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DYNAMICS OF NONLINEAR DIATOMIC LATTICES

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ABSTRACT. A model of nonlinear diatomic lattice is studied. We suppose small damping, small forcing and weak coupling between the lattices. We show the existence of breathers for undamped and unforced cases. The existence of chaos is shown for damped and forced cases. For lattices with nonsmall parameters, the existence of travelling waves is discussed.

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1. INTRODUCTION

Let us consider a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions [11, 12, 15, 16]. It represents the Bernal-Flower filaments in ice or more complex biological macromolecules in membranes, in which only the degrees of freedom that contribute predominantly to proton mobility have been conserved. In these systems, each proton lies between a pair of 'oxygens.' The proton part of the Hamiltonian is

$$H_p = \sum_n \frac{1}{2}m\dot{u}_n^2 + U(u_n) + \frac{1}{2}k_1(u_{n+1} - u_n)^2 ,$$

where u_n denotes the displacement of the nth proton with respect to the center of the oxygen pair and k_1 is the coupling between neighboring protons. Furthermore,

$$U(u) = \xi_0 \left(1 - u^2/d_0^2\right)^2$$

is the double-well potential with the potential barrier ξ_0, and $2d_0$ is the distance between its two minima. Finally, m is the mass of protons.

Similarly, the oxygen part of the Hamiltonian is

$$H_O = \sum_n \frac{1}{2}M\dot{\varrho}_n^2 + \frac{1}{2}M\Omega_0^2\varrho_n^2 + \frac{1}{2}K_1(\varrho_{n+1} - \varrho_n)^2 ,$$

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where ϱ_n is the displacement between two oxygens, M is the mass of oxygens, Ω_0 is the frequency of the optical mode and K_1 is the harmonic coupling between neighboring oxygens.

The last part in the Hamiltonian of the model arises from the dynamic interaction between two sublattices and it is given by

$$H_{int} = \sum_n \chi \varrho_n (u_n^2 - d_0^2)\,,$$

where χ measures the strength of the coupling. The Hamiltonian of the model is the sum of these three contributions

$$H = H_p + H_O + H_{int}\,.$$

We are also interested in the influence of external field and damping. For the model studied here, since a spatially homogeneous field is not coupled to the optical motion ϱ_n of the oxygens, a force term has to be considered only in the equation of motion of the protons.

Summarizing, we consider in this note the following coupled infinite chain of oscillators

$$\begin{aligned} \ddot{u}_n + \Gamma_1 \dot{u}_n &= \frac{k_1}{m}(u_{n+1} - 2u_n + u_{n-1}) + \frac{4\xi_0}{md_0^2} u_n \left(1 - \frac{u_n^2}{d_0^2}\right) \\ &\qquad - 2\frac{\chi}{m}\varrho_n u_n + \frac{F}{m}\,, \\ \ddot{\varrho}_n + \Gamma_2 \dot{\varrho}_n &= \frac{K_1}{M}(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) - \Omega_0^2 \varrho_n - \frac{\chi}{M}(u_n^2 - d_0^2)\,, \end{aligned} \tag{1.1}$$

where F is the external force on the protons and Γ_1, Γ_2 are the damping coefficients for the proton and oxygen motions.

The contents of the paper is as follows. In Section 2, we study the weakly coupled undamped and unforced chain (1.1), i. e., with $\Gamma_1 = \Gamma_2 = 0$, $F = 0$ and the coupling parameters are small. We show under certain nonresonance conditions the existence of time periodic spatially localized solutions of (1.1), the so-called breathers [5, 9, 14]. Section 3 is devoted to the case when (1.1) is weakly forced and weakly damped with again weak coupling. By using a Melnikov method [10], we show under certain conditions the existence of spatially localized Smale horseshoes in (1.1), i. e., spatially localized chaos in (1.1). The localization of the Smale horseshoes is not exponential as for the breathers. Section 4 deals with (1.1) when the involved parameters are not small. The existence of travelling waves for (1.1) and also for its continuum limit is discussed.

2. Weakly coupled undamped and unforced lattices: the existence of breathers

We assume in this section that $\Gamma_1 = \Gamma_2 = F = 0$, $k_1/m = \varepsilon\mu_1$, $K_1/M = \varepsilon\mu_2$, $-2\chi/m = \varepsilon\mu_3$, $-\chi/M = \varepsilon\mu_4$ for a small parameter ε and constants μ_i, $i = 1, 2, 3, 4$.

Hence (1.1) has the form

$$\begin{aligned} \ddot{u}_n &= \varepsilon\mu_1\left(u_{n+1} - 2u_n + u_{n-1}\right) + a^2 u_n\left(d_0^2 - u_n^2\right) + \varepsilon\mu_3\varrho_n u_n\,, \\ \ddot{\varrho}_n &= \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) - \Omega_0^2\varrho_n + \varepsilon\mu_4\left(u_n^2 - d_0^2\right)\,, \end{aligned} \tag{2.1}$$

where $a^2 = \frac{4\xi_0}{md_0^4}$. Then for $\varepsilon = 0$, we get from (2.1) the uncoupled system

$$\ddot{u}_n = a^2\left(d_0^2 - u_n^2\right)u_n, \quad \ddot{\varrho}_n = -\Omega_0^2\varrho_n\,.$$

The equation

$$\dot{u} = v, \quad \dot{v} = a^2\left(d_0^2 - u^2\right)u \tag{2.2}$$

has a hyperbolic equilibrium $u = v = 0$ and centers $u = \pm d_0$, $v = 0$ [17]. Furthermore, equation (2.2) has two symmetric periodic solutions $(\gamma_\beta(t), \dot{\gamma}_\beta(t))$ and $(-\gamma_\beta(t), -\dot{\gamma}_\beta(t))$ around $(\pm d_0, 0)$ with periods β monotonically increasing from $\frac{\sqrt{2}\pi}{ad_0}$ to $+\infty$. They accumulate on two symmetric homoclinic solutions $(\gamma(t), \dot{\gamma}(t))$ and $(-\gamma(t), -\dot{\gamma}(t))$ for

$$\gamma(t) = \sqrt{2}d_0 \operatorname{sech} ad_0 t\,.$$

We assume that $\gamma_\beta(t)$ and $\gamma(t)$ are even functions. We are interested, in this section, in spatially localized time periodic solutions of (2.1) which are called breathers. We use the approach of [5, 9, 14]. For this reason, we take the exchange $u_n \leftrightarrow u_n + d_0$ in (2.1) to get

$$\begin{aligned} &\ddot{u}_n + a^2 u_n(u_n + d_0)(u_n + 2d_0) \\ &\qquad -\varepsilon\mu_1(u_{n+1} - 2u_n + u_{n-1}) - \varepsilon\mu_3\varrho_n(u_n + d_0) = 0\,, \\ &\ddot{\varrho}_n + \Omega_0^2\varrho_n - \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) - \varepsilon\mu_4 u_n(u_n + 2d_0) = 0\,. \end{aligned} \tag{2.3}$$

Now we fix a constant $\tau > 1$ and consider the Banach spaces

$$X_i = \Big\{\{(u_n(t), \varrho_n(t))\}_{n\in\mathbb{Z}} \in C^i(\mathbb{R}, \mathbb{R})^{2\mathbb{Z}} \mid u_n, \varrho_n \text{ are even, } \beta\text{-periodic, and such that } \sup_n \tau^{|n|}(|u_n(\cdot)|_i + |\varrho_n(\cdot)|_i) < \infty\Big\}\,,$$

where $i = 0, 2$ and $|\cdot|_i$ is the usual maximum norm on $C^i([0, \beta], \mathbb{R})$. The norm on X_i for $x = \{(u_n(t), \varrho_n(t))\}_{n\in\mathbb{Z}}$ is defined as $\|x\| = \sup_n \tau^{|n|}(|u_n(\cdot)|_i + |\varrho_n(\cdot)|_i)$. The left-hand side of (2.3) defines the mapping $F : X_2 \times \mathbb{R} \to X_0$ with $\varepsilon \in \mathbb{R}$. Hence (2.3) has the form

$$F(x, \varepsilon) = 0\,. \tag{2.4}$$

We take $x_0 \in X_2$ with $u_n = 0$ for $n \neq 0$, $u_0 = \gamma_\beta(t) - d_0$ and $\varrho_n = 0$ for any n. Then $F(x_0, 0) = 0$. Clearly $F \in C^\infty(X_2 \times \mathbb{R}, X_0)$. We solve (2.4) by using the implicit function theorem. The linearization $D_xF(x_0, 0) : X_2 \to X_0$ has the form

$$\Big(\ddot u_n + 2a^2 d_0^2 u_n,\ \ddot\varrho_n + \Omega_0^2\varrho_n\Big), \quad n \neq 0\,,$$
$$\Big(\ddot u_0 + a^2(3\gamma_\beta(t)^2 - d_0^2)u_0,\ \ddot\varrho_n + \Omega_0^2\varrho_n\Big)\,.$$

The equation

$$\ddot v + a^2(3\gamma_\beta(t)^2 - d_0^2)v = 0$$

has the solutions $v_1(t) = \dot\gamma_\beta(t)$ and $v_2(t) = D_\beta\gamma_\beta(t)$. Function $v_1(t)$ is odd and β-periodic, while function $v_2(t)$ is even and satisfies the relation $v_2(t+\beta) = v_2(t) - \dot\gamma_\beta(t+\beta)$. Since $\dot\gamma_\beta(t+\beta) \neq 0$, the function $v_2(t)$ is not β-periodic. We note that the existence of an even and β-periodic solution of the equation

$$\ddot v + a^2(3\gamma_\beta(t)^2 - d_0^2)v = h(t) \tag{2.5}$$

for an even and β-periodic continuous function $h(t)$, is equivalent to the existence of a solution of the following boundary value problem

$$\begin{aligned} \ddot v + a^2(3\gamma_\beta(t)^2 - d_0^2)v &= h(t) \\ \dot v(0) = \dot v(\beta/2) &= 0\,. \end{aligned} \tag{2.6}$$

Since the homogeneous problem of (2.6) with $h(t) = 0$ has the only zero solution, we get that (2.6) is uniquely solvable. Consequently, (2.5) has a unique even and β-periodic solution.

Furthermore, we can directly check that the equations

$$\ddot u + 2a^2 d_0^2 u = h(t)\,,$$
$$\ddot\varrho + \Omega_0^2\varrho = h(t)$$

have unique even and β-periodic solutions for any even and β-periodic continuous function $h(t)$ if the following nonresonance conditions hold

$$2\sqrt{2}\frac{\sqrt{\xi_0}}{\sqrt{m}d_0}\beta = \sqrt{2}ad_0\beta \neq 2\pi k, \quad \Omega_0\beta \neq 2\pi k \tag{2.7}$$

for any $k \in \mathbb{N}$. Consequently, we see that $D_xF(x_0, 0) : X_2 \to X_0$ is continuously invertible if (2.7) hold. The implicit function theorem implies the following result.

Theorem 2.1. *Let $\tau > 1$ be given. If the conditions* (2.7) *hold, then the chain* (2.1) *has a β-periodic solution $\{(u_n(t), \varrho_n(t))\}_{n\in\mathbb{Z}}$ for any ε small such that*

$$\sup_{t,n}\Big(|u_n(t) - d_0| + |\varrho_n(t)|\Big)\tau^{|n|} < \infty\,.$$

Moreover, the relation $\varrho_n(t) = O(\varepsilon)$ is true for any n. Furthermore, $u_n(t) = d_0 + O(\varepsilon)$ for $n \neq 0$ and $u_0(t) = \gamma_\beta(t) + O(\varepsilon)$.

Under assumptions of Theorem 2.1, we get more complicated dynamics of (2.1). Namely, if we start from $x_0 \in X_2$ such that $\varrho_n = 0$ for any n, $u_n = 0$ for large n while $u_n = \gamma_\beta(t) - d_0$ for several finite numbers of n. Then we get for any ε small, under conditions (2.7), the existence of multi-site breathers. Furthermore, if we take an infinite number of excitations, i. e., $u_n = \gamma_\beta(t) - d_0$ for infinite numbers of n in the above construction of x_0, then we can repeat the proof of Theorem 2.1 for $\tau = 1$ to get β-periodic solutions of (2.1), which are not spatially localized. But they are near to $(\gamma_\beta(t), 0)$ in infinitely many modes n. Moreover, the same arguments hold when we consider $-d_0$ instead of d_0 in the above considerations, i. e., we take the exchange $u_n \leftrightarrow u_n - d_0$ in (2.1). Furthermore, if (2.7) hold for some β_0, then (2.7) hold also for any β near to β_0 and Theorem 2.1 can be applied uniformly for such β and ε small. In particular, we get under assumptions of Theorem 2.1 in (2.1) for ε small infinitely many 1-parametric families of breathers.

Finally, if we start from x_0 such that either $u_n = -d_0$, $\varrho_n = 0$ or $u_n = \gamma_\beta(t) - d_0$ and $\varrho_n = 0$, then we consider $F(x, \varepsilon) : X_2 \times \mathbb{R} \to X_0$ for $\tau = 1$, and we can repeat the proof of Theorem 2.1 to get the next result.

Theorem 2.2. *If the conditions*

$$\Omega_0\beta \neq 2\pi k \quad \textit{for any} \quad k \in \mathbb{N}$$

hold, then for any $E = \{e_n\}_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ *the chain* (2.1) *has a* β*-periodic, in general not localized spatially, solution* $\{(u_{n,E}(t), \varrho_{n,E}(t))\}_{n\in\mathbb{Z}}$ *for any* ε *small such that* $\varrho_{n,E}(t) = O(\varepsilon)$ *for any* n*, and* $u_{n,E}(t) = O(\varepsilon)$ *for* n *with* $e_n = 1$ *and* $u_{n,E}(t) = \gamma_\beta(t) + O(\varepsilon)$ *for* n *with* $e_n = 0$.

We also note that the above results can be extended to the case when for (2.3) the element x_0 is such that $\varrho_n = 0$ for any n, and either $u_n(t) = \gamma_\beta(t) - d_0$ or $u_n = 0$ or $u_n = -d_0$ or $u_n = -2d_0$. Summarizing we see that the dynamics of (2.1) is rather complicated for $\varepsilon \neq 0$ small.

3. General weakly coupled lattices: the existence of chaos

We assume in this section that $\Gamma_1 = \varepsilon\delta_1$, $\Gamma_2 = \varepsilon\delta_2$, $F = \varepsilon f(t)$, $k_1/m = \varepsilon\mu_1$, $K_1/M = \varepsilon\mu_2$, $-2\chi/m = \varepsilon\mu_3$, $-\chi/M = \varepsilon\mu_4$ for a small parameter ε, constants $\delta_1 \geq 0$, $\delta_2 > 0$, μ_i, $i = 1, 2, 3, 4$, and a C^1-smooth T-periodic function $f(t)$. Hence (1.1) has the form

$$\begin{aligned} \ddot{u}_n + \varepsilon\delta_1\dot{u}_n + a^2u_n(u_n^2 - d_0^2) &= \varepsilon\mu_1(u_{n+1} - 2u_n + u_{n-1}) \\ &\quad + \varepsilon\mu_3\varrho_n u_n + \varepsilon f(t)\,, \\ \ddot{\varrho}_n + \varepsilon\delta_2\dot{\varrho}_n + \Omega_0^2\varrho_n &= \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) \\ &\quad + \varepsilon\mu_4(u_n^2 - d_0^2)\,. \end{aligned} \tag{3.1}$$

We first consider the system

$$\begin{aligned}\ddot u+\varepsilon\delta_1\dot u+a^2u(u^2-d_0^2)&=\varepsilon\mu_3\varrho u+\varepsilon f(t)\,,\\ \ddot\varrho+\varepsilon\delta_2\dot\varrho+\Omega_0^2\varrho&=\varepsilon\mu_4(u^2-d_0^2)\,.\end{aligned}\tag{3.2}$$

We make the change of variable $\varrho\leftrightarrow\varrho-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}$ in (3.2) to get

$$\begin{aligned}\ddot u+\varepsilon\delta_1\dot u+a^2u(u^2-d_0^2)&=\varepsilon\mu_3\left(\varrho-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}\right)u+\varepsilon f(t)\,,\\ \ddot\varrho+\varepsilon\delta_2\dot\varrho+\Omega_0^2\varrho&=\varepsilon\mu_4u^2\,.\end{aligned}$$

To study a small T-periodic solution of the above system, we take its equivalent form

$$\begin{aligned}&\ddot u+\varepsilon\delta_1\dot u+a^2u(u^2-d_0^2)\\ &\quad=\varepsilon\mu_3\left(\frac{\varepsilon\mu_4}{\Omega_\varepsilon}\int_{-\infty}^t \mathrm{e}^{-\varepsilon\delta_2(t-s)/2}\sin\Omega_\varepsilon(t-s)u^2(s)\,ds-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}\right)u+\varepsilon f(t),\end{aligned}\tag{3.3}$$

where $\Omega_\varepsilon=\sqrt{\Omega_0^2-\frac{\varepsilon^2\delta_2^2}{4}}$ and $0<\varepsilon<2\Omega_0/\delta_2$. Now it is not difficult to prove in (3.3) by using the implicit function theorem the existence of a unique small T-periodic solution $u_\varepsilon(t)=O(\varepsilon)$, $\varrho_\varepsilon(t)=O(\varepsilon)$ of (3.2). Then we make in (3.1) the change of variables $u_n\leftrightarrow u_n+u_\varepsilon$, $\varrho_n\leftrightarrow\varrho_n+\varrho_\varepsilon$ to get the chain

$$\begin{aligned}\dot u_n&=v_n,\\ \dot v_n+\varepsilon\delta_1v_n-a^2u_nd_0^2+a^2u_n^3+3a^2u_n^2u_\varepsilon+3a^2u_nu_\varepsilon^2&\\ =\varepsilon\mu_1(u_{n+1}-2u_n+u_{n-1})+\varepsilon\mu_3(\varrho_nu_n+\varrho_nu_\varepsilon+\varrho_\varepsilon u_n)\,,&\\ \dot\varrho_n&=\psi_n,\\ \dot\psi_n+\varepsilon\delta_2\psi_n+\Omega_0^2\varrho_n=\varepsilon\mu_2(\varrho_{n+1}-2\varrho_n+\varrho_{n-1})+\varepsilon\mu_4(u_n^2+2u_\varepsilon u_n)\,.&\end{aligned}\tag{3.4}$$

We consider (3.4) as an ordinary differential equation in the Hilbert space

$$H:=\left\{z=\{(u_n,v_n,\varrho_n,\psi_n)\}_{n\in\mathbb{Z}}\;\Big|\;\sum\nolimits_{n\in\mathbb{Z}}(u_n^2+v_n^2+\varrho_n^2+\psi_n^2)<\infty\right\}$$

with the norm $\|z\|=\sqrt{\sum_{n\in\mathbb{Z}}(u_n^2+v_n^2+\varrho_n^2+\psi_n^2)}$. The inhomogeneous linearization of (3.4) at $z=0$ has the form

$$\begin{aligned}\dot u_n&=v_n+h_{n1}(t),\\ \dot v_n+\varepsilon\delta_1v_n+u_n(3a^2u_\varepsilon^2-a^2d_0^2-\varepsilon\mu_3\varrho_\varepsilon)&\\ -\varepsilon\mu_1(u_{n+1}-2u_n+u_{n-1})-\varepsilon\mu_3\varrho_nu_\varepsilon&=h_{n2}(t)\,,\\ \dot\varrho_n&=\psi_n+g_{n1}(t),\\ \dot\psi_n+\varepsilon\delta_2\psi_n+\Omega_0^2\varrho_n-\varepsilon\mu_2(\varrho_{n+1}-2\varrho_n+\varrho_{n-1})&\\ -2\varepsilon\mu_4u_\varepsilon u_n&=g_{n2}(t)\end{aligned}\tag{3.5}$$

with $w(t) = \{(h_{n1}(t), h_{n2}(t), g_{n1}(t), g_{n2}(t))\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R}, H)$, where $C_b(\mathbb{R}, H)$ is the Banach space of all bounded continuous functions from $\mathbb{R}$ to H with the norm $|w| = \sup_{\mathbb{R}} \|w(t)\|$. We look for a solution $z \in C_b(\mathbb{R}, H)$ of (3.4) for $\varepsilon > 0$ small. For this reason, we consider the Hilbert spaces $H_2 := H_1 \times H_1$ and

$$H_1 := \left\{ \{u_n\}_{n\in\mathbb{Z}} \;\Big|\; \sum\nolimits_{n\in\mathbb{Z}} u_n^2 < \infty \right\}$$

with the corresponding standard norms. We first study the equation

$$\begin{aligned} \dot{\varrho} &= \psi + g_1 \\ \dot{\psi} + \varepsilon\delta_2\psi + A_\varepsilon\varrho &= g_2 \end{aligned} \tag{3.6}$$

on H_2 for $(g_1, g_2) \in C_b(\mathbb{R}, H_2)$. Here,

$$A_\varepsilon\varrho = \{\Omega_0^2\varrho_n - \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1})\}_{n\in\mathbb{Z}}\,.$$

Clearly $A_\varepsilon : H_1 \to H_1$ is symmetric and positive definite for ε small. Then for any small ε, there is a symmetric and positive definite $B_\varepsilon : H_1 \to H_1$ such that

$$B_\varepsilon^2 = A_\varepsilon - \frac{\varepsilon^2\delta_2^2}{4} I\,.$$

We take the operators $\cos B_\varepsilon t$ and $\sin B_\varepsilon t$ from H_1 to H_1. For any $\varrho \in H_1$, we consider the function

$$\phi(t) := |\cos B_\varepsilon t\varrho|^2 + |\sin B_\varepsilon t\varrho|^2\;.$$

Then we have

$$\dot{\phi}(t) = -2\,\langle \cos B_\varepsilon t\varrho, B_\varepsilon \sin B_\varepsilon t\varrho\rangle + 2\,\langle \sin B_\varepsilon t\varrho, B_\varepsilon \cos B_\varepsilon t\varrho\rangle = 0\,.$$

Hence

$$|\cos B_\varepsilon t\varrho|^2 + |\sin B_\varepsilon t\varrho|^2 = \varrho\,,$$

and then $\|\cos B_\varepsilon t\| \le 1$ and $\|\sin B_\varepsilon t\| \le 1$.

Now, the equation

$$\begin{aligned} \dot{\varrho} &= \psi \\ \dot{\psi} + \varepsilon\delta_2\psi + A_\varepsilon\varrho &= 0 \end{aligned} \tag{3.7}$$

has the form $\ddot{\varrho} + \varepsilon\delta_2\dot{\varrho} + A_\varepsilon\varrho = 0$ which has the general solution

$$e^{-\varepsilon\delta_2 t/2}\Big[\cos B_\varepsilon t\varrho_1 + \sin B_\varepsilon t\varrho_2\Big]$$

for $\varrho_{1,2} \in H_1$. Consequently, the fundamental solution of (3.7) has the form

$$V_\varepsilon(t) = e^{-\varepsilon\delta_2 t/2} W_\varepsilon(t)$$

with uniformly bounded $W_\varepsilon(t)$ for $\varepsilon > 0$ small. Thus, the only bounded solution of (3.6) has the form

$$(\varrho(t), \psi(t)) = \int_{-\infty}^{t} e^{-\varepsilon\delta_2(t-s)/2} W_\varepsilon(t-s)(g_1(s), g_2(s)\,ds\,.$$

Hence

$$|(\varrho,\psi)| \le K_1|(g_1,g_2)|/\varepsilon$$

for a constant $K_1 > 0$ independent of $\varepsilon > 0$ small.

Furthermore, it is not difficult to see that the linear system

$$\begin{aligned}\dot u_n &= v_n + h_{n1}(t)\\ \dot v_n + \varepsilon\delta v_n - a^2 d_0^2 u_n &= h_{n2}(t)\end{aligned} \tag{3.8}$$

has a unique solution $\{(u_n(t), v_n(t))\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R}, H_2)$ such that

$$|\{(u_n(t), v_n(t))\}_{n\in\mathbb{Z}}| \le K_2\, |\{(h_{n1}(t), h_{n2}(t))\}_{n\in\mathbb{Z}}|$$

for a constant $K_2 > 0$ independent of $\varepsilon > 0$ small.

Now we return to (3.5). Summarizing the above arguments, we see, by using the Banach contraction mapping principle for $\varepsilon > 0$ small, that (3.5) has for any $w(t) \in C_b(\mathbb{R}, H)$ a unique solution $z \in C_b(\mathbb{R}, H)$ such that $|z| \le K_3|w|/\varepsilon$ for a constant $K_3 > 0$ independent of $\varepsilon > 0$ small. Since the system (3.5) is T-periodic, then we get from the proof of Theorem 2.1 of [4, p. 288] that (3.5) has an exponential dichotomy on $\mathbb{R}$ in the space H for any $\varepsilon > 0$ sufficiently small. Consequently, we get the next result.

Theorem 3.1. *The T-periodic solution $u_n(t) = u_\varepsilon(t)$, $\varrho_n(t) = \varrho_\varepsilon(t)$ $\forall n \in \mathbb{Z}$ of* (3.1) *is hyperbolic in H for any $\varepsilon > 0$ sufficiently small, i. e., the zero equilibrium of* (3.4) *in H is hyperbolic.*

Now we look for more complicated solutions of (3.1). For this reason, we shift in (3.4) the time $t \leftrightarrow t + \alpha$ to get the system

$$\begin{aligned}\dot u_n &= v_n,\\ \dot v_n + \varepsilon\delta_1 v_n - a^2 u_n d_0^2 &+ a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon(t+\alpha)\\ &\quad + 3a^2 u_n u_\varepsilon^2(t+\alpha) = \varepsilon\mu_1(u_{n+1} - 2u_n + u_{n-1})\\ &\quad + \varepsilon\mu_3(\varrho_n u_n + \varrho_n u_\varepsilon(t+\alpha) + \varrho_\varepsilon(t+\alpha)u_n)\,,\\ \dot\varrho_n &= \psi_n,\\ \dot\psi_n + \varepsilon\delta_2\psi_n + \Omega_0^2\varrho_n &= \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) + \varepsilon\mu_4(u_n^2 + 2u_\varepsilon(t+\alpha)u_n)\,.\end{aligned} \tag{3.9}$$

We look for a solution of (3.9) for $\varepsilon > 0$ small such that $u_n \sim 0$, $v_n \sim 0$ for $n \ne 0$ and $u_0 \sim \gamma$, $v_0 \sim \dot\gamma$.

Let $(\varrho_0, \psi_0) = \{(\varrho_n^0, \psi_n^0)\}_{n\in\mathbb{Z}}$ be the unique bounded solution of (3.6) for $g_1 = 0$ and $g_2 = \{g_{n2}\}_{n\in\mathbb{Z}}$ with $g_{n2} = 0$ for $n \ne 0$ and

$$g_{02} = \varepsilon\mu_4\left(\gamma^2 + 2u_\varepsilon(t+\alpha)\gamma\right).$$

Let us put $u_n^0 = v_n^0 = 0$ for $n \ne 0$ and $u_0^0 = \gamma$, $v_0^0 = \dot\gamma$. Now we make in (3.9) the change of variables $u_n \leftrightarrow u_n + u_n^0$, $v_n \leftrightarrow v_n + v_n^0$, $\varrho_n \leftrightarrow \varrho_n + \varrho_n^0$, $\psi_n \leftrightarrow \psi_n + \psi_n^0$ to get, for $n \neq 0$, the system

$$\begin{aligned}
\dot{u}_n &= v_n, \\
\dot{v}_n + \varepsilon\delta_1 v_n - a^2 u_n d_0^2 + a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon(t+\alpha) + 3a^2 u_n u_\varepsilon^2(t+\alpha) \\
&= \varepsilon\mu_1(u_{n+1} + u_{n+1}^0 - 2u_n + u_{n-1} + u_{n-1}^0) \\
&\quad + \varepsilon\mu_3((\varrho_n + \varrho_n^0)u_n + (\varrho_n + \varrho_n^0)u_\varepsilon(t+\alpha) \\
&\quad + \varrho_\varepsilon(t+\alpha)u_n), \\
\dot{\varrho}_n &= \psi_n, \\
\dot{\psi}_n + \varepsilon\delta_2\psi_n + \Omega_0^2\varrho_n &= \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) \\
&\quad + \varepsilon\mu_4(u_n^2 + 2u_\varepsilon(t+\alpha)u_n).
\end{aligned} \tag{3.10}$$

For the mode $n = 0$, we first note that the system

$$\begin{aligned}
\dot{u}_0 &= v_0 \\
\dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 &= h(t)
\end{aligned}$$

for $h(t) \in C_b(\mathbb{R}, \mathbb{R})$ has a solution $(u_0, v_0) \in C_b(\mathbb{R}, \mathbb{R}^2)$ (see [10]) if and only if

$$\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt = 0,$$

and such a solution is unique if $\int_{-\infty}^{\infty} u_0(t)\dot{\gamma}(t)\,dt = 0$. Consequently, for the mode $n = 0$, we get from (3.9) the equations

$$\begin{aligned}
\dot{u}_0 &= v_0, \\
\dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 &= h(t) - \dot{\gamma}(t)\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt \Big/ \int_{-\infty}^{\infty} \dot{\gamma}(t)^2\,dt, \\
\int_{-\infty}^{\infty} u_0(t)\dot{\gamma}(t)\,dt &= 0, \\
\dot{\varrho}_0 &= \psi_0, \\
\dot{\psi}_0 + \varepsilon\delta_2\psi_0 + \Omega_0^2\varrho_0 &= \varepsilon\mu_2(\varrho_1 - 2\varrho_0 + \varrho_{-1}) \\
&\quad + \varepsilon\mu_4(u_0^2 + 2u_0\gamma + 2u_\varepsilon(t+\alpha)u_0),
\end{aligned} \tag{3.11}$$

and

$$\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt = 0 \tag{3.12}$$

for

$$h(t) = -a^2(u_0^3 + 3u_0^2\gamma) - \varepsilon\delta_1\dot\gamma - 3a^2(u_0+\gamma)^2u_\varepsilon(t+\alpha) - \varepsilon\delta_1 v_0 \\ - 3a^2(u_0+\gamma)u_\varepsilon^2(t+\alpha) + \varepsilon\mu_1(u_1 - 2(u_0+\gamma) + u_{-1}) \\ + \varepsilon\mu_3\big((\varrho_0+\varrho_0^0)(u_0+\gamma) + (\varrho_0+\varrho_0^0)u_\varepsilon(t+\alpha) \\ + \varrho_\varepsilon(t+\alpha)(u_0+\gamma)\big)\,. \quad (3.13)$$

Now for $\varepsilon > 0$ small, we can solve (3.10) and (3.11) to get the solution

$$z = \Big\{\Big(u_n(t), v_n(t), \varrho_n(t), \psi_n(t)\Big)\Big\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R}, H)$$

such that $z = O(\varepsilon)$. Then we put this z into (3.13) to get the function $h_{\varepsilon,\alpha} \in C_b(\mathbb{R},\mathbb{R})$. We note $h_{\varepsilon,\alpha}(t) = O(\varepsilon)$ uniformly for $\varepsilon > 0$ small and $\alpha, t \in \mathbb{R}$. Clearly $h_{\varepsilon,\alpha}(t)$ is T-periodic in α. Then from (3.12) we get the bifurcation equation

$$Q(\varepsilon,\alpha) := \frac{1}{\varepsilon}\int_{-\infty}^{\infty} h_{\varepsilon,\alpha}(t)\dot\gamma(t)\,dt = 0\,. \quad (3.14)$$

We need to study the limit of $h_{\varepsilon,\alpha}(t)$ as $t \to 0$. If we put

$$\lim_{\varepsilon\to 0} u_\varepsilon(t)/\varepsilon = w(t), \quad \lim_{\varepsilon\to 0}\varrho_\varepsilon(t)\varepsilon = \zeta(t),$$

then from (3.2) we get

$$\ddot w - a^2d_0^2 w = f(t), \quad \ddot\zeta + \Omega_0^2\zeta = -\mu_4 d_0^2\,.$$

Hence $\zeta = -\mu_4 d_0^2/\Omega_0^2$ and

$$w(t) = -\frac{1}{2ad_0}\int_{-\infty}^{t} \mathrm{e}^{-ad_0(t-s)} f(s)\,ds - \frac{1}{2ad_0}\int_{t}^{\infty} \mathrm{e}^{ad_0(t-s)} f(s)\,ds\,. \quad (3.15)$$

It is clear that $w(t)$ is T-periodic. Furthermore, since $\gamma(t) \to 0$ as $t \to \pm\infty$ exponentially, from the formula over (3.8) of the bounded solution of (3.6) we see that $\lim_{\varepsilon\to 0}(\varrho_0,\psi_0)/\varepsilon = \{(\varrho_{0n},\psi_{0n})\}_{n\in\mathbb{Z}}$ with $\varrho_{0n} = \psi_{0n} = 0$ for $n \neq 0$ and

$$\ddot\varrho_{00} + \Omega_0^2\varrho_{00} = \mu_4\gamma(t)^2\,,$$

i. e.,

$$\varrho_{00}(t) = \frac{\mu_4}{\Omega_0}\int_{-\infty}^{t}\sin\Omega_0(t-s)\gamma(s)^2\,ds\,.$$

Summarizing, from (3.13) we get

$$M(\alpha) := Q(0,\alpha) = \int_{-\infty}^{\infty}\Big[-\delta_1\dot\gamma(t) - 3a^2\gamma(t)^2 w(t+\alpha) \\ - 2\mu_1\gamma(t)\Big]\dot\gamma(t)\,dt = -\frac{4}{3}\delta_1 a d_0^3 + a^2\int_{-\infty}^{\infty}\gamma(t)^3\dot w(t+\alpha)\,dt\,. \quad (3.16)$$

Clearly, $M(\alpha)$ is T-periodic. We note that similarly we can prove that

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\alpha}Q(\varepsilon,\alpha)/\varepsilon = M'(\alpha)$$

uniformly for $\alpha \in \mathbb{R}$. Summarizing, we get the next result.

Theorem 3.2. *Let M be given by* (3.16). *If there is a simple zero α_0 of M, i. e., $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, then* (3.1) *has for any $\varepsilon > 0$ small a bounded solution $z(t)$ with small u_n, ϱ_n for $n \neq 0$ and (u_0, ϱ_0) near $(\gamma(t-\alpha_0), 0)$.*

Now, it is not difficult to prove, as in the finite-dimensional case [10], that

$$\left(z(t) - \left\{(u_\varepsilon(t), \dot u_\varepsilon(t), \varrho_\varepsilon(t), \dot\varrho_\varepsilon(t))\right\}_{n\in\mathbb{Z}}\right) \to 0$$

exponentially fast as $t \to \pm\infty$ in H. Moreover, near $z(t)$, we can construct the Smale horseshoe. Consequently, we get in this case chaos in (3.1) with corresponding infinitely many periodic orbits with arbitrarily large periods. This Smale horseshoe of (3.1) is spatially localized but not exponentially as in breathers.

To be more concrete, we take

$$f(t) = \Upsilon\cos\omega t$$

for $\Upsilon > 0$. Then (3.15) gives

$$w(t) = -\frac{\Upsilon}{\omega^2 + a^2 d_0^2}\cos\omega t\,,$$

and the formula (3.16) has now the form

$$M(\alpha) = -\frac{4}{3}\delta_1 a d_0^3 + \frac{\omega\Upsilon\pi\sqrt{2}}{a}\operatorname{sech}\frac{\omega\pi}{2ad_0}\sin\omega\alpha\,.$$

Consequently, if

$$4\sqrt{2}\delta_1\sqrt{\xi_0}d_0 < 3\sqrt{m}\omega\Upsilon\pi\operatorname{sech}\frac{\omega\pi}{2ad_0}\,, \tag{3.17}$$

then $M(\alpha)$ has a simple zero, so then (3.1) is chaotic for any $\varepsilon > 0$ small. We note that the inequality (3.17) gives sufficient conditions between the magnitude of the forcing Υ and the damping δ_1 in order to get chaos in (3.1) for $\varepsilon > 0$ small. So chaos is generated by the proton part of (3.1). If $\delta_1 = 0$, then (3.1) is always chaotic for $f(t) = \Upsilon\cos\omega t$. Furthermore, if $f(t) = 0$, i. e., there is no forcing, then it is not difficult to prove that (3.1) has no nonconstant periodic solutions in space H.

Finally, we note that similarly we can study the case when more than one modes are excited. We do not carry out such computations here.

4. Concluding remarks

In this note, we have studied weakly coupled diatomoc lattices presented by model (1.1). In the case where (1.1) is unforced and undamped, the existence of infinitely many time-periodic and spatially localized solutions (the so-called breathers) has been shown. For small damping and forcing, we have proved the existence of infinitely many spatially localized Smale horseshoes, i. e., the existence of chaos in (1.1). The localization of the Smale horseshoes is not exponential as is the case for breathers. The proofs of these results are based on the use of the implicit function theorem. This would allow us to establish profiles of these solutions by using analytic-numeric methods as in [7].

For general 'non-small' parameters involved in (1.1), the study of dynamics of (1.1) is rather difficult. The existence of travelling waves could give some answers. There are two possibilities to handle this problem.

The first one is to consider directly in (1.1) travelling wave solutions

$$u_n(t) = \phi(vt - n), \quad \varrho_n(t) = \psi(vt - n)$$

to get the system

$$\begin{aligned} v^2\ddot{\phi}(t) + \Gamma_1 v\dot{\phi}(t) &= \frac{k_1}{m}(\phi(t+1) - 2\phi(t) + \phi(t-1)) + \frac{4\xi_0}{md_0^2}\phi(t)\left(1 - \frac{\phi^2(t)}{d_0^2}\right) \\ &\quad - 2\frac{\chi}{m}\psi(t)\phi(t) + \frac{F}{m}, \\ v^2\ddot{\psi}(t) + \Gamma_2 v\dot{\psi}(t) &= \frac{K_1}{M}(\psi(t+1) - 2\psi(t) + \psi(t-1)) \\ &\quad - \Omega_0^2\psi(t) - \frac{\chi}{M}(\phi(t) - d_0^2), \end{aligned} \tag{4.1}$$

where now F is a constant external force. Equations of types similar to (4.1) are studied in [6].

The second one is to take the continuum limit of (1.1) to obtain the partial differential equation

$$\begin{aligned} u_{tt} + \Gamma_1 u_t &= \frac{k_1}{m}b^2 u_{xx} + \frac{4\xi_0}{md_0^2}u\left(1 - \frac{u^2}{d_0^2}\right) - 2\frac{\chi}{m}\varrho u + \frac{F}{m}, \\ \varrho_{tt} + \Gamma_2\varrho_t &= \frac{K_1}{M}b^2\varrho_{xx} - \Omega_0^2\varrho - \frac{\chi}{M}(u^2 - d_0^2), \end{aligned} \tag{4.2}$$

where again F is a constant external force, $x = bn$ is the continuum space variable and $b > 0$ is the lattice spacing. The travelling wave solutions

$$u(x,t) = \phi(vt - x), \quad v(x,t) = \psi(vt - x)$$

of (4.2) satisfy the system

$$\ddot{\phi}\left(v^2 - \frac{k_1}{m}b^2\right) + \Gamma_1 v\dot{\phi} = \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) - 2\frac{\chi}{m}\psi\phi + \frac{F}{m}\,,$$
$$\ddot{\psi}\left(v^2 - \frac{K_1}{M}b^2\right) + \Gamma_2 v\dot{\psi} = -\Omega_0^2\psi - \frac{\chi}{M}(\phi^2 - d_0^2)\,. \tag{4.3}$$

System (4.3) is numerically studied in [11, 16]. System (4.1) was not yet investigated. But for certain values of parameters in (4.1-4.3), the bifurcation methods as in this paper can be applied to find analytically either periodic, homoclinic or heteroclinic solutions of (4.1-4.3).

For instance, if

$$k_1/m = K_1/M \quad \text{and} \quad v^2 \sim K_1 b^2/M\,,$$

then by using the theory of singularly perturbed ordinary differential equations [3], the dynamics of (4.3) is reduced to the system

$$\Gamma_1 v\dot{\phi} = \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) - 2\frac{\chi}{m}\psi\phi + \frac{F}{m}\,,$$
$$\Gamma_2 v\dot{\psi} = -\Omega_0^2\psi - \frac{\chi}{M}(\phi^2 - d_0^2) \tag{4.4}$$

for

$$v^2 = K_1 b^2/M.$$

System (4.4) admits the Lyapunov function

$$L(\phi,\psi) = \frac{\xi_0}{md_0^2}\left(1 - \frac{\phi^2}{d_0^2}\right)^2 - \frac{\chi}{M}\psi(\phi^2 - d_0^2) - \Omega_0^2\frac{\psi^2}{2} + \frac{F}{m}\phi\,.$$

Hence the limit sets of (4.4) are its equilibria. In general they are hyperbolic. So for

$$v^2 \sim K_1 b^2/M,$$

they persist in (4.3) and consequently, on bounded sets, these hyperbolic equilibria are limit sets of (4.3).

Similarly, we can study the case

$$v^2 \sim K_1 b^2/M \quad \text{when} \quad k_1/m \neq K_1/M\,.$$

This was numerically studied in [11]. The reduced system is now

$$\ddot{\phi}\left(v^2 - \frac{k_1}{m}b^2\right) + \Gamma_1 v\dot{\phi} = \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) - 2\frac{\chi}{m}\psi\phi + \frac{F}{m}\,,$$
$$\Gamma_2 v\dot{\psi} = -\Omega_0^2\psi - \frac{\chi}{M}(\phi^2 - d_0^2) \tag{4.5}$$

for $v^2 = K_1 b^2/M$. According to [3], the dynamics of bounded solutions of (4.3) is now approximated by (4.5). To study bounded solutions of (4.5), we consider its equivalent form given by the inegro-differential equation

$$\ddot{\phi}\left(v^2 - \frac{k_1}{m}b^2\right) + \Gamma_1 v\dot{\phi} = \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) + \frac{2\chi^2}{vMm\Gamma_2}\phi\int_{-\infty}^{t} e^{-\frac{\Omega_0^2(t-s)}{\Gamma_2 v}}(\phi(s)^2 - d_0^2)\,ds + \frac{F}{m}. \quad (4.6)$$

Equation (4.6) can be studied similarly to (3.3) but we do not investigate it any further in this note. We postpone this to another paper.

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