Monotone positive solution of nonlinear third-order two-point boundary value problem

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MONOTONE POSITIVE SOLUTION OF NONLINEAR THIRD-ORDER TWO-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we are concerned with the existence of monotone positive solution to the nonlinear third-order two-point boundary value problem

$$\begin{cases} x'''(t) + f(x(t)) = 0, & t \in (0,1), \\ x(0) = x''(0) = x'(1) = 0. \end{cases}$$

Under suitable assumptions on f and by using a fixed point theorem of cone expansion and compression of functional type due to Avery, Anderson and Krueger, we establish the existence of at least one positive solution of the boundary value problem.

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1. INTRODUCTION

In this paper, we are concerned with the existence of single monotone positive solution to nonlinear third-order two-point boundary value problem (BVP):

$$\begin{cases} x'''(t) + f(x(t)) = 0, & t \in (0,1), \\ x(0) = x''(0) = x'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0, \infty) \to [0, \infty)$ is continuous. Here, by a positive solution u^* of BVP (1.1), we mean a function u^* satisfies (1.1) and $u^*(t) > 0$, for $0 < t < 1$.

Third-order equations arise in a variety of different areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity-driven flows and so on [17]. The existence of positive solutions of third-order equations with two-point or three-point boundary conditions have been studied by several authors using the fixed point index theorem, Leray-Schauder continuation theorem, nonlinear alternative of Leray, upper solution and lower solutions method, and comparison principles and so

on. One may see [3,4,7–11,13–16,18,20–24,26] and the references therein for some related results. In a recent paper [25], by applying a monotone iterative method, Sun studied the boundary value problem

$$
\begin{cases}
x'''(t) + f(t, x(t), x'(t)) = 0, & t \in (0, 1), \\
x(0) = x''(0) = x'(1) = 0,
\end{cases}
$$

where $f : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, and $q : (0, 1) \to [0, \infty)$ is a continuous function such that $0 < \int_0^1 (1-s)q(s)ds < \infty$. The iterative schemes for approximating the solutions are obtained.

One of the most frequently used tools for proving the existence of positive solutions to the boundary value problems is Krasnosel'skii's theorem on cone expansion and compression and its norm-type version due to Guo [19]. Many authors have used this approach or a variation to obtain positive solution of boundary value problem; see [1, 2]. Recently, Anderson and Avery in [5], Avery, Anderson and Krueger in [6] generalized the fixed point theorem of cone expansion and compression of norm type by replacing the norms with two functionals satisfying certain conditions to produce a fixed point theorem of cone expansion and compression of functional type, and they applied the fixed point to verify the existence of one positive solution to a discrete second-order conjugate boundary value problem and the right focal second order boundary value problem. Ehrke [12] applied this fixed point to verify the existence of one positive solution to the left focal second order boundary value problem. According to [5], the flexibility of using functionals instead of norms allows the theorem to be used in a wider variety of situations especially in applications to boundary value problems it allows for improved sufficiency conditions for the existence of a positive solution.

Motivated greatly by the above-mentioned works, in this paper we will try using this fixed point theorem to consider the existence of monotone positive solution to BVP (1.1). This paper is organized as follows. In Section 2, we present some definitions and background results on cones and completely continuous operators. We also state the fixed point theorem of cone expansion and compression of functional type. Some lemmas and the main result will be given in Section 3. In Section 4, an explicit example is given to illustrate our main result.

2. PRELIMINARIES

In this section, we present some definitions and background results on cones and completely continuous operators. We also state the fixed point theorem of cone expansion and compression of functional type.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* of E if it satisfies the following two conditions:

(1) $x \in P$, $\lambda > 0$ implies $\lambda x \in P$;

(2) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \le y$ if and only if $y - x \in P$.

Definition 2. A cone P of a real Banach space E is said to be normal if there exists $\delta > 0$ such that $\|x + y\| \ge \delta$ for all $x, y \in P$ with $\|x\| = \|y\| = 1$.

It is well known that for a cone P in a real Banach space E, P is normal if and only if the norm of the Banach space E is semimonotone. That is, there exists a constant $N > 0$ such that $0 \le x \le y$ implies that $\|x\| \le N\|y\|$. A proof of this result can be found in [19].

Definition 3. Let E be a real Banach space. An operator $T : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into precompact sets. Additionally, the operator T is called increasing on a domain D, provided that $Tx_1 \le Tx_2$ for all $x_1, x_2 \in D$ with $x_1 \le x_2$.

Let α and γ are non-negative continuous functionals on P; then, for positive real numbers r and R, we define the following sets:

$$P(\alpha, r) = \{x \in P : \alpha(x) < r\},$$
$$P(\gamma, R) = \{x \in P : \gamma(x) < R\},$$
$$P(\gamma, \alpha, r, R) = \{x \in P : r < \alpha(x) \text{ and } \gamma(x) < R\}.$$

Now we state the fixed point theorem of cone expansion and compression of functional type due to Anderson, Avery, and Krueger in [6].

Theorem 1. *Let P be a normal cone in a real Banach space E, and let α and γ are nonnegative continuous functionals on P. Assume $P(\gamma, \alpha, r, R)$ as above is a nonempty bounded subset of P,*

$$T : \overline{P(\gamma, \alpha, r, R)} \to P$$

is a completely continuous operator and

$$\overline{P(\alpha, r)} \subseteq P(\gamma, R).$$

If one of the two conditions,

(H1) $\alpha(Tx) \le r$ *for all* $x \in \partial P(\alpha, r)$, $\gamma(Tx) \ge R$ *for all* $x \in \partial P(\gamma, R)$,

$$\inf_{x \in \partial P(\gamma, R)} \|Tx\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \ge 1$, *and* $\mu \in (0, 1]$, *the functionals satisfy the properties*

$$\alpha(\lambda y) \ge \lambda \alpha(y), \ \gamma(\mu z) \le \mu \gamma(z), \text{ and } \alpha(0) = 0,$$

$(H2)$ $\alpha(Tx) \geq r$, for all $x \in \partial P(\alpha, r)$, $\gamma(Tx) \leq R$, for all $x \in \partial P(\gamma, R)$,

$$\inf_{x \in \partial P(\alpha, r)} \|Tx\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$, *and* $\mu \geq 1$, *the functionals satisfy the properties*

$$\alpha(\lambda y) \leq \lambda \alpha(y), \ \gamma(\mu z) \geq \mu \gamma(z), \ and \ \gamma(0) = 0,$$

is satisfied, then T *has at least one fixed point* x^* *such that*

$$r \leq \alpha(x^*) \ and \ \gamma(x^*) \leq R.$$

Moreover, suppose there exist x_l, $x_u \in P$ *such that* $\overline{P(\gamma, \alpha, r, R)} \subseteq [x_l, x_u]$.

(E1) *If there exists an increasing, completely continuous operator* $U : [x_l, x_u] \to P$, *such that* $Tx \leq Ux$ *for all* $x \in [x_l, x_u]$ *and* $U^2 x_u \leq U x_u$, *then*

$$x^* \leq x_u^* \leq U^n x_u,$$

where $n \in N$ *and* $x_u^* = \lim_{n \to \infty} U^n x_u$.

(E2) *If there exists an increasing, completely continuous operator* $L : [x_l, x_u] \to P$, *such that* $Lx \leq Tx$ *for all* $x \in [x_l, x_u]$ *and* $Lx_l \leq L^2 x_l$, *then*

$$L^n x_l \leq x_l^* \leq x^*,$$

where $n \in N$ *and* $x_l^* = \lim_{n \to \infty} L^n x_l$.

3. MAIN RESULT

In this section, we discuss the existence of positive solution of BVP (1.1). We shall consider the Banach space $E = C[0, 1]$ equipped with norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Denote

$$C^+[0, 1] = \{u \in C[0, 1] : u(t) \geq 0, \ t \in [0, 1]\}.$$

In arriving our result, we need the following three preliminary lemmas.

Lemma 1 ([23]). *Let* $x(t) \in C[0, 1]$, *then the BVP:*

$$\begin{cases} u'''(t) + f(x(t)) = 0, & 0 \leq t \leq 1, \\ u(0) = u''(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) f(x(s)) ds,$$

where

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1, \\ t(1 - s) - \dfrac{1}{2}(t - s)^2, & 0 \leq s \leq t \leq 1. \end{cases} \tag{3.2}$$

By simple computation we can get the following results.

Lemma 2. *For any $\eta \in (0,1)$, the Green's fuction $G(t,s)$ defined by (3.2) have the following properties:*

$$\int_0^1 G(t,s)ds = -\frac{1}{6}t^3 + \frac{1}{2}t, \qquad t \in [0,1];$$

$$\int_\eta^1 G(t,s)ds = \begin{cases} \frac{1}{2}(1-\eta)^2 t, & t \in [0,\eta], \\ -\frac{1}{6}(t-\eta)^3 + \frac{1}{2}(1-\eta)^2 t, & t \in [\eta,1]. \end{cases}$$

Particularly,

$$\int_0^1 G(1,s)ds = \frac{1}{3}, \quad \int_\eta^1 G(\eta,s)ds = \frac{1}{2}\eta(1-\eta)^2.$$

Define the cone P by

$$P = \{x \in C^+[0,1] : x(t) \text{ is concave and nondecreasing}\}.$$

Then P is a normal cone of E. Define the integral operator $T : P \to C^+[0,1]$ by

$$(Tx)(t) = \int_0^1 G(t,s)f(x(s))ds. \tag{3.3}$$

By Lemma 1, BVP (1.1) has a positive solution $x^* = x^*(t)$ if and only if x^* is a fixed point of T.

Lemma 3. *For any $x \in P$, $\eta \in (0,1)$, the following properties hold:*
(1) $x(t) \geq t\|x\|$, $t \in [0,1]$;
(2) $x(t) \geq \eta\|x\|$, $t \in [\eta,1]$;
(3) $\|x\| = x(1)$.

Lemma 4. *Suppose that $f \in C([0,\infty),[0,\infty))$, then $T : P \to P$ is completely continuous.*

Proof. For $x \in P$, let $w(t) = (Tx)(t)$. Then $w(0) = w''(0) = w'(1) = 0$ and $w'''(t) = -f(x(t)) \leq 0$. Hence $w''(t)$ is decreasing and since $w''(0) = 0$, we know $w''(t) \leq 0$ for all $t \in [0,1]$, thus $w(t)$ is concave and $w'(t)$ is decreasing. By the boundary conditions $w'(1) = 0$ we have $w'(t) \geq 0$ for all $t \in [0,1]$, thus $w(t)$ is increasing and $w(t) \geq 0$ by the boundary condition $w(0) = 0$. Hence $w(t) \in P$. So $T : P \to P$. The operator T is completely continuous by an application of Ascoli-Arzela Theorem (see [27]). $\square$

Define continuous functionals $\alpha, \gamma : P \to [0,\infty)$ by

$$\alpha(x) := \min_{t \in [\eta,1]} x(t) = x(\eta), \text{ and } \gamma(x) := \max_{t \in [0,1]} x(t) = \|x\| = x(1).$$

It is clear that

$$\alpha(\lambda y) = \lambda\alpha(y), \ \gamma(\mu z) = \mu\gamma(z), \text{ and } \gamma(0) = 0,$$

for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$, and $\mu \geq 1$.

Now we discuss the existence of at least one monotone positive solution for the BVP (1.1). We obtain the following existence result.

Theorem 2. *Suppose that* $f \in C([0, \infty), [0, \infty))$, $\eta \in (0, 1)$ *and there exist two positive real numbers* r, R *with* $r < \eta R$ *such that the following conditions are satisfied:*

(A1) $f(w) \geq \dfrac{2r}{\eta(1 - \eta)^2}$, *for* $w \in [r, R]$,

(A2) $f(w) \leq 3R$, *for* $w \in [0, R]$.

Then BVP (1.1) *has at least one monotone positive solution* $x^* \in P$ *such that*

$$\frac{rt}{\eta} \leq x^*(t) \leq \frac{1}{2}(-t^3 + 3t)R, \quad t \in [0, \eta],$$

and

$$\left(-\frac{(t - \eta)^3}{3\eta(1 - \eta)^2} + \frac{t}{\eta}\right) r \leq x^*(t) \leq \frac{1}{2}(-t^3 + 3t)R, \quad t \in [\eta, 1].$$

Proof. From Lemma 4 we know $T : P \to P$ is completely continuous. Also, the cone P is normal since the norm of the Banach space E is semi-monotone. We verify a series of claims for the result.

Claim 1. $\overline{P(\alpha, r)} \subset P(\gamma, R)$.

Let $x \in \overline{P(\alpha, r)}$, from Lemma 3 we have

$$r \geq \alpha(x) = x(\eta) \geq \eta \|x\| = \eta \gamma(x).$$

Thus $R > \frac{1}{\eta} r \geq \gamma(x)$, i.e. $x \in P(\gamma, R)$, Therefore, $\overline{P(\alpha, r)} \subset P(\gamma, R)$.

Claim 2. For $x \in \partial P(\alpha, r)$ (i.e. $\alpha(x) = x(\eta) = r$) we have $\alpha(Tx) \geq r$.

Let $x \in \partial P(\alpha, r)$, then $x(s) \geq \alpha(x) = r$, $s \in [\eta, 1]$. Thus it follows from (A1) and Lemma 2,

$$\alpha(Tx) = \min_{t \in [\eta, 1]} (Tx)(t) = (Tx)(\eta) = \int_0^1 G(\eta, s) f(x(s)) ds \geq \int_\eta^1 G(\eta, s) f(x(s)) ds$$

$$\geq \frac{2r}{\eta(1 - \eta)^2} \int_\eta^1 G(\eta, s) ds = \frac{2r}{\eta(1 - \eta)^2} \cdot \frac{\eta(1 - \eta)^2}{2} = r.$$

The above arguments also yield

$$\inf_{x \in \partial P(\alpha, r)} \|Tx\| \geq r > 0.$$

Claim 3. For $x \in \partial P(\gamma, R)$ (i.e. $\gamma(x) = x(1) = R$) we have $\gamma(Tx) \leq R$.

Let $x \in \partial P(\gamma, R)$, then $x(s) \leq \gamma(x) = R$, $s \in [0, 1]$. Thus from $(A2)$ and Lemma 2 we have

$$\gamma(Tx) = \max_{t \in [0,1]} (Tx)(t) = (Tx)(1) = \int_0^1 G(1, s) f(x(s)) ds$$

$$\leq 3R \int_0^1 G(1, s) ds = 3R \cdot \frac{1}{3} = R.$$

By Theorem 1 we know that T has a fixed point $x^* \in P$ such that

$$r \leq \alpha(x^*) \quad \text{and} \quad \gamma(x^*) \leq R.$$

Now define the increasing completely continuous operators $L : P \to P$ and $U : P \to P$ by

$$(Lx)(t) := \int_\eta^1 G(t, s) \frac{2r}{\eta(1-\eta)^2} ds$$

and

$$(Ux)(t) := \int_0^1 G(t, s) 3R ds$$

respectively. Then for all $x \in P(\gamma, \alpha, r, R)$, we have

$$Lx \leq Tx \leq Ux.$$

Moreover, if we define x_l, $x_u \in P$ by

$$x_l(t) := \begin{cases} \dfrac{rt}{\eta}, & t \in [0, \eta], \\ r, & t \in [\eta, 1], \end{cases} \quad \text{and} \quad x_u(t) := R, \quad t \in [0, 1],$$

then as a consequence of the concavity of P, we have $P(\gamma, \alpha, r, R) \subset [x_l, x_u]$. Clearly, by the the definition of U and L, for any $t \in [0, 1]$, we have

$$(Ux_u)(t) = \int_0^1 G(t, s) 3R ds \leq 3R \int_0^1 G(1, s) ds = R = x_u(t),$$

and

$$(Lx_l)(t) = \int_\eta^1 G(t,s) \frac{2r}{\eta(1-\eta)^2} ds = \frac{2r}{\eta(1-\eta)^2} \int_\eta^1 G(t,s) ds$$

$$= \frac{2r}{\eta(1-\eta)^2} \begin{cases} \frac{1}{2}(1-\eta)^2 t, & t \in [0,\eta], \\ -\frac{1}{6}(t-\eta)^3 + \frac{1}{2}(1-\eta)^2 t, & t \in [\eta,1] \end{cases}$$

$$\geq \frac{2r}{\eta(1-\eta)^2} \begin{cases} \frac{1}{2}(1-\eta)^2 t, & t \in [0,\eta], \\ \frac{1}{2}\eta(1-\eta)^2, & t \in [\eta,1] \end{cases}$$

$$= \begin{cases} \frac{rt}{\eta}, & t \in [0,\eta], \\ r, & t \in [\eta,1] \end{cases} = x_l(t).$$

This gives

$$(Lx_l)(t) = \frac{2r}{\eta(1-\eta)^2} \int_\eta^1 G(t,s) ds \leq x^*(t) \leq 3R \int_0^1 G(t,s) ds = (Ux_u)(t).$$

It follows from Lemma 2,

$$\frac{rt}{\eta} \leq x^*(t) \leq \frac{1}{2}(-t^3 + 3t)R, \quad t \in [0,\eta],$$

and

$$\left(-\frac{(t-\eta)^3}{3\eta(1-\eta)^2} + \frac{t}{\eta}\right) r \leq x^*(t) \leq \frac{1}{2}(-t^3 + 3t)R, \quad t \in [\eta,1].$$

This completes the proof. $\square$

4. EXAMPLE

In this section, in order to illustrate our result, we consider one simple example.

Example 1. Consider the BVP

$$\begin{cases} x'''(t) + \frac{1}{64}x^2(t) + \frac{1}{8}x(t) + \frac{32\sqrt[3]{x(t)}}{1+\sqrt[3]{x(t)}} = 0, & t \in (0,1), \\ x(0) = x''(0) = x'(1) = 0. \end{cases} \tag{4.1}$$

Set $f(x) = \frac{1}{64}x^2 + \frac{1}{8}x + \frac{32\sqrt[3]{x}}{1+\sqrt[3]{x}}$. Then $f \in C([0,\infty), [0,\infty))$.

Let $\eta = \frac{1}{2}$, $r = 1$, $R = 8$. For $x \in [0,R]$, we have

$$f(x) \leq f(R) = f(8) = \frac{70}{3} < 24 = 3R,$$

and for $x \in [r, R]$, we have

$$f(x) \geq f(r) = f(1) = \frac{1033}{64} > 16 = \frac{2r}{\eta(1-\eta)^2}.$$

Hence, by Theorem 2, BVP 4.1 has at least one monotone positive solution $x^*(t)$ such that

$$2t \leq x^*(t) \leq -4t^3 + 12t, \quad t \in [0, \frac{1}{2}],$$

and

$$-\frac{8}{3}(t - \frac{1}{2})^3 + 2t \leq x^*(t) \leq -4t^3 + 12t, \quad t \in [\frac{1}{2}, 1].$$

Remark 1. In this example, the function f satisfy condition $f(0) = 0$, thus the problem has a trivial solution.

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