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M. E. Özdemir, Mustafa Gürbüz, and Çetin Yildiz

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INEQUALITIES FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE QUASI-CONVEX OR h-CONVEX FUNCTIONS

M. EMİN ÖZDEMİR, MUSTAFA GÜRBÜZ, AND ÇETİN YILDIZ

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Abstract. In this paper, we establish some new integral inequalities for quasi-convex and $h-$convex functions by using a kernel and fairly elementary analysis.

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1. INTRODUCTION

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$
be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

For recent results, refinements, counterparts, generalizations and new *Hermite-Hadamard* type inequalities see [9], [2],[4] and [11].

Definition 1 (See [7])**.** A function $f : [a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\}, \qquad (QC)$$

holds for all $x, y \in [a,b]$ and $\lambda \in [0,1]$.

Definition 2 (see [10])**.** Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an $h-$convex function or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0,1]$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

In recent years, several authors have been studied on integral inequalities. One of the well known of these inequalities is Simpson's inequality as following:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see the papers [8], [1], [3], [5].

In [6], Özdemir *et al.* proved the following lemma:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$ and $r\in\mathbb{R}^{+}$ then the following equality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx \tag{1.1}$$

$$=(b-a)^2\int_0^1 k(t)f''(tb+(1-t)a)\,dt$$

where

$$k(t)=\begin{cases}\frac{t}{r}\left(\frac{1}{r+1}-t\right), & t\in\left[0,\frac{1}{2}\right)\\ (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right), & t\in\left[\frac{1}{2},1\right]\end{cases}.$$

In this article, by using functions whose second derivatives' absolute values are quasi-convex or $h-$convex, we obtained new inequalities related to the left hand side of Simpson inequality. Also new inequalities are proved.

2. Results for Quasi-Convex Functions

Theorem 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$ and $|f''|^q$ is quasi-convex, where $a,b\in I$ with $a<b$, $r\geq1$ and $q\geq1$, then one has the following inequality;*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq\frac{r^3-3r+6}{12r(r+1)^3}(b-a)^2\left(\sup\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{2.1}$$

Proof. Suppose that $q=1$, from Lemma 1 and by using quasi-convexity of $|f''|$, we obtain

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f(tb+(1-t)a)\,dt\right|$$

$$\leq (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|dt\right\}$$

$$\leq (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\sup\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\sup\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}dt\right\}$$

$$=\frac{r^3-3r+6}{12r(r+1)^3}(b-a)^2\sup\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}.$$

Now suppose that $q>1$, from Lemma 1 we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f(tb+(1-t)a)\,dt\right|$$

$$\leq (b-a)^2\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|dt$$

By using Power mean inequality we get

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f(tb+(1-t)a)\,dt\right|$$

$$\leq (b-a)^2\left(\int_0^1|k(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is quasi-convex, we have

$$\left| \frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r}\int_0^1 f(tb+(1-t)a)\,dt \right|$$
$$\leq \frac{r^3-3r+6}{12r(r+1)^3}(b-a)^2\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 1. *Under the assumptions of Theorem 2, if we choose $r=1$ in (2.1), we have the following inequality:*

$$\left| \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx \right|$$
$$\leq \frac{(b-a)^2}{24}\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Corollary 2. *If we take $q=1$ and $r=1$ in (2.1), we obtain an inequality which includes the left-hand side of the Corollary 3 in [6]:*

$$\left| \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx \right|$$
$$\leq \frac{(b-a)^2}{24}\sup\left\{|f''(a)|, |f''(b)|\right\}$$
$$\leq \frac{(b-a)^2}{24}\left(|f''(a)| + |f''(b)|\right)$$

Corollary 3. *If we choose $r=2$ in (2.1), we obtain*

$$\left| \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{81}\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{81}\left(|f''(a)|^q + |f''(b)|^q\right)^{\frac{1}{q}}$$

Let $a_1 = |f''(a)|^q$, $b_1 = |f''(b)|^q$. Here $0 < \frac{1}{q} < 1$ for $q > 1$, using the fact that $\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}(a_k)^s + \sum_{k=1}^{n}(b_k)^s$ *for* $0 \leq s \leq 1$, $a_1, a_2, ..., a_n \geq 0$,

$b_1, b_2, ..., b_n \geq 0$, *we get*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{81}\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$. If $|f''|^p$ is quasi-convex on $[a,b]$,where $a,b \in I$ with $a < b$, $r \geq 1$ and $p > 1$, then the following inequality hold*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{2.2}$$
$$\leq \frac{(b-a)^2}{4}\left(\sup\left\{\left|f''(a)\right|^p, \left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{r(r+1)^{\frac{q+2}{q}}}\right)\left\{\frac{(r-1)^{q+1}\left[2(q+2)-(r-1)(q+1)+2^{2+q}\right]}{(q+1)(q+2)}\right\}^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 1 and triangle inequality, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(raca+b2)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)^2\left\{\left|\int_0^{\frac{1}{2}}\frac{t}{r}\left(\frac{1}{r+1}-t\right)f''(tb+(1-t)a)\,dt\right|\right.$$
$$\left.+\left|\int_{\frac{1}{2}}^1(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)f''(tb+(1-t)a)\,dt\right|\right\}$$
$$=(b-a)^2\left\{\left|\int_0^{\frac{1}{2}}f''(tb+(1-t)a)\left(\frac{1}{r+1}-t\right)\frac{t}{r}dt\right|\right.$$

$$+\left|\int_{\frac{1}{2}}^{1} f''(tb+(1-t)a)\left(\frac{t}{r}-\frac{1}{r+1}\right)(1-t)dt\right|\Bigg\}.$$

By using weighted version of Hölder's inequality which is described as

$$\left|\int_I f(s)g(s)h(s)ds\right| \leq \left(\int_I |f(s)|^p h(s)ds\right)^{\frac{1}{p}} \left(\int_I |g(s)|^q h(s)ds\right)^{\frac{1}{q}}$$

for $p>1$, $p^{-1}+q^{-1}=1$, h is non-negative on I and provided all the other integrals exist and are finite; we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)^2$$
$$\times\left\{\left(\int_0^{\frac{1}{2}} \left|f''(tb+(1-t)a)\right|^p \frac{t}{r}dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|\frac{1}{r+1}-t\right|^q \frac{t}{r}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1} \left|f''(tb+(1-t)a)\right|^p (1-t)dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left|\frac{t}{r}-\frac{1}{r+1}\right|^q (1-t)dt\right)^{\frac{1}{q}}\right\}.$$

By using the quasi-convexity of $|f''|^p$, we get

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)^2\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^{\frac{1}{2}} \frac{t}{r}dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|\frac{1}{r+1}-t\right|^q \frac{t}{r}dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\frac{t}{r}-\frac{1}{r+1}\right|^{q}(1-t)dt\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{(b-a)^2}{4}\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{r(r+1)^{\frac{q+2}{q}}}\right)\left\{\frac{(r-1)^{q+1}\left[2(q+2)-(r-1)(q+1)+2^{2+q}\right]}{(q+1)(q+2)}\right\}^{\frac{1}{q}}.$$

So the proof is complete. □

Corollary 4. *Under the assumptions of Theorem 3, if we choose $r=1$ in the inequality (2.2), we have the following inequalities:*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)^2}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$
$$\leq\frac{(b-a)^2}{16}\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$

where $\left(\frac{1}{2}\right)^{\frac{1}{q}}\leq 1$, $q\in(1,\infty)$.

Corollary 5. *If we choose $r=2$ in (2.2), we obtain*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{(b-a)^2}{24}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{3}\right)^{\frac{2}{q}}\left(\frac{(q+3)+2^{q+2}}{(q+1)(q+2)}\right)\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}.$$

3. Results for h-Convex Functions

Theorem 4. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$ and $|f''|$ is $h-$convex, where $a,b\in I$ with $a<b$ and $r\geq 1$, then one has the following inequality;*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{3.1}$$

$$\leq (b-a)^2 K(h,r)\left(\left|f''(a)\right|+\left|f''(b)\right|\right)$$

where

$$K(h,r)=\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_0^1 h^2(t)dt.$$

Proof. From Lemma 1 and properties of absolute value, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq (b-a)^2\int_0^1 |k(t)|\left|f''(tb+(1-t)a)\right|dt$$

$$=(b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|dt\right\}.$$

Since $|f''|$ is $h-$convex, we get

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq (b-a)^2\left[\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|h(t)f''(b)+h(1-t)f''(a)\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|h(t)f''(b)+h(1-t)f''(a)\right|dt\right]$$

$$\leq (b-a)^2\left[\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left\{h(t)\left|f''(b)\right|+h(1-t)\left|f''(a)\right|\right\}dt\right.$$

$$+\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left\{h(t)\left|f''(b)\right|+h(1-t)\left|f''(a)\right|\right\}dt\Bigg]$$
$$=(b-a)^2\left[J_1+J_2\right]. \tag{3.2}$$

So we can write

$$J_1=\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left\{h(t)\left|f''(b)\right|+h(1-t)\left|f''(a)\right|\right\}dt$$
$$=\left|f''(a)\right|\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|h(1-t)\,dt+\left|f''(b)\right|\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|h(t)\,dt.$$

Since $cd\le\frac{1}{2}\left(c^2+d^2\right)$ for $c,d\in\mathbb{R}$ we have

$$J_1\le\left|f''(a)\right|\left\{\frac{1}{2}\left[\int_0^{\frac{1}{2}}\left(\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right)^2dt+\int_0^{\frac{1}{2}}h^2(1-t)dt\right]\right\}$$
$$+\left|f''(b)\right|\left\{\frac{1}{2}\left[\int_0^{\frac{1}{2}}\left(\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right)^2dt+\int_0^{\frac{1}{2}}h^2(t)dt\right]\right\}$$
$$=\left|f''(a)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_{\frac{1}{2}}^{1}h^2(t)dt\right]\right\} \tag{3.3}$$
$$+\left|f''(b)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_0^{\frac{1}{2}}h^2(t)dt\right]\right\}.$$

By similar calculation we get

$$J_2\le\left|f''(a)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_0^{\frac{1}{2}}h^2(t)dt\right]\right\} \tag{3.4}$$

$$+\left|f''(b)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_{\frac{1}{2}}^{1}h^2(t)dt\right]\right\}.$$

If we use the inequalities (3.3) and (3.4) in (3.2) we get the desired result. □

Corollary 6. *In the inequality (3.1), if we choose $h(t)=1$ and $r=1$, we get*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right| \le \frac{37.13}{2^5.3.5}(b-a)^2\left(\frac{|f''(a)|+|f''(b)|}{2}\right).$$

Corollary 7. *In Theorem 6, if we choose $h(t)=t$ and $r=2$ we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{11.131}{2^5.3^3.5}(b-a)^2\left(\frac{|f''(a)|+|f''(b)|}{2}\right).$$

Theorem 5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$. If $|f''|^p$ is $h-$convex on $[a,b]$,where $a,b\in I$ with $a<b$, $r\ge 1$ and $p>1$, then the following inequality hold*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{3.5}$$
$$\le \frac{(b-a)^2}{4r}\left\{\frac{2^{q+2}+(r-1)^{q+1}(qr+q+r+3)}{2(r+1)^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}$$
$$\times\left\{\left[\left|f''(b)\right|^p A(h,t)+\left|f''(a)\right|^p B(h,t)\right]^{\frac{1}{p}}+\left[\left|f''(b)\right|^p B(h,t)+\left|f''(a)\right|^p A(h,t)\right]^{\frac{1}{p}}\right\}$$

where $A(h,t)=\frac{1}{24}+\int_0^{\frac{1}{2}}h^2(t)\,dt$ and $B(h,t)=\frac{1}{24}+\int_{\frac{1}{2}}^{1}h^2(t)\,dt$.

Proof. By using Lemma 1, triangle inequality and weighted version of Hölder's inequality we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)^2$$
$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|f''(tb+(1-t)a)\right|^p\frac{t}{r}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q\frac{t}{r}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\left|f''(tb+(1-t)a)\right|^p(1-t)dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\frac{t}{r}-\frac{1}{r+1}\right|^q(1-t)dt\right)^{\frac{1}{q}}\right\}$$

By using $h-$convexity of $|f''|^p$ we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)^2$$
$$\times\left\{\left(\int_0^{\frac{1}{2}}\left[h(t)\left|f''(b)\right|^p+h(1-t)\left|f''(a)\right|^p\right]\frac{t}{r}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q\frac{t}{r}dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{2}}^1\left[h(t)\left|f''(b)\right|^p+h(1-t)\left|f''(a)\right|^p\right](1-t)dt\right)^{\frac{1}{p}}$$
$$\left.\left(\int_{\frac{1}{2}}^1\left|\frac{t}{r}-\frac{1}{r+1}\right|^q(1-t)dt\right)^{\frac{1}{q}}\right\}$$

$$=(b-a)^2$$

$$\times\left\{\left(\frac{|f''(b)|^p}{r}\int_0^{\frac{1}{2}} th(t)\,dt+\frac{|f''(a)|^p}{r}\int_0^{\frac{1}{2}} th(1-t)\,dt\right)^{\frac{1}{p}}\right.$$

$$\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q t\,dt\right)^{\frac{1}{q}}$$

$$+\left(\left|f''(b)\right|^p\int_{\frac{1}{2}}^{1}(1-t)h(t)\,dt+\left|f''(a)\right|^p\int_{\frac{1}{2}}^{1}(1-t)h(1-t)\,dt\right)^{\frac{1}{p}}$$

$$\left.\left(\int_{\frac{1}{2}}^{1}\left|\frac{t}{r}-\frac{1}{r+1}\right|^q(1-t)dt\right)^{\frac{1}{q}}\right\}.$$

Since $cd\leq\frac{1}{2}\left(c^2+d^2\right)$ for $c,d\in\mathbb{R}$ we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq(b-a)^2\left\{\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q t\,dt\right)^{\frac{1}{q}}\right.$$

$$\times\left[\frac{|f''(b)|^p}{2r}\left(\int_0^{\frac{1}{2}} t^2dt+\int_0^{\frac{1}{2}} h^2(t)\,dt\right)\right.$$

$$\left.+\frac{|f''(a)|^p}{2r}\left(\int_0^{\frac{1}{2}} t^2dt+\int_0^{\frac{1}{2}} h^2(1-t)\,dt\right)\right]^{\frac{1}{p}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\frac{t}{r}-\frac{1}{r+1}\right|^{q}(1-t)dt\right)^{\frac{1}{q}}\times\left[\frac{|f''(b)|^{p}}{2}\left(\int_{\frac{1}{2}}^{1}(1-t)^{2}dt+\int_{\frac{1}{2}}^{1}h^{2}(t)\,dt\right)\right.$$

$$\left.\left.+\frac{|f''(a)|^{p}}{2}\left(\int_{\frac{1}{2}}^{1}(1-t)^{2}dt+\int_{\frac{1}{2}}^{1}h^{2}(1-t)\,dt\right)\right]^{\frac{1}{p}}\right\}$$

If we simplify the expression, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_{a}^{b}f(x)\,dx\right|$$

$$\leq\frac{(b-a)^{2}}{4r}\left\{\frac{2^{q+2}+(r-1)^{q+1}(qr+q+r+3)}{2(r+1)^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}$$

$$\times\left\{\left[\left|f''(b)\right|^{p}\left(\frac{1}{24}+\int_{0}^{\frac{1}{2}}h^{2}(t)\,dt\right)+\left|f''(a)\right|^{p}\left(\frac{1}{24}+\int_{\frac{1}{2}}^{1}h^{2}(t)\,dt\right)\right]^{\frac{1}{p}}\right.$$

$$\left.+\left[\left|f''(b)\right|^{p}\left(\frac{1}{24}+\int_{\frac{1}{2}}^{1}h^{2}(t)\,dt\right)+\left|f''(a)\right|^{p}\left(\frac{1}{24}+\int_{0}^{\frac{1}{2}}h^{2}(t)\,dt\right)\right]^{\frac{1}{p}}\right\}$$

then the proof is complete. □

Corollary 8. *If we choose $h(t)=1$, $r=1$ in the inequality (3.5) we get*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_{a}^{b}f(x)\,dx\right|$$

$$\leq\frac{(b-a)^{2}}{4}\left\{\frac{1}{(q+1)(q+2)}\right\}^{\frac{1}{q}}\left\{\frac{13}{12}\left(\left|f''(a)\right|^{p}+\left|f''(b)\right|^{p}\right)\right\}^{\frac{1}{p}}.$$

Corollary 9. *If we choose $h(t)=1$, $r=2$ in (3.5), we get*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{4}\left\{\frac{2^{q+2}+(3q+5)}{2.3^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}\left\{\frac{13}{24}\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)\right\}^{\frac{1}{p}}.$$

Corollary 10. *If we choose $h(t)=t$, $r=2$ in the inequality (3.5), we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{8}\left\{\frac{2^{q+2}+(3q+5)}{2.3^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}$$
$$\times\left\{\left(\frac{|f''(a)|^p}{12}+\frac{|f''(b)|^p}{3}\right)^{\frac{1}{p}}+\left(\frac{|f''(a)|^p}{3}+\frac{|f''(b)|^p}{12}\right)^{\frac{1}{p}}\right\}.$$

Corollary 11. *In addition to the conditions given above, if we choose $p=q=2$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{48\sqrt{2}}\left\{\left(\frac{|f''(a)|^2}{12}+\frac{|f''(b)|^2}{3}\right)^{\frac{1}{2}}+\left(\frac{|f''(a)|^2}{3}+\frac{|f''(b)|^2}{12}\right)^{\frac{1}{2}}\right\}.$$

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Authors' addresses

M. Emin Özdemir
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
E-mail address: emos@atauni.edu.tr

Mustafa Gürbüz
Graduate School of Natural and Applied Sciences, Ağri İbrahim Çeçen University, Ağri, Turkey
E-mail address: mgurbuz@agri.edu.tr

Çetin Yildiz
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
E-mail address: cetin@atauni.edu.tr