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STRONG CONVERGENCE OF AN EXPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. In this paper, we study strong convergence of an iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. We prove some strong convergence theorems for this iterative algorithm. Our results improve and extend the corresponding results of Kettapun et al. [2].

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Keywords: iteration process, convex metric spaces, asymptotically quasi-nonexpansive mappings, common fixed point, strong convergence.

1. INTRODUCTION AND BASIC DEFINITIONS

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $I = \{1, 2, ..., k\}$, the set of first k natural numbers. Denote by $F(T)$ the set of fixed points of T and by $F := \bigcap_{i=1}^{k} F(T_i)$ the set of common fixed points of a finite family $\{T_i : i \in I\}$.

Takahashi [10] introduced initially a notion of convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. Later on, many authors discussed the existence of the fixed point and the convergence of the iterative processes for various mappings in convex metric spaces.

We recall some definitions in a metric space (X, d).

Definition 1. Let (X, d) be a metric space and $T : X \to X$ be a mapping. The map T is said to be:

(1) Nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.
(2) Quasi-nonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$.
(3) Asymptotically nonexpansive [1] if there exists $k_n \in [0, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, T^n y) \leq (1 + k_n)\, d(x, y)\ \forall x, y \in X$.
(4) Asymptotically quasi-nonexpansive if there exists $k_n \in [0, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, p) \leq (1 + k_n)\, d(x, p)\ \forall x \in X, \forall p \in F(T)$.

From above definitions, the following diagram of implications holds and none of arrows is reversible (see, for example, [1, 2, 6, 12]).

$$\begin{array}{lcl} Nonexpansive & \Longrightarrow & Quasi-nonexpansive \\ \Downarrow & & \Downarrow \\ Asymptotically\ nonexpansive & \Longrightarrow & Asymptotically \\ & & quasi-nonexpansive \end{array}$$

Definition 2 ([10]). A convex structure in a metric space (X,d) is a mapping $W : X \times X \times [0,1] \to X$ satisfying, for all $x, y, u \in X$ and all $\lambda \in [0,1]$,

$$d\left(u, W\left(x, y; \lambda\right)\right) \leq \lambda d\left(u, x\right) + (1-\lambda) d\left(u, y\right).$$

A metric space together with a convex structure is called a convex metric space.

A nonempty subset C of X is said to be convex if $W(x, y; \lambda) \in C$ for all $(x, y; \lambda) \in C \times C \times [0,1]$.

Definition 3 ([3]). Let (X,d) be a metric space and q be a fixed element of X. A q-starshaped structure in X is a mapping $W : X \times X \times [0,1] \to X$ satisfying, for all $x, y \in X$ and all $\lambda \in [0,1]$,

$$d\left(q, W\left(x, y; \lambda\right)\right) \leq \lambda d\left(q, x\right) + (1-\lambda) d\left(q, y\right).$$

A metric space together with a q-starshaped structure is called a q-starshaped metric space.

Clearly, a convex metric space is a q-starshaped metric space but the converse is not true in general.

The convex metric space is a more general space and each normed linear space is a special example of a convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed linear space.

Example 1 ([7]). Let $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\right\}$. For each $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ and $\lambda \in [0,1]$, we define a mapping $W : X \times X \times [0,1] \to X$ by

$$W(x, y; \lambda) = \left(\lambda x_1 + (1-\lambda) y_1, \frac{\lambda x_1 x_2 + (1-\lambda) y_1 y_2}{\lambda x_1 + (1-\lambda) y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then (X, d, W) is a convex metric space, but it is not a normed space.

In 2008, Khan et al. [4] introduced an iterative process for a finite family of mappings as follows:

Let C be a convex subset of a Banach space X and let $\{T_i : i \in I\}$ be a family of self-mappings of C. Suppose that $\alpha_{in} \in [0,1]$, for all $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$x_{n+1} = (1-\alpha_{kn}) x_n + \alpha_{kn} T_k^n y_{(k-1)n},$$

$$\begin{aligned} y_{(k-1)n} &= \left(1-\alpha_{(k-1)n}\right)x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= \left(1-\alpha_{(k-2)n}\right)x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n} \\ y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all n. The iterative process (1.1) is the generalized form of the modified Mann (one-step) iterative process by Schu [8], the modified Ishikawa (two-step) iterative process by Tan and Xu [11], and the three-step iterative process by Xu and Noor [13].

In [3], Khan and Ahmed transformed this process into convex metric spaces.

Recently, Kettapun et al. [2] introduced a new iteration process for a finite family of mappings as follows:

$$\begin{aligned} x_1 \in C,\ x_{n+1} &= (1-\alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= \left(1-\alpha_{(k-1)n}\right)y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= \left(1-\alpha_{(k-2)n}\right)y_{(k-3)n} + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1-\alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1-\alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{1.2}$$

where $y_{0n} = x_n$ for all n.

Equivalence of the iteration schemes (1.1) and (1.2) is a problem worth investigating, assuming that the same point $x_1 \in C$ is initiated for various mappings.

Now, we transform iteration process (1.2) to the case of a family of asymptotically quasi-nonexpansive mappings in convex metric spaces as follows:

Definition 4. Let (X,d) be a convex metric space with convex structure W and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings. Suppose that $\alpha_{in} \in [0,1]$, for all $n = 1,2,\ldots$ and $i = 1,2,\ldots,k$. For any given $x_1 \in X$, we define iteration process $\{x_n\}$ as follows.

$$\begin{aligned} x_{n+1} &= W\left(T_k^n y_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}\right), \\ y_{(k-1)n} &= W\left(T_{k-1}^n y_{(k-2)n}, y_{(k-2)n}, \alpha_{(k-1)n}\right), \\ y_{(k-2)n} &= W\left(T_{k-2}^n y_{(k-3)n}, y_{(k-3)n}, \alpha_{(k-2)n}\right), \\ &\vdots \\ y_{2n} &= W\left(T_2^n y_{1n}, y_{1n}, \alpha_{2n}\right), \\ y_{1n} &= W\left(T_1^n y_{0n}, y_{0n}, \alpha_{1n}\right), \end{aligned} \tag{1.3}$$

where $y_{0n} = x_n$ for all n.

Lemma 1 ([5]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty, \ \sum_{n=0}^{\infty} c_n < \infty, \ a_{n+1} = (1 + b_n) a_n + c_n, \ n \geq 0.$$

Then

$i)$ $\lim_{n\to\infty} a_n$ exists,
$ii)$ if $\liminf_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} a_n = 0$.

Remark 1. It is easy to verify that Lemma 1 (ii) holds under the hypothesis $\limsup_{n\to\infty} a_n = 0$ as well. Therefore, the condition (ii) in Lemma 1 can be reformulated as follows:

$ii)'$ if either $\liminf_{n\to\infty} a_n = 0$ or $\limsup_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.
The following proposition was proved by Khan and Ahmed [3] for one family of asymptotically quasi-nonexpansive mappings.

Proposition 1 ([3]). *Let X be a convex metric space and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F := \bigcap_{i=1}^{k} F(T_i)$. Then, there exist a point $p \in F$ and a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that*

$$d\left(T_i^n x, p\right) \leq (1 + r_n)\, d(x, p),$$

for all $x \in K$, for each $i \in I$.

2. Main results

We begin with a technical result.

Lemma 2. *Let (X, d, W) be a convex metric space with convex structure W and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \varnothing$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). Then*

(i) $d(x_{n+1}, p) \leq (1 + r_n)^k\, d(x_n, p)$, for all $p \in F$ and for each $n \in N$,
(ii) there exists a constant $M > 0$ such that, for all $n, m \in N$ and for every $p \in F$, $d(x_{n+1}, p) \leq M d(x_n, p)$.

Proof. (i) For all $p \in F$, we have from Proposition 1 that

$$\begin{aligned} d(y_{1n}, p) = d\left(W\left(T_1^n y_{0n}, y_{0n}, \alpha_{1n}\right), p\right) &\leq \alpha_{1n} d\left(T_1^n y_{0n}, p\right) + (1 - \alpha_{1n})\, d(y_{0n}, p) \\ &\leq \alpha_{1n}(1 + r_n)\, d(x_n, p) + (1 - \alpha_{1n})\, d(x_n, p) \\ &\leq (1 + \alpha_{1n} r_n)\, d(x_n, p) \\ &\leq (1 + r_n)\, d(x_n, p). \end{aligned}$$

Assume that $d\left(y_{jn},p\right) \leq (1+r_n)^j\, d\left(x_n,p\right)$ holds for some $1 \leq j \leq k-2$. Then

$$\begin{aligned} d\left(y_{(j+1)n},p\right) &= d\left(W\left(T_{j+1}^n y_{jn}, y_{jn}, \alpha_{(j+1)n}\right),p\right) \\ &\leq \alpha_{(j+1)n} d\left(T_{j+1}^n y_{jn},p\right) + \left(1-\alpha_{(j+1)n}\right) d\left(y_{jn},p\right) \\ &\leq \alpha_{(j+1)n}\left(1+r_n\right) d\left(y_{jn},p\right) + \left(1-\alpha_{(j+1)n}\right) d\left(y_{jn},p\right) \\ &\leq \left(1+r_n\right) d\left(y_{jn},p\right) \\ &\leq \left(1+r_n\right)\left(1+r_n\right)^j d\left(x_n,p\right) \\ &= \left(1+r_n\right)^{j+1} d\left(x_n,p\right). \end{aligned}$$

Therefore, by mathematical induction, we obtain

$$d\left(y_{in},p\right) \leq \left(1+r_n\right)^i d\left(x_n,p\right), \text{ for } i = 1,2,\ldots,k-1. \tag{2.1}$$

Now, we have from (2.1) that

$$\begin{aligned} d\left(x_{n+1},p\right) &= d\left(W\left(T_k^n y_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}\right),p\right) \\ &\leq \alpha_{kn} d\left(T_k^n y_{(k-1)n},p\right) + \left(1-\alpha_{kn}\right) d\left(y_{(k-1)n},p\right) \\ &\leq \alpha_{kn}\left(1+r_n\right) d\left(y_{(k-1)n},p\right) + \left(1-\alpha_{kn}\right) d\left(y_{(k-1)n},p\right) \\ &\leq \left(1+r_n\right) d\left(y_{(k-1)n},p\right) \\ &\leq \left(1+r_n\right)\left(1+r_n\right)^{k-1} d\left(x_n,p\right) \\ &= \left(1+r_n\right)^k d\left(x_n,p\right). \end{aligned}$$

(ii) If $t \geq 0$, then $1+t \leq e^t$ and so, $(1+t)^k \leq e^{kt}$, $k = 1,2,\ldots$. Thus, from part (i), we get

$$\begin{aligned} d\left(x_{n+m},p\right) &\leq \left(1+r_{n+m-1}\right)^k d\left(x_{n+m-1},p\right) \\ &\leq \exp\left\{k r_{n+m-1}\right\} d\left(x_{n+m-1},p\right) \leq \cdots \leq \exp\left\{k \sum_{i=1}^{n+m-1} r_i\right\} d\left(x_n,p\right) \\ &\leq \exp\left\{k \sum_{i=1}^{\infty} r_i\right\} d\left(x_n,p\right). \end{aligned}$$

Setting $M = \exp\left\{k \sum_{i=1}^{\infty} r_i\right\} < \infty$, completes the proof. □

Lemma 3. *Let (X,d,W) be a convex metric space with convex structure W and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \varnothing$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). If $\lim_{n\to\infty} d\left(x_n,F\right) = 0$ where $d\left(x,F\right) = \inf\{d\left(x,p\right) : p \in F\}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. We use Lemma 2 (ii) to prove that $\{x_n\}$ is a Cauchy sequence. From $\lim_{n\to\infty} d(x_n, F) = 0$, for each $\varepsilon > 0$ there exists $n_1 \in N$ such that

$$d(x_n, F) < \frac{\varepsilon}{M+1} \ \forall n \geq n_1.$$

Thus, there exists $q \in F$ such that

$$d(x_n, q) < \frac{\varepsilon}{M+1} \ \forall n \geq n_1. \tag{2.2}$$

Using Lemma 2 (ii) and (2.2), we obtain

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, q) + d(x_n, q) \leq M d(x_n, q) + d(x_n, q) \\ &= (M+1) d(x_n, q) \\ &< (M+1)\left(\frac{\varepsilon}{M+1}\right) = \varepsilon \end{aligned}$$

for all $n, m \geq n_1$. Therefore $\{x_n\}$ is a Cauchy sequence. □

We now state and prove the main theorem of this section.

Theorem 1. *Let (X, d, W) be a convex metric space with convex structure W and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \varnothing$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). Then if $\lim_{n\to\infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{d(x, p) : p \in F\}$, then $\{x_n\}$ is a Cauchy sequence.*

(i) $\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0$ if converges to a unique point in F.

(ii) $\{x_n\}$ converges to a unique point in F if X is complete and either $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$.

Proof. (i) Let $p \in F$. Since $\{x_n\}$ converges to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d(x_n, p) < \varepsilon \ \forall n \geq n_0.$$

Taking infimum over $p \in F$, we have

$$d(x_n, F) < \varepsilon \ \forall n \geq n_0.$$

This means $\lim_{n\to\infty} d(x_n, F) = 0$ so that

$$\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0.$$

(ii) Suppose that X is complete and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. Then, we have from Lemma 1 (ii) and Remark 1 that $\lim_{n\to\infty} d(x_n, F) = 0$. From the completeness of X and Theorem 3, we get that $\lim_{n\to\infty} x_n$ exists and equals $q \in X$, say. Moreover, since the set F of fixed points of asymtotically quasi-nonexpansive mappings is closed $q \in F$ from $\lim_{n\to\infty} d(x_n, F) = 0$. That is, $q \in F$. Hence $\{x_n\}$ converges to a unique point in F. □

Remark 2. Since a Banach space and each of its convex subsets are convex metric spaces, so Theorem 1 reduces toTheorem 3.2 in [2]. Also, Corollary 3.3 of Kettapun, Kananthai and Suantai [2] is special case of Theorem 1.

3. Applications

Now, we give a couple of applications of Theorem 1.

Theorem 2. *Let (X, d, W) be a complete convex metric space with convex structure W and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \varnothing$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). Assume that the following two conditions hold.*

$$i) \lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \tag{3.1}$$

ii) the sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ implies

$$\lim \inf_{n\to\infty} d(y_n, F) = 0 \text{ or } \lim \sup_{n\to\infty} d(y_n, F) = 0. \tag{3.2}$$

Then $\{x_n\}$ converges to a unique point in F.

Proof. From (3.1) and (3.2), we have that

$$\lim \inf_{n\to\infty} d(x_n, F) = 0 \text{ or } \lim \sup_{n\to\infty} d(x_n, F) = 0.$$

Therefore, we obtain from Theorem 1 (ii) that the sequence $\{x_n\}$ converges to a unique point in F. □

Theorem 3. *Let (X, d, W) be a complete convex metric space with convex structure W and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings satisfying $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ with $F \neq \varnothing$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). If one of the following is true, then the sequence $\{x_n\}$ converges to a unique point in F.*

i) If there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0,\infty)$ such that $d(x_n, T_i x_n) \geq g(d(x_n, F))$ for all $n \geq 1$ (see Senter and Dotson [9])

ii) There exists a function $f : [0,\infty) \to [0,\infty)$ which is right continuous at 0, $f(0) = 0$ and $f(d(x_n, T_i x_n)) \geq d(x_n, F)$ for all $n \geq 1$.

Proof. First suppose that (i) holds. Then

$$\lim_{n\to\infty} g(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, T_i x_n).$$

So, $\lim_{n\to\infty} g(d(x_n, F)) = 0$; and properties of g imply $\lim_{n\to\infty} d(x_n, F) = 0$.

Now all the conditions of Theorem 1 are satisfied, therefore by its conclusion $\{x_n\}$ converges to a point of F.

Next, assume (ii). Then

$$\lim_{n\to\infty} d\left(x_n, F\right) \leq \lim_{n\to\infty} f\left(d\left(x_n, T_i x_n\right)\right) = f\left(\lim_{n\to\infty} d\left(x_n, T_i x_n\right)\right) = f(0) = 0.$$

Again in this case, $\lim_{n\to\infty} d\left(x_n, F\right) = 0$. Thus $\liminf_{n\to\infty} d\left(x_n, F\right) = 0$ or $\limsup_{n\to\infty} d\left(x_n, F\right) = 0$. By Theorem 1, $\{x_n\}$ converges to a point of F. □

Remark 3. In the Banach space setting, Theorem 3 reduces to the improvement of Theorem 4.1 in [2].

Remark 4. We can prove all the results obtained so far in the context of a q-starshaped metric space with suitable changes. We leave the details to the reader.

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