Miskolc Mathematical Notes
Vol. 4 (2003), No 2, pp. 151-155

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2003.65

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Miskolc Mathematical Notes HU ISSN 1586-8850
Vol. 4 (2003), No. 2, pp. 151–155

A GENERALIZATION OF THE WALLIS FORMULA

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[Received: March 21, 2003]

Abstract. We prove a generalization of the Wallis formula which is connected with the generalized π (denoted by $\hat{\pi}(n)$, $n > 0$ is real) introduced by Elbert in [1].

Mathematics Subject Classification: Primary 34C10; Secondary 40A20

Keywords: Wallis formula, generalized π

1. Introduction

The infinite product of the form

$$\frac{\pi}{2} = \begin{cases} \lim_{k\to\infty} \frac{2^2}{1\cdot 3} \cdot \frac{4^2}{3\cdot 5} \cdot \cdots \cdot \frac{(2k)^2}{(2k-1)\cdot(2k+1)} \\ \lim_{k\to\infty} 2 \cdot \frac{2\cdot 4}{3^2} \cdot \frac{4\cdot 6}{5^2} \cdot \cdots \cdot \frac{(2k-2)\cdot 2k}{(2k-1)^2} \end{cases} \tag{1}$$

is known as the Wallis formula (or product) [5]. This formula is the most remarkable expression for the number π. The formula is proved in an elementary way on the basis of recurrence formula of $\int \sin^\alpha(t)\,dt$.

In this paper we give a generalization and prove it similarly as in the original case.

2. The main result

Theorem (Generalized Wallis formula)**.** *Let $n > 0$ be an arbitrary real number. Then*

$$\frac{\hat{\pi}(n)}{2} = \begin{cases} \lim_{k\to\infty} \frac{1}{n} \prod_{i=1}^{k} \frac{i^2(n+1)^2}{i^2(n+1)^2-n^2} \\ \lim_{k\to\infty} \frac{n+1}{n} \prod_{i=1}^{k-1} \frac{i\,(i+1)(n+1)^2}{[(i+1)(n+1)-1][(i+1)(n+1)-n]} \end{cases} \tag{2}$$

where

$$\hat{\pi}(n) = 2 \cdot \frac{\frac{\pi}{n+1}}{\sin\frac{\pi}{n+1}}. \tag{2a}$$

In the case where $n = 1$, (2) is the original Wallis formula (1).

Research supported by the Hungarian National Foundation for Scientific Research, Grant T/026138.

Proof. Consider the half-linear differential equation

$$y''|y'|^{n-1} + y^{n^*} = 0, \quad y = y(x), \quad (-\infty < x < \infty), \tag{3}$$

where $y^{n^*} = |y|^n \cdot \operatorname{sign} y$, $n > 0$.

In [1] Elbert introduced the generalized sine function $S_n(x)$ as the solution of the differential equation (3) satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$. The function $S_n(x)$ is periodic with the period $2\hat{\pi}$, and

$$0 < S_n(x) < 1 \quad \text{for } 0 < x < \frac{\hat{\pi}}{2}$$

and

$$S_n\left(\frac{\hat{\pi}}{2}\right) = 1, \quad S'_n\left(\frac{\hat{\pi}}{2}\right) = 0.$$

It is proved that the generalized Pythagorean formula holds

$$|S_n(x)|^{n+1} + |S'_n(x)|^{n+1} = 1. \tag{4}$$

First we show the following simple integral formula

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha}(t)\,dt = \frac{\alpha - n}{\alpha} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha-(n+1)}(t)\,dt \qquad (\alpha \geq n+1). \tag{5}$$

From the differential equation (3), we have

$$\int S_n^{n^*}(t)\,dt = -\frac{1}{n} S_n'^{n^*} + \text{const.} \tag{6}$$

By partial integration we obtain from (6) and (4) that

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha}(t)\,dt = \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha^*}(t)\,S_n^{\alpha-n}(t)\,dt =$$
$$= \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha}(t)\,dt = \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha^*}(t)\,S_n^{\alpha-n}(t)\,dt =$$
$$= \frac{\alpha - n}{n} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha-(n+1)}(t)\,dt + \frac{\alpha - n}{n} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha}(t)\,dt. \tag{7}$$

Rearranging (7) we obtain (5).

From (6), we get

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{n^*}(t)\,dt = \frac{1}{n}. \tag{8}$$

In the cases where $\alpha = k(n+1)$ and $\alpha = k(n+1)+n$,$(k = 1, 2, \dots)$, we obtain from (5) the relations

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)}(t)\,dt = \frac{k - \nu}{k} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{(k-1)(n+1)}(t)\,dt \tag{9}$$

and

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)+n}(t)\,dt = \frac{k}{k+\nu}\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{(k-1)(n+1)+n}(t)\,dt, \tag{10}$$

where $\nu = \frac{n}{n+1}$, $\nu \in (0,1)$.

Repeating (9) and (10) and using (8) we get

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)}(t)\,dt \ = \ \frac{\hat{\pi}(n)}{2}\prod_{i=1}^{k}\frac{i-\nu}{i}, \qquad (k = 1,2,\dots) \tag{11}$$

and

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)+n}(t)\,dt \ = \ \frac{1}{n}\prod_{i=1}^{k}\frac{i}{i+\nu}. \tag{12}$$

In the case where $n = 1$, formulas (11) and (12) are the well-known recurrence formulas ($k(n+1)$ is the even, $k(n+1)+n$ is the odd equivalent of the index).

Since $0 < S_n(x) < 1$ on $\left(0, \frac{\hat{\pi}(n)}{2}\right)$, we have

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)+n}(t)\,dt \ < \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)}(t)\,dt$$
$$< \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{(k-1)(n+1)+n}(t)\,dt \qquad (k = 1,2,\dots). \tag{13}$$

By (11) and (12), formula (13) can be written in the form

$$A_k < \frac{\hat{\pi}(n)}{2} < B_k \qquad (k = 1,2,\dots), \tag{14}$$

where

$$A_k = \frac{1}{n}\prod_{i=1}^{k}\frac{i^2}{i^2-\nu^2} \tag{15}$$

and

$$B_k = \frac{1}{\nu}\prod_{i=1}^{k-1}\frac{i\,(i+1)}{\left[i+1-\frac{\nu}{n}\right][i+1-\nu]}. \tag{16}$$

Clearly,

$$B_k = \frac{k+\nu}{k}A_k. \tag{17}$$

The sequence A_k is strictly increasing because

$$\frac{i^2(n+1)^2}{i^2(n+1)^2-n^2} > 1 \qquad (i = 1,2,\dots,k).$$

Hence, by (14), the limit $\lim_{k\to\infty} A_k$ exists.

The sequence B_k is strictly decreasing because

$$\frac{i\,(i+1)}{\left[i+1-\frac{\nu}{n}\right][i+1-\nu]} = \frac{i\,(i+1)}{i\,(i+1)+\frac{\nu^2}{n}} < 1 \qquad (i = 1, 2, \ldots, k-1).$$

By (14), the limit $\lim_{k\to\infty} B_k$ exists.

We estimate the difference of B_k and A_k, (14) and (17)

$$0 < B_k - A_k = \frac{\nu}{k} A_k < \frac{\nu}{k}\frac{\hat{\pi}(n)}{2}. \tag{18}$$

With increasing n, $B_k - A_k$ can be diminished, so $\frac{\hat{\pi}(n)}{2}$ can be approximated with optional accuracy with the A_k, B_k sequence, that is

$$\lim_{k\to\infty} A_k = \frac{\hat{\pi}(n)}{2} = \lim_{k\to\infty} B_k,$$

which proves our theorem. □

Remark. The integrals (11) and (12) may be computed by using the gamma-functions.

Second Proof. We take

$$I(n,\alpha) = \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha}(t)\,dt. \tag{19}$$

Consider $\varphi_n(t)$, the inverse function of $S_n(t)$. From (4),

$$\frac{d\varphi_n(t)}{dt} = \frac{1}{\sqrt[n+1]{1-t^{n+1}}} \qquad (0 \leqq t < 1). \tag{20}$$

From (19), using the substitution $\tau = S_n(t)^{n+1}$, we get

$$I(n,\alpha) = \frac{1}{n+1}\int_0^1 \tau^{\frac{\alpha-n}{n+1}}(1-\tau)^{-\frac{1}{n+1}}\,d\tau. \tag{21}$$

Consider the Euler first integral [2]

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{22}$$

where x, y are positive. This integral can be evaluated using gamma-function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{23}$$

In the right side of (20), there is the Euler first integral. Using (23), we get

$$I(n,\alpha) = \frac{1}{n+1}\frac{\Gamma\left(\frac{\alpha+1}{n+1}\right)\Gamma\left(\frac{n}{n+1}\right)}{\Gamma\left(\frac{\alpha}{n+1}\right)} \qquad (\alpha > 0). \tag{24}$$

In the case $\alpha = k(n+1)+n$, $(k = 1, 2, ,)$, we have

$$I(n, k(n+1)+n) = \frac{1}{n+1} \frac{\Gamma(k+1)\Gamma\left(\frac{n}{n+1}\right)}{\Gamma\left(k+\frac{n}{n+1}\right).}$$

Using the elementary formula for the gamma-function

$$x\Gamma(x) = \Gamma(x+1), \tag{25}$$

we get

$$I(n, k(n+1)+n) = \frac{1}{n} \prod_{i=1}^{k} \frac{i}{i+\nu},$$

which is equivalent to (12). If $\alpha = k(n+1)$, applying (22), we obtain

$$I(n, k(n+1)) = \frac{\Gamma\left(k+\frac{1}{n+1}\right)\Gamma\left(\frac{n}{n+1}\right)}{\Gamma(n+1).}$$

In view of (25), we have

$$I(n, k(n+1)) = \frac{1}{n+1}\Gamma\left(\frac{1}{n+1}\right)\Gamma\left(\frac{n}{n+1}\right)\prod_{i=1}^{k} \frac{i-\nu}{i}. \tag{26}$$

With the help of the Euler functional equation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

and (2a), we get

$$\frac{1}{n+1}\Gamma\left(\frac{n}{n+1}\right)\Gamma\left(\frac{n}{n+1}\right) = \frac{\hat{\pi}(n)}{2}.$$

Substituting this into (26), we obtain (11). □

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