_{u \in \mathcal{B}_{loc}(\mathbb{R}^n)} \mathcal{P}_v(u) \subset \mathcal{B}_{loc}(\mathbb{R}^n) \quad (5.10)$$

and formulate the following two lemmas.

Lemma 2.18. *The set* $\operatorname{conv} A_v \subset \mathcal{B}_{loc}(\mathbb{R}^n)$ *is compact and convex in $\mathcal{B}_{loc}(\mathbb{R}^n)$ for any* $v \in \mathrm{BSC}(\mathbb{R}^n)$.

Lemma 2.19. *The set $\mathcal{B} \setminus \left(\bigcup_{u \in A_v} \mathcal{P}_v u\right)$ is compact in $\mathcal{B}_{loc}(\mathbb{R}^n)$ for any* $v \in \mathrm{BSC}(\mathbb{R}^n)$.

In connection with these lemmas, we introduce the following sets, $\overline{A} = A_v$ and $\mathcal{B} = \mathcal{B}_{loc}(\mathbb{R}^n)$, and formulate the following statement.

Proposition 2.20. *The fixed point problem* (5.9) *with an arbitrary, but fixed, function* $v \in \mathrm{BSC}(\mathbb{R}^n)$ *restricted to the topological space* $\mathcal{B}_{loc}(\mathbb{R}^n)$*, is uniquely solvable, with the solution belonging to the space* $\mathrm{BSC}(\mathbb{R}^n)$*. Moreover, this solution satisfies the standard viscosity condition*

$$v(x) = \lim_{u \to 0^+} \overline{u}(x,t)$$

for all $(x, y) \in \Lambda$.

Thereby, the Hamilton-Jacobi equation (5.1) with Cauchy data from the functional space $\mathrm{BSC}(\mathbb{R}^n)$ possesses a unique solution representable as the fixed point problem (5.3), (5.4), which is solvable for any $v \in \mathrm{BSC}(\mathbb{R}^n)$.

In the case where the Cauchy data belong to the space $\mathrm{BSC}(\mathbb{S}^n)$, where $\mathbb{S}^n$ is the N-dimensional sphere imbedded smoothly into the space $\mathbb{R}^{N+1}$, and $\nabla u \in T^*(\mathbb{S}^N)$, the Hamilton-Jacobi equation (1.2) on the sphere $\mathbb{S}^n$ defines [12] the evolution of a function $u \in \mathrm{BSC}(\mathbb{S}^N)$, which can be also represented in the Lax inf-type form. Below we will analyse this problem making use of some results from Section 1.

2.7. Solution of extremality problem on $\mathbb{S}^N$. As is well-known, the following extended finite-dimensional Hamiltonian system is closely related to the Hamilton-Jacobi equation (I.1.2) on the sphere $\mathbb{S}^N$:

$$dq/d\tau = \partial H(q,p)/\partial p, \quad dp/d\tau = -\partial H(q,p)/\partial q, \tag{6.1}$$

where $H(q,p) = 1/2\,\|p\|^2\,\|q\|^2$, $(q,p) \in T^*(\mathbb{R}^{N+1})$, $\tau \in \mathbb{R}_+$. This system is considered together with the following constraints:

$$(q,p) \in T^*(\mathbb{S}^N) := \left\{ q \in \mathbb{R}^{N+1} : \|q\|^2 - 1 = 0,\ \langle q, p\rangle = 0 \right\}.$$

From (6.1) we find that, for $\tau \in (0,t]$, on $T^*(\mathbb{R}^{N+1})$,

$$dq/d\tau = p\,\|q\|^2, \quad dp/d\tau = -\,\|p\|^2\,q. \tag{6.2}$$

Having taken the Cauchy data $q|_{\tau=0^+} = y \in \mathbb{S}^N$, $q|_{\tau=t} = x \in \mathbb{S}^N$, from (6.2), one easily obtains that, for all $t \in \mathbb{R}_+$, the expression

$$\|p\| = t^{-1} \arccos\langle y, x\rangle \tag{6.3}$$

is independent of $\tau \in (0,t]$. Consider now the following extremality expression for the Hamilton-Jacobi equation (1.2), Section 1, extended in a natural way on the entire space $\mathbb{R}^{N+1}$:

$$\begin{aligned} \tilde{u}(x,t) &= \inf_{\substack{\{q(\tau)\in\mathbb{S}^N :\, q|_{\tau=0^+}=y\in\mathbb{S}^N,\\ q|_{\tau=t}=x\in\mathbb{S}^N\}}} \left\{ v(y) + \int_0^t d\tau(\langle p, dq/dt\rangle - H(q,p)) \right\} \\ &= \inf_{y\in\mathbb{S}^N} \left\{ v(y) + \frac{t}{2}\,\|p(t;x,t|y)\|^2 \right\}, \end{aligned} \tag{6.4}$$

where we have put $q = q(\tau; x, t|y)$, $p = p(\tau; x, t|y)$ for $\tau \in (0, t]$, $x, y \in \mathbb{S}^N$, and made use of the equality $d\,\|p\|/d\tau = 0$ implied by equations (6.2).

On the other hand, for the quantity $\|p\| \in \mathbb{R}^{N+1}$, we have formula (6.3), which, together with (6.4), leads us to the inf-type Lax expression

$$\tilde{u}(x,t) = \inf_{y \in \mathbb{S}^N} \left\{ v(y) + \frac{1}{2t} \arccos^2 \langle y, x \rangle \right\}, \tag{6.5}$$

which we suggest as a "candidate" for a solution of the Hamilton-Jacobi equation (1.2), Section 1, on the sphere $\mathbb{S}^N$ with the Cauchy data $v \in \mathrm{BSC}(\mathbb{S}^N)$. The required equality

$$\tilde{u}(x,t) = u(x,t),$$

for almost all $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$, can be obtained by an argument similar to that in Section 1. Here, we shall consider the case where $v \in \mathrm{BSC}(\mathbb{S}^N)$.

2.8. Analysis of the extremality problem on $\mathbb{S}^N$. Here we shall prove the equality

$$\tilde{u}(x,t) = u(x,t)$$

for all $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$, based on the properties of the Cauchy data for (1.2), Section 1, and the exact inf-type expression (6.5). It is easy to show [3, 4] that there exists a point $\tilde{\xi}(x,t) \in \mathbb{S}^N$, such that

$$\tilde{u}(x,t) = v(\tilde{\xi}(x,t)) + \frac{1}{2t} \arccos^2 \left\langle \tilde{\xi}(x,t), x \right\rangle, \tag{7.1}$$

for any $x \in \mathbb{S}^N$ and a fixed $t \in \mathbb{R}_+$. At the same time, one can show that the exact solution $u : \mathbb{S}^N \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the following expression involving differential forms:

$$du(x,t) = \langle p(x,t), dx \rangle - 1/2\, \|p(x,t)\|^2\, dt, \tag{7.2}$$

which is completely equivalent to the Hamilton-Jacobi equation (1.2), Section 1, where $p : \mathbb{S}^N \times \mathbb{R}_+ \to \mathbb{R}^{N+1}$ fulfils (6.2). Thus, based on (6.2) and (7.2), one gets easily that

$$\begin{aligned} u(x,t) = v(\xi(x,t)) + \int_0^t (du(q(\tau; x, t|y)), t)/d\tau) d\tau = \\ = v(\xi(x,t)) + 1/2 \int_0^t d\tau\, \|p(\tau; x, t|y)\|^2, \end{aligned} \tag{7.3}$$

that is, for all $x \in \mathbb{S}^N$ and fixed $t \in \mathbb{R}_+$, we have

$$u(x,t) = v(\xi(x,t)) + \frac{1}{2t} \arccos^2 \langle \xi(x,t), x \rangle. \tag{7.4}$$

Here, $\xi : \mathbb{S}^N \times \mathbb{R}_+ \to \mathbb{S}^N$ is a mapping defined as follows:

$$(x - \xi \langle x, \xi \rangle) \langle \xi, x \rangle = t p_0(\xi) (1 - \langle x, \xi \rangle)^{1/2}, \tag{7.5}$$

where the function $p_0 : \mathbb{S}^N \to \mathbb{R}^{N+1}$ satisfies the following relation implied by (7.3):

$$\nabla u(\xi, t)|_{t=0^+} := p_0(\xi), \tag{7.6}$$

depending only on the Cauchy data $v \in \mathrm{BSC}(\mathbb{S}^N)$. Therefore, it is sufficient to prove the equality for all $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$, and that will yield the required equality

$$\tilde{u}(x, t) = u(x, t).$$

It is easy to establish the following lemma.

Lemma 2.21. *Expression* (7.1) *is* $\mathbb{S}^N \ni x$*-differentiable for each* $t \in \mathbb{R}_+$*, and the following equality holds:*

$$\nabla \tilde{u}(\tilde{\xi}, t)|_{t=0^+} = \tilde{p}_0(\tilde{\xi}), \tag{7.7}$$

where the relation

$$\left(x - \tilde{\xi}\langle x, \tilde{\xi}\rangle\right)\left\langle \tilde{\xi}(x, t), x\right\rangle = t\tilde{p}_0(\tilde{\xi})\left(1 - \langle x, \tilde{\xi}\rangle\right)^{1/2}, \tag{7.8}$$

is fulfilled for any $x \in \mathbb{S}^N$ *and* $t \in \mathbb{R}_+$*.*

In view of (7.8) and (7.6), we conclude that, with

$$p_0(x) = \tilde{p}_0(x),$$

for any $x \in \mathbb{S}^N$, the equality $\tilde{\xi}(x, t) = \xi(x, t)$ holds for all $t \in \mathbb{R}_+$. This means, by (7.1) and (7.2), that the inf-type Lax expression (6.4) solves the Hamilton-Jacobi equation (1.2), Section 1, on the sphere $\mathbb{S}^N$.

The argument above proves the following statement.

Proposition 2.22. *The* inf*-type expression*

$$u(x, t) = \inf_{y \in \mathbb{S}^N} \left\{ v(y) + \frac{1}{2t} \arccos^2 \langle x, y\rangle \right\},$$

solves the Hamilton-Jacobi equation (1.2)*, Section* 1*, on the sphere* $\mathbb{S}^N$ *with the Cauchy data* $v \in \mathrm{BSC}(\mathbb{S}^N)$ *having the standard limit solution property.*

It is also easy to see that the method suggested above can be applied to the problem of finding an inf-type solution of an oscillator Hamilton-Jacobi equation on the sphere $\mathbb{S}^N$. This problem will produce an inf-type solution based on properties of the well-known K. Neumann N-oscillator dynamical system, whose complete Liouville-Arnold integrability was proved by J. Moser [10, 12].

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Authors' Addresses

N. K. Prykarpatska:
IAPMM, NAS of Ukraine, Lviv 290601, Ukraine
Current address: AGH University of Science and Technology, Department of Applied Mathematics, Krakow 30059, Poland
E-mail address: pryk.anat@ua.fm, natpats@yahoo.com

D. L. Blackmore:
NJIT, Department of Mathematical Sciences, Newark, NJ 07102, USA
E-mail address: deblac@m.njit.edu

A. K. Prykarpatsky:
IAPMM, NAS of Ukraine, Lviv 290601, Ukraine, and AGH University of Science and Technology, Department of Applied Mathematics, Krakow 30059 Poland
E-mail address: pryk.anat@ua.fm, prykanat@cybergal.com

M. Pytel-Kudela:
AGH University of Science and Technology, Department of Applied Mathematics, Krakow 30059, Poland
E-mail address: imath@mat.agh.edu.pl