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AN EXISTENCE THEOREM FOR AN ORDINARY DIFFERENTIAL EQUATION IN MENGER PROBABILISTIC METRIC SPACE

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Abstract. In this article an existence theorem for an ordinary differential equation in probabilistic metric space is proved.

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1. INTRODUCTION

It is well known that the theory of probabilistic metric space is a new frontier branch between probability theory and functional analysis and has an important background, which contains the common metric space as a special case. Chang et al. [2] generalized contraction mapping principle in probabilistic metric space and as an application of this generalization, an existence theorem of solution of some kind of ordinary differential equations in probabilistic metric space is proved. Here, we consider an ordinary differential equations in the Menger probabilistic metric space, then we prove the existence of solution for this problem. Due to do this and for the sake of convenience, some definitions and notations are recalled from [1, 3–9].

Definition 1. A mapping $F : R \to R^+$ is called a *distribution function* if it is nondecreasing and left-continuous and it has the following properties:

(1) $\inf_{t\in R} F(t) = 0$,
(2) $\sup_{t\in R} F(t) = 1$.

Let D^+ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left-continuous mapping from R into $[0,1]$ such that $\sup_{t\in R} F(t) = 1$). Also denote by H the distribution function

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Definition 2. A *probabilistic metric space* (briefly, *PM-space*) is an ordered pair (S,F) where S is a nonempty set and $F : S\times S \to D^+\Big(F(p,q)$ is denoted by $F_{p,q}$ for every $(p,q)\in S\times S\Big)$ satisfies the following conditions:

(1) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$ $(p,q\in S)$.
(2) $F_{p,q}(t) = F_{q,p}(t)$ for all $p,q\in S$ and $t\in R$.
(3) If $F_{p,q}(t_1) = 1$ and $F_{q,r}(t_2) = 1$ then $F_{p,r}(t_1+t_2) = 1$ for $p,q,r\in S$ and $t_1,t_2\in R^+$.

If only (2) and (3) hold, the ordered pair (S,F) is known as a *probabilistic semimetric space*.

Definition 3. A mapping $\varDelta : [0,1]\times[0,1]\to[0,1]$ is called a *triangular norm* (abbreviated, *t-norm*) if the following conditions are satisfied:

(i) $\varDelta(a,1) = a$ for every $a\in[0,1]$,
(ii) $\varDelta(a,b) = \varDelta(b,a)$ for every $a,b\in[0,1]$,
(iii) $a\ge b,\ c\ge d \to \varDelta(a,c)\ge\varDelta(b,d)$ $(a,b,c,d\in[0,1])$,
(iv) $\varDelta(a,\varDelta(b,c)) = \varDelta(\varDelta(a,b),c)$ $(a,b,c\in[0,1])$

Definition 4. A *Menger probabilistic metric space* (briefly, *Menger PM-space*) (see [6]) is a triple $(S,F,\varDelta)$, where (S,F) is a probabilistic metric space, $\varDelta$ is a t-norm and the following inequality holds:

$$F_{p,q}(t_1+t_2)\ge\varDelta(F_{p,r}(t_1),F_{r,q}(t_2)), \tag{1.1}$$

for all $p,q,r\in S$ and every $t_1>0,\ t_2>0$.

Definition 5. A triple $(S,F,\varDelta)$ is called a *Menger probabilistic normed space* (briefly, *Menger PN-space*) (see [2]) if S is a real vector space, F is a mapping from S into D (for $x\in S$, the distribution function $F(x)$ is denoted by F_x and $F_x(t)$ is the value of F_x and $t\in R$) and $\varDelta$ is a t-norm satisfying the following conditions:

(i) $E_x(0) = 0$,
(ii) $F_x(t) = H(t)$ for all $t>0$ if and only if $x = 0$,
(iii) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha\in R, \alpha\ne 0$,
(iv) $F_{x+y}(t_1+t_2)\ge\varDelta(F_x(t_1),F_y(t_2))$ for all $x,y\in E$ and $t_1,t_2\in R^+$.

Schweizer, Sklar and Thorp [7] proved that if $(S,F,\varDelta)$ is a Menger PM-space with $\sup_{0<t<1}\varDelta(t,t) = 1$, then $(S,F,\varDelta)$ is a Hausdorff topological space in the topology τ induced by the family of $(\epsilon,\lambda)-$neighborhoods

$$\{U_p(\epsilon,\lambda) : p\in S, \epsilon>0, \lambda>0\},$$

where

$$U_p(\epsilon,\lambda) = \{u\in S : F_{u,p}(\epsilon) > 1-\lambda\}.$$

Definition 6. Let (S, F, Δ) be a Menger PM-space with $\sup_{0<t<1} \Delta(t,t) = 1$.

(1) A sequence $\{u_n\}$ in S is said to be $\tau-$convergent to $u \in S$ (we write $u_n \to u$) if for any given $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u}(\epsilon) > 1-\lambda$ whenever $n \geq N$.

(2) A sequence $\{u_n\}$ in S is called a $\tau-$Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u_m}(\epsilon) > 1-\lambda$, whenever $n, m \geq N$.

(3) A Menger PM-space (S, F, Δ) is said to be τ-complete if each τ-Cauchy sequence in S is τ-convergent to some point in S.

Finally, we state a fixed point theorem form [2]. This theorem prove the existence of a fixed point in a Menger PM-space.

Theorem 1. *Let (S, F, Δ) be a complete Menger PM-space with a t-norm Δ satisfying $\Delta(t,t) \geq t$ for all $t \in [0,1]$. Let $T : S \to S$ be a mapping satisfying the following conditions:*

$$F_{Tx,Ty}(t) \geq F_{x,y}(\frac{t}{k(\alpha,\beta)}),$$

for all $x, y \in S, t \geq 0$ and $\alpha, \beta \in (0, +\infty)$ with $F_{x,y}(\alpha) > 0$ and $F_{x,y}(\beta) < 1$, where $k(\alpha,\beta) : (0,+\infty)^2 \to (0,1)$ is a function. Then T has exactly one fixed point in S.

In the next section, the existence of a solution for an ordinary differential equation in the Menger PN-space is proved.

2. AN EXISTENCE THEOREM

In this section we prove some lemmas to obtain the existence of solution for an ordinary differential equation in the Menger PN-space.

Lemma 1. *If (R, F, Δ) is a Menger PN-space, then*

$$|x| < |y| \Longrightarrow F_x(t) > F_y(t),$$

for all $x, y \in R$ and $t \geq 0$.

Proof. Note that

$$F_x(t) = F_{\frac{x}{y}y}(t) = F_y(\frac{t}{|\frac{x}{y}|}) \geq F_y(t).$$

The last inequality holds, because $F_x(.)$ is a nondecreasing function. □

Definition 7. Let (R, F, Δ) be a Menger PN-space and $(E, \|.\|_E)$ be a normed real vector space, we define a mapping $\tilde{F} : E \to D^+$ by

$$\tilde{F}_x(t) = F_{\|x\|_E}(t)$$

and denote the distribution function by $\tilde{F}_x$ and the value of $\tilde{F}_x$ at $t \in R$ by $\tilde{F}_x(t)$.

Proposition 1. *Let (R, F, Δ) be a Menger PN-space, then $(E, \tilde{F}, \Delta)$ is also a Menger PN-space.*

Proof. Note that

$$\tilde{F}_x(t) = F_{\|x\|_E}(t) \in D^+.$$

Secondly, $\tilde{F}_x(t)$ satisfies the next four conditions of the Definition 5.

(i) $\tilde{F}_x(0) = F_{\|x\|_E}(0) = 0,$

(ii) when $t > 0$, $\tilde{F}_x(t) = 1 \Longleftrightarrow F_{\|x\|_E}(t) = 1 \Longleftrightarrow \|x\|_E = 0 \Longleftrightarrow x = 0,$

(iii) $\tilde{F}_{\alpha x}(t) = F_{\|\alpha x\|_E} = F_{|\alpha|\|x\|_E}(t) = F_{\|x\|_E}(\frac{t}{|\alpha|}) = \tilde{F}_x(\frac{t}{|\alpha|}),$

and finally, by Lemma 1

$$\begin{aligned}(iv) \qquad \tilde{F}_{x+y}(t_1+t_2) &= F_{\|x+y\|_E}(t_1+t_2) \geq F_{\|x\|_E+\|y\|_E}(t_1+t_2)\\ &\geq \Delta(F_{\|x\|_E}(t_1), F_{\|y\|_E}(t_2))\\ &= \Delta(\tilde{F}_x(t_1), \tilde{F}_y(t_2)).\end{aligned}$$

This ends the proof. □

Proposition 2. *Let (R, F, Δ) be a Menger PN-space, then (R, G, Δ) is a PM-space, where*

$$G_{x,y}(t) = F_{x-y}(t)$$

Proposition 3. *Let (R, F, Δ) be a complete Menger PN-space with a continuous t-norm Δ. Then $(C([0,T]), R), \tilde{F}, \Delta)$ is also a complete Menger PN-space.*

Now, we can state the existence theorem for the solution of an ordinary differential equation in the probabilistic metric space as follow:

Theorem 2. *Let (R, F, Δ) be a complete Menger PN-space with a t-norm Δ satisfying $\Delta(t,t) \geq t$ for all $t \in [0,1]$. Let f be Lipschitz mapping from $R \times \overline{U_{x_0}(\epsilon, \lambda)}$ into R. Then the following differential equation*

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0. \end{cases}$$

has a solution in $C([0, \delta_0], R)$, where δ_0 is small enough.

Proof. Note that

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases}$$

is equivalent to the following integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds.$$

Let $C([0, \delta_0], R)$ be the same as above, and

$$Tx(t) = x_0 + \int_0^t f(s, x(s))ds,$$

for all $x(.) \in C([0,\delta_0],\overline{U_{x_0}(\epsilon,\lambda)})$. Note that T is continuous and $T : C([0,\delta_0],\overline{U_{x_0}(\epsilon,\lambda)}) \to C([0,\delta_0],\overline{U_{x_0}(\epsilon,\lambda)})$ is contraction. In order to do this, we have

$$\begin{aligned}\tilde{F}_{Tx-Ty}(t) &= \tilde{F}_{\int_0^t (f(s,x(s))-f(s,y(s)))ds}(t)\\ &= F_{\|\int_0^t f(s,x(s))-f(s,y(s))ds\|_\infty}(t)\\ &\geq F_{\delta_0 k\|x-y\|_\infty}(t) = F_{\|x-y\|_\infty}(\frac{t}{\delta_0 K}) = \tilde{F}_{x(t)-y(t)}(\frac{t}{\delta_0 K})\end{aligned}$$

It means that

$$G_{Tx,Ty}(t) \geq G_{x,y}(\frac{t}{\delta_0 K}).$$

In addition, Proposition 3 shows $(C([0,\delta_0],R),G,\Delta)$ is a complete Menger PM-space and T is contraction, so by Theorem 1, T has exactly one fixed point in $C([0,\delta_0],R)$, i.e. there exists a unique $x(.) \in C([0,\delta_0],R)$ such that

$$Tx(t) = x(t).$$

It means that

$$x(t) = x_0 + \int_0^t f(s,x(s))ds.$$

Thus $x(t)$ is a solution for

$$\begin{cases} \dot{x}(t) = f(t,x(t)),\\ x(0) = x_0, \end{cases}$$

Moreover, this solution is unique. □

Example 1. Consider

$$\begin{cases} \dot{x}(t) = t + x(t),\\ x(0) = x_0, \end{cases} \tag{2.1}$$

By Theorem 2 the ordinary differential equation (2.1) for any $\delta_0 < 1$ has a unique solution.

Example 2. Consider

$$\begin{cases} \dot{x}(t) = t + sin x(t),\\ x(0) = x_0, \end{cases} \tag{2.2}$$

By Theorem 2 the ordinary differential equation (2.2) for any $\delta_0 < 1$ has a unique solution.

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