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SOME SHARP FUNCTION ESTIMATES FOR VECTOR-VALUED MULTILINEAR INTEGRAL OPERATOR

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Abstract. In this paper, we establish some sharp estimates for certain vector-valued multilinear integral operators. The operators include Littlewood-Paley operators, Marcinkiewicz operators and the Bochner-Riesz operator. As an application, we obtain the (L^p, L^q)-norm inequality for the vector-valued multilinear operators.

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Keywords: vector-valued multilinear operator, Littlewood-Paley operator, Marcinkiewicz operator, Bochner-Riesz operator, Sharp estimate, BMO

1. INTRODUCTION

As a part of the development of singular integral operators, their commutators and multilinear operators have been well studied in [2–5, 14–18]. In [8], Hu and Yang obtained a variant sharp estimate for the multilinear singular integral operators. In [18], C. Pérez and R. Trujillo-Gonzalez obtained a sharp weighted estimate for the vector-valued singular integral operators and their commutators. The main purpose of this paper is to prove the sharp estimates for some multilinear operators related to certain integral operators. The integral operators include Littlewood-Paley operator, Marcinkiewicz operator and the Bochner-Riesz operator. As an applications, we obtain the (L^p, L^q)-norm inequalities for the multilinear operators.

2. NOTATIONS AND THEOREMS

Let m_j $(j = 1,\cdots,l)$ be positive integers with $m_1 + \cdots + m_l = m$, $A_j : R^n \to \mathcal{C}(j = 1,\cdots,l)$, $F_t : R^n \times R^n \times [0,+\infty) \to \mathcal{C}$ and $(x,y,t) \mapsto F_t(x,y)$ be some locally integrable functions. Set

$$F_t(f)(x) := \int_{R^n} F_t(x,y) f(y)dy$$

and

$$F_t^A(f)(x) := \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} F_t(x,y) f(y)dy$$

for every bounded and compactly supported function f, where

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

In addition F_t satisfies the following condition: for fixed $\varepsilon > 0$ and $0 \le \delta < n$,

$$||F_t(x,y)|| \le C|x-y|^{-n+\delta}$$

and

$$||F_t(y,x) - F_t(z,x)|| \le C|y-z|^\varepsilon |x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \le |x-z|$. For $1 < r < \infty$ and for the vector-valued multilinear operators $|T^A(f)|_r$ and $|T(f)|_r$ the bounded and compactly supported functions on $R^n \to H$ related to F_t^A are defined by

$$|T^A(f)(x)|_r := \left(\sum_{i=1}^{\infty} (T^A(f_i)(x))^r\right)^{1/r} \text{ and } |T(f)(x)|_r := \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^r\right)^{1/r},$$

where

$$T^A(f_i)(x) := ||F_t^A(f_i)(x)||, \;\; T(f_i)(x) := ||F_t(f_i)(x)||.$$

Here $||\cdot||$ is the norm of the Banach space $H := \{\text{continous and bounded } h : R^n \to \mathscr{C}\}$ with norm $||h||$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Set

$$|f(x)|_r := \left(\sum_{i=1}^{\infty} |f_i(x)|^r\right)^{1/r}.$$

Suppose that $|T|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$.

If $m = 0$, T^A is just the vector-valued multilinear commutator of T and A (see [15]). If $m > 0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–5, 14]). The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear integral operators T^A. As an application, we obtain a (L^p, L^q) -norm inequality for the vector-valued multilinear operators. In Section 4, we give some examples.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function $f : R^n \to \mathscr{C}$, define $f_Q = |Q|^{-1} \int_Q f(x)dx$ and

$$f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

and set $||f||_{BMO} := ||f^{\#}||_{L^\infty}$. We write $f \in BMO(R^n)$ if $||f||_{BMO} < \infty$. It is well-known that (see [7])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

We shall prove the following theorems.

Theorem 1. *If $1 < r < \infty$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j (j = 1,\cdots,l)$, then there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(R^n)$, $1 < s < n/\delta$ and $\tilde{x} \in R^n$,*

$$(|T^A(f)|_r)^{\#}(\tilde{x}) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Theorem 2. *If $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j (j = 1,\cdots,l)$, then $|T^A|_r : L^p(R^n) \to L^q(R^n)$ is bounded for $1 < r < \infty$, $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, that is*

$$|||T^A(f)|_r||_{L^q} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |||f|_r||_{L^p}.$$

3. Proofs of the Theorems

To prove the theorems, we need the following lemmas.

Lemma 1 (see [4]). *Let $A : R^n \to \mathcal{C}$ be a locally integrable function and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha| = m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2 (see [1, 6]). *Suppose that $1 < r < \infty$, $0 \le \delta < n$, $1 \le s < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then*

$$||M_{\delta,s}(|f|_r)||_{L^q} \le C|||f|_r||_{L^p}.$$

Proof of Theorem 1. It suffices to prove for $f=\{f_i\}\in C_0^\infty(R^n)$ and some constant C_0, the following inequality:

$$\frac{1}{|Q|}\int_Q ||T^A(f)(x)|_r - C_0|dx \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}_j(x)=A_j(x)-\sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(A_j;x,y)$ $=R_{m_j}(\tilde{A}_j;x,y)$ and $D^\alpha\tilde{A}_j=D^\alpha A_j-(D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. We split $f=g+h=\{g_i\}+\{h_i\}$ for $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
F_t^A(f_i)(x) &= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)f_i(y)dy\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)h_i(y)dy\\
&+\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)g_i(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)g_i(y)dy\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)g_i(y)dy\\
&+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)g_i(y)dy,
\end{aligned}$$

then, by Minkowski' inequality,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left||T^A(f)(x)|_r-|T^{\tilde{A}}(h)(x_0)|_r\right|dx\\
&\le\frac{1}{|Q|}\int_Q\left|||F_t^A(f)(x)||_r-||F_t^{\tilde{A}}(h)(x_0)||_r\right|dx\\
&\le\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^\infty||F_t^A(f_i)(x)-F_t^{\tilde{A}}(h_i)(x_0)||^r\right)^{1/r}dx
\end{aligned}$$

$$\le\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^\infty\left|\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|F_t^{\tilde{A}}(h_i)(x)-F_t^{\tilde{A}}(h_i)(x_0)\right|\right|^r\right)^{1/r}dx$$

$$:=I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, we get by Lemma 1,

$$R_m(\tilde{A}_j;x,y)\le C|x-y|^m\sum_{|\alpha_j|=m}||D^{\alpha_j}A_j||_{BMO},$$

thus, by the (L^s,L^q)-boundedness of $|T|_r$ with $1<s<n/\delta$ and $1/q=1/s-\delta/n$, we obtain

$$\begin{aligned}I_1&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\frac{1}{|Q|}\int_Q|T(g)(x)|_r dx\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_r^q dx\right)^{1/q}\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1/q}\left(\int_{\tilde{Q}}|f(x)|_r^s dx\right)^{1/s}\end{aligned}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

For I_2, denoting $s=pq$ for $1<p<n/\delta$, $q>1$, $1/q+1/q'=1$ and $1/u=1/p-\delta/n$, we get, by Hölder's inequality,

$$\begin{aligned}
I_2 &\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r dx\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r^u dx\right)^{1/u}\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/u}\left(\int_{R^n}(|D^{\alpha_1}\tilde{A}_1(x)||g(x)|_r)^p dx\right)^{1/p}\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{pq'}dx\right)^{1/pq'}\\
&\times\left(\frac{1}{|Q|^{1-s\delta/n}}\int_{\tilde{Q}}|f(x)|_r^{pq}dx\right)^{1/pq}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

Similarly, for I_4, denoting $s=pq_3$ for $1<p<n/\delta$, $q_1,q_2,q_3>1$, $1/q_1+1/q_2+1/q_3=1$ and $1/u=1/p-\delta/n$, we obtain

$$\begin{aligned}
I_4 &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)(x)|_r dx\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)(x)|_r^u dx\right)^{1/u}
\end{aligned}$$

$$\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/t}\left(\int_{R^n}(|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)||g(x)|_r)^p dx\right)^{1/p}$$

$$\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{pq_1}dx\right)^{1/pq_1}$$

$$\left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{pq_2}dx\right)^{1/pq_2} \times \left(\frac{1}{|Q|^{1-s\delta/n}}\int_{\tilde{Q}} |f(x)|_r^{pq_3}dx\right)^{1/pq_3}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^{\alpha}A_j||_{BMO}\right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

For I_5, we write

$$F_t^{\tilde{A}}(h_i)(x) - F_t^{\tilde{A}}(h_i)(x_0)$$

$$= \int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy$$

$$+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}F_t(x_0,y)h_i(y)dy$$

$$+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}F_t(x_0,y)h_i(y)dy$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}F_t(x,y)\right.$$

$$\left.-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}F_t(x_0,y)\right]\times D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}F_t(x,y)\right.$$

$$\left.-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]\times D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$

$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}F_t(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]$$

$$\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$

$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 1 and the following inequality (see [19])

$$|b_{Q_1} - b_{Q_2}| \leq C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}$,

$$|R_m(\tilde{A}; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|)$$
$$\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.$$

Note that $|x-y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on F_t,

$$\begin{aligned}
||I_5^{(1)}|| &\le C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) \\
&\quad \times \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y)||h_i(y)|dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \\
&\quad \times \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f_i(y)|dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \\
&\quad \times \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k \tilde{Q}|^{1-\delta/n}} \int_{2^k \tilde{Q}} |f_i(y)|dy,
\end{aligned}$$

thus, by Minkowski' inequality,

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} ||I_5^{(1)}||^r \right)^{1/r} &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \\
&\quad \times \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k \tilde{Q}|^{1-\delta/n}} \int_{2^k \tilde{Q}} |f(y)|_r dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,1}(|f|_r)(\tilde{x}).
\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [4]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

thus

$$\left(\sum_{i=1}^{\infty}||I_5^{(2)}||^r\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(y)|_r dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,1}(|f|_r)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(3)}||^r\right)^{1/r}\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,1}(|f|_r)(\tilde{x}).$$

For $I_5^{(4)}$, we get

$$\left(\sum_{i=1}^{\infty}||I_5^{(4)}||^r\right)^{1/r}$$
$$\le C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|\left|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0,y)}{|x_0-y|^m}\right|\right|$$
$$\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy$$
$$+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|$$
$$\times\frac{|(x_0-y)^{\alpha_1}|||F_t(x_0,y)||}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy$$
$$\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{s'}dy\right)^{1/s'}\left(\frac{1}{|2^k\tilde{Q}|^{1-s\delta/n}}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(5)}||^r\right)^{1/r}\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

For $I_5^{(6)}$, taking $q_1,q_2>1$ such that $1/s+1/q_1+1/q_2=1$, then

$$\left(\sum_{i=1}^{\infty}||I_5^{(6)}||^r\right)^{1/r}$$
$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left\|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0,y)}{|x_0-y|^m}\right\|$$
$$\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_r dy$$
$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\left(\frac{1}{|2^k\tilde{Q}|^{1-s\delta/n}}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}$$
$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{q_1}dy\right)^{1/q_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{q_2}dy\right)^{1/q_2}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

Thus

$$I_5\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. We choose $1<s<p$ in Theorem 1 and by using Lemma 2, we get

$$||T^A(f)||_{L^q}\le C||(T^A(f))^{\#}||_{L^q}\le C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)||M_{\delta,s}(|f|_r)||_{L^q}$$

$$\leq C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)|||f|_r||_{L^p}.$$

This finishes the proof. □

4. APPLICATIONS

Now we give some applications of the Theorems:

Application 1. Littlewood-Paley operators.

Fixed $0\leq\delta<n$, $\varepsilon>0$ and $\mu>(3n+2-2\delta)/n$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx=0$,

(2) $|\psi(x)|\leq C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y)-\psi(x)|\leq C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y|<|x|$.

We denote $\Gamma(x)=\{(y,t)\in R_+^{n+1}:|x-y|<t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x):=\left(\int_0^\infty|F_t^A(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x):=\left[\int\int_{\Gamma(x)}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x):=\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x):=\int_{R^n}\frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy,$$

$$F_t^A(f)(x,y):=\int_{R^n}\frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m}f(z)\psi_t(y-z)dz$$

and $\psi_t(x):=t^{-n+\delta}\psi(x/t)$ for $t>0$. Set $F_t(f)(y)=f*\psi_t(y)$. We also define

$$g_\psi(f)(x):=\left(\int_0^\infty|F_t(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x):=\left(\int\int_{\Gamma(x)}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x):=\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see[20]). Let H be the space

$$H := \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}$$

or

$$H := \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(y,t)|^2 dy dt/t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left|\left| \chi_{\Gamma(x)} F_t^A(f)(x,y) \right|\right|, \quad S_\psi(f)(x) = \left|\left| \chi_{\Gamma(x)} F_t(f)(y) \right|\right|$$

and

$$g_\mu^A(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^A(f)(x,y) \right|\right|,$$

$$g_\mu(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t(f)(y) \right|\right|.$$

It is easily to see that g_ψ, S_ψ and g_μ satisfy the conditions of Theorems 1 and 2 (see [9–11]), thus Theorems 1 and 2 hold for g_ψ^A, S_ψ^A and g_μ^A.

Application 2. Marcinkiewicz operators.

Fixed $0 \le \delta < n$, Fix $\lambda > \max(1, 2n/(n+2-2\delta))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) := \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S^A(f)(x) := \left[\int \int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) := \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x) := \int_{|x-y| \le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy$$

and

$$F_t^A(f)(x,y) := \int_{|y-z|\le t} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)dz.$$

Set

$$F_t(f)(x) := \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define

$$\mu_\Omega(f)(x) := \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) := \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) := \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators (see[21]). Let H be the space

$$H := \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$$

or

$$H := \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad \mu_S(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)||$$

and

$$\mu_\lambda^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right|\right|,$$

$$\mu_\lambda(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right|\right|.$$

It is easily to see that μ_Ω, μ_S and μ_λ satisfy the conditions of Theorems 1 and 2 (see [12, 21]), thus Theorems 1 and 2 hold for μ_Ω^A, μ_S^A and μ_λ^A.

Application 3. Bochner-Riesz operator .

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) := \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy,$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) := \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define

$$B_{\delta,*}(f)(x) := \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the maximal Bochner-Riesz operator (see [13]). Let H be the space $H := \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \ \ B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^A$ satisfies the conditions of Theorems 1 and 2 (see [22]), thus Theorems 1 and 2 hold for $B_{\delta,*}^A$.

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