The method of lower and upper solutions to the Darboux problem for partial differential inclusions

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THE METHOD OF LOWER AND UPPER SOLUTIONS TO THE DARBOUX PROBLEM FOR PARTIAL DIFFERENTIAL INCLUSIONS

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ABSTRACT. The method of lower and upper solutions combined with a fixed point theorem for condensing multivalued maps due to Martelli is used to investigate the existence of solutions to an hyperbolic differential inclusion.

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1. INTRODUCTION

This note deals with the existence of solutions to the following hyperbolic differential inclusion (Darboux problem):

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x, y, u(x, y)), \quad (x, y) \in J \times J, \tag{1.1}$$

$$u(x, 0) = f(x), \quad u(0, y) = g(y) \tag{1.2}$$

where $J = [0, T]$, $F : J \times J \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact and convex valued multivalued map and $f, g : J \to \mathbb{R}$.

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order partial differential equations. This method has been used only in the context of single-valued partial differential equations. We refer to the books by Carl and Heikkilae [6], Heikkilae and Lakshmikantham [10], Ladde et al. [15], to the papers by Agarwal [1], Agarwal and Sheng [2], Blakley and Pandit [5], Lakshmikantham and Pandit [14], Pandit [18] and the references cited therein.

In this note we shall establish an existence result for the problem (1.1), (1.2). Our approach is based on the existence of upper and lower solutions and on a fixed point theorem for condensing multivalued maps due to Martelli [17]. Other existence results for the Darboux problem for partial differential equations and inclusions by

the aid of other tools can be found in the recent monograph by Kamont [12] and in the papers [4, 7, 8, 13, 19].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. In the sequel, we set $\mathcal{J} = J \times J$.

In what follows, $C(\mathcal{J}, \mathbb{R})$ is the Banach space of continuous functions from $\mathcal{J}$ into $\mathbb{R}$ with the norm $\|z\|_\infty := \sup\{|z(x, y)| : (x, y) \in \mathcal{J}\}$ for each $z \in C(\mathcal{J}, \mathbb{R})$.

The condition

$$u \le \bar{u} \quad \text{if and only if} \quad u(x, y) \le \bar{u}(x, y) \quad \text{for all} \quad (x, y) \in \mathcal{J}$$

defines a partial ordering in $C(\mathcal{J}, \mathbb{R})$. If $\underline{u},\ \overline{u} \in C(\mathcal{J}, \mathbb{R})$ and $\underline{u} \le \overline{u}$, we put

$$[\underline{u}, \overline{u}] = \{u \in C(\mathcal{J}, \mathbb{R}) : \underline{u} \le u \le \overline{u}\}.$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u. s. c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u. s. c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In what follows, $CC(X)$ denotes the set of all nonempty compact and convex subsets of X.

A multivalued map $G : \mathcal{J} \to CC(X)$ is said to be measurable if for each $w \in X$ the function $Y : \mathcal{J} \to \mathbb{R}$ defined by

$$Y(x, y) = d(w, G(x, y)) = \inf\{|w - v| : v \in G(x, y)\}$$

is measurable.

Definition 2.1. A multivalued map $F : \mathcal{J} \times \mathbb{R} \to 2^\mathbb{R}$ is said to be an L^1-Carathéodory map if

(i) $(x, y) \mapsto F(x, y, u)$ is measurable for each $u \in \mathbb{R}$;
(ii) $u \mapsto F(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in \mathcal{J}$;
(iii) For each $k > 0$, there exists $h_k \in L^1(\mathcal{J}, \mathbb{R}_+)$ such that

$$\|F(x, y, u)\| = \sup\{|v| : v \in F(x, y, u)\} \le h_k(x, y)$$

for all $|u| \le k$ and for almost all $(x, y) \in \mathcal{J}$.

An upper semi-continuous map $G : X \to 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [3].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books by Deimling [9] and Hu and Papageorgiou [11].

Definition 2.2. By a solution of (1.1), (1.2) we mean a function $u(\cdot, \cdot) \in C(\mathcal{J}, \mathbb{R})$ such that there exists $v \in L^1(\mathcal{J}, \mathbb{R})$ for which we have

$$u(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds \ \text{ for each } (x, y) \in \mathcal{J}$$

and with $v(t, s) \in F(t, s, u(t, s))$ a. e. on $\mathcal{J}$.

Now, we introduce the concept of lower and upper solutions to (1.1), (1.2). It will be the basic tool in the approach that follows.

Definition 2.3. A function $\underline{u}(\cdot, \cdot) \in C(\mathcal{J}, \mathbb{R})$ is said to be a lower solution of (1.1), (1.2) if there exists $v_1 \in L^1(\mathcal{J}, \mathbb{R})$ such that $v_1(t, s) \in F(t, s, \underline{u}(t, s))$ a. e. on $\mathcal{J}$, and for which we have

$$\underline{u}(x, y) \leq f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t, s) dt ds \ \text{ for each } (x, y) \in \mathcal{J}.$$

A function $\overline{u}(\cdot, \cdot) \in C(\mathcal{J}, \mathbb{R})$ is said to be an upper solution of (1.1), (1.2) if there exists $v_2 \in L^1(\mathcal{J}, \mathbb{R})$ such that $v_2(t, s) \in F(t, s, \overline{u}(t, s))$ a. e. on $\mathcal{J}$, and for which we have

$$\overline{u}(x, y) \geq f(x) + g(y) - f(0) + \int_0^x \int_0^y v_2(t, s) dt ds \ \text{ for each } (x, y) \in \mathcal{J}.$$

For the multivalued map F and for each $u \in C(\mathcal{J}, \mathbb{R})$ we define $S^1_{F,u}$ by

$$S^1_{F,u} = \{v \in L^1(\mathcal{J}, \mathbb{R}) : v(t, s) \in F(t, s, u(t, s)) \text{ for a. e. } (t, s) \in \mathcal{J}\}.$$

Our considerations are based on the following lemmas.

Lemma 2.4 ([16])**.** *Let F be a multivalued map satisfying* (H1) *and let Γ be a linear continuous mapping from $L^1(\mathcal{J}, \mathbb{R})$ to $C(\mathcal{J}, \mathbb{R})$, then the operator*

$$\Gamma \circ S_F : C(\mathcal{J}, \mathbb{R}) \to CC(C(\mathcal{J}, \mathbb{R})), \quad u \longmapsto (\Gamma \circ S_F)(u) := \Gamma(S_{F,u})$$

is a closed graph operator in $C(\mathcal{J}, \mathbb{R}) \times C(\mathcal{J}, \mathbb{R})$.

Lemma 2.5 ([17])**.** *Let X be a Banach space and $N : X \to CC(X)$ be an u. s. c. and condensing map. If the set*

$$\Omega := \{u \in X : \lambda u \in N(u) \ \text{for some} \ \lambda > 1\}$$

is bounded, then N has a fixed point.

We will need the following hypotheses:

(H1) $F : \mathcal{J} \times \mathbb{R} \to CC(\mathbb{R})$ is an L^1- Carathéodory multivalued map.

(H2) there exist $\underline{u}$ and $\overline{u}$ in $C(\mathcal{J}, \mathbb{R})$ as lower and upper solutions, respectively, to the problem (1.1), (1.2) such that $\underline{u} \leq \overline{u}$.

(H3) The functions f, $g : J \to \mathbb{R}$ are continuous with $f(0) = g(0)$.

3. Main result

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Assume that hypotheses* (H1)–(H3) *hold. Then problem* (1.1), (1.2) *has at least one solution on* $\mathcal{J}$.

Proof. Let us transform problem (1.1), (1.2) into a fixed point problem. Consider the following modified problem

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} \in F(x, y, (\tau u)(x, y)), \quad (x, y) \in \mathcal{J} \tag{3.1}$$

$$u(x, 0) = f(x), \quad u(0, y) = g(y) \tag{3.2}$$

where $\tau : C(\mathcal{J}, \mathbb{R}) \to C(\mathcal{J}, \mathbb{R})$ is the truncation operator defined by

$$(\tau u)(x, y) = \begin{cases} \underline{u}(x, y) & \text{if } u(x, y) < \underline{u}(x, y); \\ u(x, y) & \text{if } \underline{u}(x, y) \leq u \leq \overline{u}(x, y); \\ \overline{u}(x, y)) & \text{if } \overline{u}(x, y) < u(x, y). \end{cases}$$

Set

$$C_0(\mathcal{J}, \mathbb{R}) := \{u \in C(\mathcal{J}, \mathbb{R}) : u(x, 0) = f(x), \ u(0, y) = g(y)\}.$$

A solution to (3.1), (3.2) is a fixed point of the operator $N : C_0(\mathcal{J}, \mathbb{R}) \to 2^{C_0(\mathcal{J}, \mathbb{R})}$ defined by

$$N(u) := \left\{ h \in C_0(\mathcal{J}, \mathbb{R}) : h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds, \right.$$
$$\left. v \in \tilde{S}_{F, \tau u}^1 \right\}$$

where

$$\tilde{S}_{F, \tau u}^1 = \{v \in S_{F, \tau u}^1 : v(t, s) \geq v_1(t, s) \text{ a. e. on } A_1 \text{ and } v(t, s) \leq v_2(t, s) \text{ a. e. on } A_2\},$$
$$S_{F, \tau u}^1 = \{v \in L^1(\mathcal{J}, \mathbb{R}) : v(t, s) \in F(t, s, (\tau u)(t, s)) \text{ for a. e. } (t, s) \in \mathcal{J}\},$$
$$A_1 = \{(t, s) \in \mathcal{J} : u(t, s) < \underline{u}(t, s) \leq \overline{u}(t, s)\},$$
$$A_2 = \{(t, s) \in \mathcal{J} : \underline{u}(t, s) \leq \overline{u}(t, s) < u(t, s)\}.$$

Remark 3.2.   (i) For every $u \in C(\mathcal{J}, \mathbb{R})$, the set $S_{F, u}^1$ is nonempty (see Lasota and Opial [16]).

(ii) For every $u \in C(\mathcal{J}, \mathbb{R})$, the set $\tilde{S}^1_{F,\tau u}$ is nonempty. Indeed, by (i) there exists $v \in S^1_{F,u}$. Set $w = v_1 \chi_{A_1} + v_2 \chi_{A_2} + v \chi_{A_3}$, where

$$A_3 = \{(t, s) \in \mathcal{J} : \underline{u}(t, s) \leq u(t, s) \leq \overline{u}(t, s)\}.$$

Then, by decomposability, $w \in \tilde{S}^1_{F,\tau u}$.

We shall show that N is a completely continuous multivalued map, u. s. c. with convex closed values. The proof will be divided into several steps.

Remark 3.3. It is clear that the fixed points of N are solutions to (3.1), (3.2).

STEP 1: *$N(u)$ is convex for each $u \in C(\mathcal{J}, \mathbb{R})$.*

Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v^1, v^2 \in \tilde{S}_{F,\tau u}$ such that for each $(x, y) \in \mathcal{J}$ we have

$$h_i(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v^i(t, s) dt ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for each $(x, y) \in \mathcal{J}$ we have

$$(\alpha h_1 + (1 - \alpha)h_2)(x, y) = f(x) + g(y) - f(0)$$
$$+ \int_0^x \int_0^y [\alpha v^1(t, s) + (1 - \alpha)v^2(t, s)] ds.$$

Since $\tilde{S}_{F,\tau u}$ is convex (because F has convex values), we have $\alpha h_1 + (1 - \alpha)h_2 \in N(u)$.

STEP 2: *N is bounded on bounded sets of $C(\mathcal{J}, \mathbb{R})$.*

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(u), u \in B_r = \{u \in C(\mathcal{J}, \mathbb{R}) : \|u\|_\infty \leq r\}$ one has $\|h\|_\infty \leq c$.

If $h \in N(u)$, then there exists $v \in \tilde{S}_{F,\tau u}$ such that for each $(x, y) \in \mathcal{J}$ we have

$$h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds.$$

By (H1) we have for each $(x, y) \in \mathcal{J}$ that

$$|h(x, y)| \leq |f(x)| + |g(y)| + |f(0)| + \int_0^x \int_0^y h_r(t, s) dt ds.$$

Then

$$\|h\|_\infty \leq \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^T \int_0^T h_r(t, s) dt ds = c.$$

STEP 3: *N sends bounded sets of $C(\mathcal{J}, \mathbb{R})$ into equicontinuous sets.*

Let $(x_1, y_1), (x_2, y_2) \in \mathcal{J}$, $x_1 < x_2$, $y_1 < y_2$ and B_r be a bounded set of $C(\mathcal{J}, \mathbb{R})$. For each $u \in B_r$ and $h \in N(u)$, there exists $v \in \tilde{S}_{F,\tau u}$ such that

$$h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds.$$

Thus we obtain

$$|h(x_2, y_2) - h(x_1, y_1)| \le |f(x_2) - f(x_1)| + |g(y_2) - g(y_1)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} |v(t, s)| dt ds$$

$$\le |f(x_2) - f(x_1)| + |g(y_2) - g(y_1)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} h_r(t, s) dt ds.$$

As $(x_2, y_2) \to (x_1, y_1)$, the right-hand side of the above inequality tends to zero.

Taking into account Step 2, Step 3, and the Arzelà–Ascoli theorem, we conclude that N is completely continuous and therefore a condensing multivalued map.

STEP 4: *N has closed graph.*

Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We shall prove that $h_* \in N(u_*)$. The inclusion $h_n \in N(u_n)$ means that there exists $v_n \in \tilde{S}_{F, \tau u_n}$ such that

$$h_n(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_n(t, s) dt ds, \ (x, y) \in \mathcal{J}.$$

We must prove that there exists $g_* \in \tilde{S}_{F, \tau u_*}$ such that

$$h_*(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_*(t, s) dt ds, \ (x, y) \in \mathcal{J}.$$

Now, we consider the linear continuous operator $\Gamma : L^1(\mathcal{J}, \mathbb{R}) \to C(\mathcal{J}, \mathbb{R})$,

$$v \longmapsto \Gamma(v)(x, y) = \int_0^x \int_0^y v(t, s) dt ds, \ (x, y) \in \mathcal{J}.$$

From Lemma 2.4, it follows that $\Gamma \circ \tilde{S}_F$ is a closed graph operator.

Clearly we have

$$\|(h_n(x, y) - f(x) - g(y) + f(0)) - (h_*(x, y) - f(x) - g(y) + f(0))\|_\infty \to 0$$

as $n \to \infty$. Moreover, from the definition of Γ, we have

$$(h_n(x, y) - f(x) - g(y) + f(0)) \in \Gamma(\tilde{S}_{F, \tau u_n}).$$

Since $u_n \to u_*$, it follows from Lemma 2.4 that, for some $v_* \in \tilde{S}_{F, \tau u_*}$,

$$h_*(x, y) - f(x) - g(y) + f(0) = \int_0^x \int_0^y v_*(t, s) dt ds, \ (x, y) \in \mathcal{J}.$$

STEP 5: *The set*

$$\Omega := \{u \in C(\mathcal{J}, \mathbb{R}) : \lambda u \in N(u) \ for some \ \lambda > 1\}$$

is bounded.

Let $u \in \Omega$. Then $\lambda u \in N(u)$ for some $\lambda > 1$. Thus there exists $v \in \tilde{S}_{F, \tau u}$ such that

$$u(x, y) = \lambda^{-1} f(x) + \lambda^{-1} g(y) - \lambda^{-1} f(0) + \lambda^{-1} \int_0^x \int_0^y v(t, s) dt ds, \ (x, y) \in \mathcal{J}.$$

This implies that for each $(x, y) \in \mathcal{J}$ we have

$$|u(x, y)| \le |f(x)| + |g(y)| + |f(0)| + \int_0^x \int_0^y |v(t, s)| dt ds.$$

The definition of τ implies that there exists $\phi \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that $\|F(x, y, \tau u)\| =$ $\sup\{|v| : v \in F(x, y, \tau u)\} \le \phi(x, y)$ for each $u \in C(\mathcal{J}, \mathbb{R})$. Thus we obtain

$$\|u\|_\infty \le \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^T \int_0^T \phi(t, s) dt ds = K.$$

This shows that Ω is bounded. Set $X := C(\mathcal{J}, \mathbb{R})$. As a consequence of Lemma 2.5 we deduce that N has a fixed point which is a solution of (3.1), (3.2) on $\mathcal{J}$.

STEP 6: *The solution u of* (3.1), (3.2) *satisfies the relations*

$$\underline{u}(x, y) \le u(x, y) \le \overline{u}(x, y) \text{ for all } (x, y) \in \mathcal{J}.$$

Let u be a solution to (3.1), (3.2). We prove that $\underline{u}(x, y) \le u(x, y)$ for all $(x, y) \in \mathcal{J}$.

Let us suppose the contrary. Then there exist $x_1, x_2, y_1, y_2 \in J$, $x_1 < x_2$, $y_1 < y_2$ such that $\underline{u}(x_1, y_1) = u(x_1, y_1)$ and $\underline{u}(x, y) > u(x, y)$ for all $(x, y) \in (x_1, x_2) \times (y_1, y_2)$. In view of the definition of τ there exists $v(\cdot, \cdot) \in F(\cdot, \cdot, \underline{u}(\cdot, \cdot))$ a. e. on $\mathcal{J}$ with $v(x, y) \ge v_1(x, y)$ a. e. on $(x_1, x_2) \times (y_1, y_2)$ such that

$$u(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds, \quad (x, y) \in (x_1, x_2) \times (y_1, y_2).$$

Since $\underline{u}$ is a lower solution to (1.1), (1.2), then

$$\underline{u}(x, y) \le f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t, s) dt ds \quad (x, y) \in (x_1, x_2) \times (y_1, y_2).$$

It follows from the facts $u(x_1, y_1) = \underline{u}(x_1, y_1)$, $v(x, y) \ge v_1(x, y)$ that

$$\underline{u}(x, y) \le u(x, y) \text{ for all } (x, y) \in (x_1, x_2) \times (y_1, y_2),$$

which is a contradiction since $u(x, y) < \underline{u}(x, y)$ for all $(x, y) \in (x_1, x_2) \times (y_1, y_2)$. Consequently, $\underline{u}(x, y) \le u(x, y)$ for all $(x, y) \in \mathcal{J}$.

Analogously, we can prove that $u(x, y) \le \overline{u}(x, y)$ for all $(x, y) \in \mathcal{J}$. This shows that problem (3.1), (3.2) has a solution in the interval $[\underline{u}, \overline{u}]$. Since $\tau(u) = u$ for all $u \in [\underline{u}, \overline{u}]$, we see that u is a solution to (1.1), (1.2). $\qquad \square$

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