Common fixed point theorems for strict occasionally weakly compatible mappings in compact metric spaces

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COMMON FIXED POINT THEOREMS FOR STRICT OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN COMPACT METRIC SPACES

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Abstract. We prove a common fixed point theorem for four two pairs of hybrid mappings in compact metric space satisfying an implicit relations using the concept of strict occasionally weak compatibility which generalize theorems of [1, 4, 7, 28]. As an application we obtain a general fixed point theorem for hybrid pairs satisfying a contractive condition of integral type, which is a new result in compact metric spaces.

1. INTRODUCTION

Let (X, d) be a metric space. Denote by $B(X)$ the set of all nonempty sets of X. As in [10, 11] we define the functions $D(A, B)$ and $\delta(A, B)$ by:

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

for $A, B \in B(X)$.

If A consists of a single point "a" we write $\delta(A, B) = \delta(a, B)$.

If B consists also of a single point "b" we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of δ that

$$\delta(A, B) = \delta(B, A) \geq 0, \forall A, B \in B(X),$$

$$\delta(A, B) = 0 \text{ implies } A = B = \{a\}.$$

Definition 1 ([10, 11])**.** A sequence $\{A_n\}$ of nonempty sets of (X, d) is said to be convergent to a set A of X if

(i) each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for all $n \in \mathbb{N}$,

(ii) for any arbitrary $\varepsilon > 0$, there exists an integer $m > 0$ such that $A_n \subset A_\varepsilon$ for $n > m$, where A_ε denote the set of all points $x \in X$ for which there exists a point $a \in X$, depending on x, such that $d(x, a) < \varepsilon$.

A is said to be the limit of the sequence $\{A_n\}$.

Lemma 1 ([10])**.** *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ convergent to A and B, respectively, then $\delta(A_n, B_n) \to \delta(A, B)$.*

Lemma 2 ([10])**.** *Let $\{A_n\}$ be a sequence in $B(X)$ and $y \in X$ such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Definition 2. A set valued mapping $F : X \to B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\} \in B(X)$ converges to $\{Fx\}$, whenever $\{x_n\}$ is a sequence in X converging to x in X.

F is said to be continuous at X if it is continuous at every point in X.

Let A and S be self mappings of a metric space (X, d). Jungck [12] defined A and S to be compatible if $\lim_{n \to \infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Ax = Sx$. We denote by $C(A, S)$ the set of all coincidence points of A and S.

In [23], Pant defined A and S to be pointwise R - weakly commuting if for all $x \in X$, there exists $R > 0$ such that $d(SAx, ASx) \le Rd(Ax, Sx)$. It has been proved in [24] that pointwise R - weakly commuting is equivalent to commutativity at coincidence points.

Definition 3 ([17])**.** A and S is said to be weakly compatible if $SAu = ASu$ for $u \in C(A, S)$.

Definition 4 ([2])**.** A and S is said to be occasionally weakly compatible mappings (briefly owc) if $ASu = SAu$ for some $u \in C(A, S)$.

Remark 1. If A and S are weakly compatible and $C(A, S) \ne 0$ then A and S are owc, but the converse is not true (Example, [2]).

Some fixed point theorems for occasionally weakly compatible mappings are proved in [2, 6–8, 16, 22, 30–32] and in other papers.

Definition 5. Let $f : (X, d) \to (X, d)$ and $F : (X, d) \to B(X)$ be. Then:
1) a point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$. We denote by $C(f, F)$ the set of all coincidence points of f and F.
2) a point $x \in X$ is said to be a strict coincidence point of f and F if $\{fx\} = Fx$. We denote by $SC(f, F)$ the set of all strict coincidence points of f and F.
3) a point $x \in X$ is said to be a fixed point of F if $x \in Fx$.
4) a point $x \in X$ is said to be a strict fixed point of F if $\{x\} = Fx$.

Definition 6 ([14])**.** The mappings $f : X \to X$ and $F : X \to B(X)$ is said to be δ - compatible if $\lim_{n \to \infty} \delta(Ffx_n, fFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $Fx_n \in B(X)$, $fx_n \to t$, $Fx_n \to \{t\}$ for some $t \in X$.

Definition 7 ([15])**.** The hybrid pair $f : X \to X$ and $F : X \to B(X)$ is weakly compatible if for each $x \in SC(f, F)$, $Ffx = fFx$.

Remark 2. If the pair (f, F) is δ - compatible, then it is weakly compatible but the converse is not true [15].

Definition 8. The hybrid pair $f : X \to X$ and $F : X \to B(X)$ is strict occasionally weakly compatible (briefly sowc) if there exists $x \in SC(f, F)$ such that $Ffx = fFx$.

Remark 3. If $C(f, F) \neq \phi$ and the pair (f, F) is weakly compatible then the pair (f, F) is owc.

There exists sowc pairs which are not weakly compatible.

Example 1 ([6])**.** Let $X = [0, 2]$ with usual metric. Define $f : X \to X$ and $F : X \to B(X)$ by

$$fx = \begin{cases} x, x = 0 \\ 2 - x, x \neq 0 \end{cases}$$

and

$$Fx = \begin{cases} [0, x], x \leq 1 \\ [0, 2x], x > 1 \end{cases}$$

Clearly, $C(f, F) = \{0, 1\}$, $SC(f, F) = \{0\}$, $Ff0 = fF0 = \{0\}$ and $Ffx \neq fFx$ for all $x \in (0, 2]$. Hence, the pair (f, F) is sowc, but it is not weakly compatible.

Remark 4. It is obviously $\{f0\} = F0 = \{0\}$ and $F1 = [0, 1]$. Therefore 0 and 1 are fixed points for f and F and only 0 is a strict point of f and F.

2. PRELIMINARIES

In [9], Branciari established the following result

Theorem 1. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that for all $x, y \in X$*

$$\int_0^{d(fx, fy)} h(t)dt \leq c \int_0^{d(x, y)} h(t)dt,$$

where $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with a finite integral) on each compact subset of $[0, \infty)$ such that for $\varepsilon > 0$, $\int_0^\varepsilon h(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n \to \infty} f^n x = z$.

Recently, Kumar et al. [20] extended Theorem 1 for two compatible mappings.

Theorem 2. *Let $f, g : (X, d) \to (X, d)$ compatible mappings satisfying the following conditions:*

  1) *g is continuous,*
  2) *$f(X) \subset g(X)$ and*

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(gx,gy)} h(t)dt,$$

for all $x, y \in X$, $c \in (0, 1)$, where h is as in Theorem 1.
Then f and g have a unique common fixed point.

Definition 9. Let X be a nonempty set. A symmetric on X is a nonnegative real valued function D on $X \times X$ such that

  (i) $D(x, y) = 0$ if and only if $x = y$,
  (ii) $D(x, y) = D(y, x)$ for any $x, y \in X$.

Some fixed point theorems in metric and symmetric spaces for compatible, weakly compatible and occasionally weakly compatible mappings satisfying a contractive condition of integral type have been established in [3, 12, 19, 21, 29, 35] and in other papers.

Let (X, d) be a metric space and $D(x, y) = \int_0^{d(x,y)} h(t)dt$, where $h(t)$ is as in Theorem 1. It is proved in [21] and [29] that $D(x, y)$ is a symmetric on X. It has also been proved in [21] and [29] that the study of fixed points for mappings satisfying a contractive condition of integral type is reduced to the study of fixed points in symmetric spaces.

The method is not applicable for hybrid pairs.

Definition 10. An altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies:
  $(\psi_1) : \psi$ is increasing and continuous,
  $(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

In [18] a fixed point result involving altering distances have been obtained. Fixed point problem involving altering distances have been studied in [28, 37, 38] and in other papers.

Definition 11. A weakly altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies:
  $(\psi_1) : \psi$ is increasing,
  $(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

Lemma 3. *The function $\psi(t) = \int_0^t h(x)dx$, where $h(x)$ is as in Theorem 1 is a weakly altering distance.*

Proof. The proof follows from Lemma 2.5 [30].   □

Several classical fixed point theorems and common fixed point theorems have recently unified by considering a general condition expressed by an implicit relation [25, 26] and other papers.

Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, convex metric spaces, reflexive metric spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single valued functions, hybrid pairs of functions, set - valued functions.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces and intuitionistic metric spaces.

In [30] a general fixed point theorem for compatible mappings satisfying an implicit relation has been proved.

In [13] the results from [30] have been improved relaxing compatibility to weak compatibility.

In [27] a general fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation is proved.

In [28] a common fixed point theorem for four weakly compatible mappings in compact metric spaces involving an altering distance was proved, which extends the main results of [4] and [37].

Theorem 3 ([28]). *Let f, g, S and T be self mappings of a compact metric space (X, d) such that*

    a) *$f(X) \subset T(X)$ and $g(X) \subset S(X)$,*
    b) *the pairs (f, T) is compatible and the pair (g, S) is weakly compatible,*
    c) *f and S are continuous,*
    d) *$\psi(d(fx, gy)) \leq a\psi(d(Sx, Ty)) + b[\psi(d(fx, Sx) + \psi(d(gy, Ty)] + +c[\psi(d(Sx, gy) \cdot \psi(d(fx, Ty)]^{1/2}$ for all $x, y \in X$, $a, b, c \geq 0$, $a + 2b < 1$, $a + c < 1$, and ψ is an altering distance.*

Then f, g, S and T have a unique common fixed point in X.

Recently, in [5] the authors have proved a new fixed point theorem for mappings satisfying a new type of implicit relation.

The results from [5] are extended in [30] for owc mappings involving altering distances.

In [1] the following theorem is proved.

Theorem 4. *Let I, J be two single valued functions from a compact metric space (X, d) into itself and $F, G : X \to B(X)$ two set-valued functions with $\cup G(X) \subset I(X)$ and $\cup F(X) \subset J(X)$ such that*

$$\psi(\delta(Fx, Gy)) < \max\{\psi(d(Ix, Jy)), \psi(\delta(Ix, Fx)), \psi(\delta(Jy, Gy)\},$$
$$\min\{\psi(D(Ix, Gy)), \psi(D(Jy, Fx))\}$$
$$-\omega(\max\{\psi(d(Ix, Jy)), \psi(\delta(Ix, Fx)), \psi(\delta(Jy, Gy)\},$$

$$\min\{\psi(D(Ix, Gy)), \psi(D(Jy, Fx))\}\}$$

for all $x, y \in X$, where the right hand side of inequality is positive, ψ is an altering distance and $\omega : [0, \infty) \to [0, \infty)$ is a continuous function satisfying $0 < \omega(r) < r$ for $r > 0$.

If the pairs (I, F) and (J, G) are weakly compatible and the functions F, I are continuous, then there exists a unique point $p \in X$ such that $\{p\} = \{Ip\} = Fp = \{Jp\} = Gp$.

Remark 5. In the proof of this theorem is used the fact that the function $r - \omega(r)$ is a non-decreasing function.

Some fixed point theorems for hybrid pair in compact metric spaces are proved in [33, 34, 36] and in other papers.

The purpose of this paper is to extend Theorem 3, Theorem 4 and Theorem 2 [4] for strictly owc mappings satisfying implicit relations and to transfer the study of fixed points for hybrid pairs of mappings satisfying a contractive condition of integral type in compact metric spaces to the study of fixed points in compact metric spaces by altering distances.

3. IMPLICIT RELATIONS

Let $\mathcal{F}_c$ be the family of all real functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1) F is increasing in variable t_1 and nonincreasing in variables t_2 and t_4,
(ϕ_2) If $u \geq 0$, $v > 0$, $w \geq 0$ such that
(ϕ_{2a}) $F(u, v, v, u, w, 0) \leq 0$ or
(ϕ_{2b}) $F(u, v, u, v, 0, w) \leq 0$,
      then $u < v$ and $u = 0$ if $v = 0$.
(ϕ_3) $F(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.

Example 2. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 t_6)^{1/2}$, where $a > 0$, $b, c \geq 0$, $a + 2b < 1$ and $a + c < 1$.

(ϕ_1) : Obviously.
(ϕ_2) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u - av - b(u + v) \leq 0$. Then
      $u \leq \frac{a+b}{1-b} v < v$. Similarly, $F(u, v, u, v, 0, w) \leq 0$ implies $u < v$. If $u = 0$,
      $v > 0$, $w > 0$, then $u < v$.
(ϕ_3) : $F(t, t, 0, 0, t, t) = t(1 - (a + c)) > 0$, $\forall t > 0$.

Example 3. $F(t_1, ..., t_6) = t_1^2 - at_2^2 - b\frac{t_3^2 + t_4^2}{1 + \min\{t_5, t_6\}}$, where $a > 0$ and $a + 2b < 1$.

(ϕ_1) : Obviously.
(ϕ_2) : Let $u > 0$, $v > 0$, $w > 0$ and $F(u, v, v, u, w, 0) = u^2 - av^2 - b(u^2 + v^2) \leq 0$
      which implies $u^2 \leq \frac{a+b}{1-b} v^2$, hence $u < v$. Similarly, $F(u, v, u, v, 0, w) \leq 0$
      implies $u < v$. If $u = 0$, $v > 0$ then $u < v$. If $v = 0$ then $u = 0$.

(ϕ_3) : $F(t,t,0,0,t,t) = t^2(1-a) > 0$, $\forall t > 0$.

Example 4. $F(t_1,...,t_6) = t_1^2 - at_2^2 - b\frac{t_3 t_4}{1+t_5 t_6}$, where $a > 0$ and $a + b < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $u \geq 0, v > 0, w \geq 0$ and $F(u,v,v,u,w,0) = u^2 - av^2 - buv \leq 0$. Then $f(t) = t^2 - bt - a \leq 0 \leq$ where $t = \frac{u}{v}$. Since $f(0) = -a < 0$ and $f(1) = 1 - (a+b) > 0$, there exists $h \in (0,1)$ such that $f(t) < 0$ for $t < h$. Hence $u < v$. Similarly, $F(u,v,u,v,0,w) \leq 0$ implies $u < v$. If $v = 0$ then $u = 0$.

(ϕ_3) : $F(t,t,0,0,t,t) = t^2(1-a) > 0$, $\forall t > 0$.

Let $\omega : [0,\infty) \to [0,\infty)$ with $0 < \omega(r) < r$ for $r > 0$, $\omega(0) = 0$ and $r - \omega(r)$ is non decreasing.

Example 5.

$$F(t_1,...,t_6) = t_1 - \max\{t_2,t_3,t_4,\min\{t_5,t_6\}\} + \omega(\max\{t_2,t_3,t_4,\min\{t_5,t_6\}\}).$$

(ϕ_1) : It follows from the fact that $t - \omega(t) > 0$ is a non decreasing function.

(ϕ_2) : Let $u \geq 0, v > 0, w \geq 0$ and $F(u,v,v,u,w,0) = u - \max\{u,v\} + \omega(\max\{u,v\}) \leq 0$ which implies $u - \max\{u,v\} < 0$. If $v = 0$ then $u - [u - \omega(u)] \leq 0$ which implies $\omega(u) \leq 0$, a contradiction if $u > 0$, hence $u = 0$. Similarly, $F(u,v,u,v,0,w) \leq 0$ implies $u < v$ if $v > 0$. If $v = 0$ then $u = 0$.

(ϕ_3) : $F(t,t,0,0,t,t) = t - (t - \omega(t)) = \omega(t) > 0$, $\forall t > 0$.

4. FIXED POINTS FOR SOWC MAPPINGS IN COMPACT METRIC SPACES

Theorem 5. *Let $I : (X,d) \to (X,d)$ and $F : (X,d) \to B(X)$ be sowc mappings. If I and F have a unique point of strict coincidence $\{z\} = \{Ix\} = Fx$, then z is the unique common fixed point of I and F which is a strict fixed point for F.*

Proof. Since I and F are sowc, there exists a point $x \in X$ such that $\{z\} = \{Ix\} = Fx$ implies $IFx = FIx$. Then, $\{Iz\} = \{IIx\} \in IFx = FIx$. Then $u = Ix$ is a point of strict coincidence of I and F. By hypothesis $u = z$ and $\{z\} = \{Iz\} = Fz$. Hence z is a common fixed point for I and F. Suppose that $v \neq z$ is another common fixed point of I and F, which is a strict fixed point for F. Then $\{v\} = \{Iv\} = Fv$. Hence v is a point of strict coincidence of I and F, by hypothesis $v = z$. $\square$

Theorem 6. *Let (X,d) be a metric space and let $I, J : X \to X$ and $F, G : X \to B(X)$ such that*

$$\phi(\psi(\delta(Fx,Gy)), \psi(d(Ix,Jy)), \psi(\delta(Ix,Fx)),$$
$$\psi(\delta(Jy,Gy)), \psi(D(Ix,Gy)), \psi(D(Jy,Fx))) \leq 0 \qquad (4.1)$$

holds for all $x,y \in X$, where ϕ satisfies condition (ϕ_3) and ψ is weakly altering distance. Suppose that there exists $x,y \in X$ such that $\{u\} = \{Ix\} = Fx$ and $\{v\} = \{Jy\} = Gy$. Then u is the unique point of strict coincidence of I and F and v is the unique point of strict coincidence of J and G.

Proof. First we prove that $Ix = Jy$. Suppose that $Ix \neq Jy$. Then by (4.1) we obtain

$$\phi(\psi(d(Ix, Jy)), \psi(d(Ix, Jy)), 0, 0, \psi(d(Ix, Jy)), \psi(d(Ix, Jx))) \leq 0,$$

a contradiction of (ϕ_3). Hence $d(Ix, Jy) = 0$ which implies $Ix = Jy$. Thus $\{u\} = \{Ix\} = Fx = Gy = \{Jy\}$. Suppose that $z \in X, z \neq x$ such that $\{w\} = \{Iz\} = Fz$. Then by (4.1) we obtain

$$\phi(\psi(d(Iz, Jy)), \psi(d(Iz, Jy)), 0, 0, \psi(d(Iz, Jy)), \psi(d(Iz, Jx))) \leq 0,$$

a contradiction of (ϕ_3). Hence $\{w\} = \{Iz\} = Fz = \{Jy\} = Gy = Fx = \{Ix\} = \{u\}$ and u is the unique point of strict coincidence of I and F. Similarly, v is the unique point of strict coincidence of J and G.   □

Theorem 7. *Let (X, d) be a compact metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ satisfying the inequality (4.1) for all $x, y \in X$, $\phi \in \mathcal{F}_c$ satisfies condition (ϕ_3) and ψ is weakly altering distance such that $Fx \subset J(X)$ and $Gx \subset I(X)$, $\forall x \in X$ and the functions I and F are continuous. Then*

3) F and I have a strict coincidence point,

4) G and J have a strict coincidence point.

Moreover, if the pairs (I, F) and (J, G) are strict owc, then I, J, F and G have an unique common fixed point which is a strict fixed point for F and G.

Proof. Let $m = \inf\{\delta(Ix, Fx) : x \in X\}$. Because (X, d) is compact and F and I are continuous as in [1, 33, 34] there exists $x_0 \in X$ such that $\delta(Ix_0, Fx_0) = m$. We prove that $m = 0$. Suppose that $m > 0$. Since $Fx \subset JX, \forall x \in X$, there exists $Jy_0 \in Fx_0$ and $d(Ix_0, Jy_0) \leq \delta(Ix_0, Fx_0) = m$.

By (4.1) we have

$$\phi(\psi(\delta(Fx_0, Gy_0)), \psi(d(Ix_0, Jy_0)), \psi(\delta(Ix_0, Fx_0)),$$
$$\psi(\delta(Jy_0, Gy_0)), \psi(D(Ix_0, Gy_0)), \psi(D(Jy_0, Fx_0))) \leq 0.$$

By (ϕ_1) we obtain

$$\phi(\psi(d(Jy_0, Gy_0)), \psi(m), \psi(m),$$
$$\psi(\delta(Jy_0, Gy_0)), \psi(D(Ix_0, Gy_0)), 0) \leq 0. \qquad (4.2)$$

Since $\psi(m) > 0$, by (ϕ_{2a}) we obtain

$$\psi(\delta(Jy_0, Gy_0)) < \psi(m).$$

Since $Gx \subset IX, \forall x \in X$, there exists a point $z_0 \in X$ such that $Iz_0 \in Gy_0$ and $d(Iz_0, Jy_0) \leq m$. We obtain $\psi(m) \leq \psi(\delta(Iz_0, Fz_0)) \leq \psi(\delta(Fz_0, Gy_0))$.

Then by (4.1) we have

$$\phi(\psi(\delta(Fz_0, Gy_0)), \psi(d(Iz_0, Jy_0)), \psi(\delta(Iz_0, Fx_0)),$$
$$\psi(\delta(Jy_0, Gy_0)), \psi(D(Iz_0, Gy_0)), \psi(D(Jy_0, Fz_0))) \leq 0.$$

By (ϕ_1) we obtain

$$\phi(\psi(\delta(Iz_0, Fz_0)), \psi(m), \psi(\delta(Iz_0, Fz_0)),$$
$$\psi(m), 0, \psi(D(Jy_0, Fz_0))) \leq 0.$$

By (ϕ_{2b}) we have

$$\psi(\delta(Iz_0, Fz_0)) < \psi(m).$$

Hence, $\psi(m) \leq \psi(\delta(Iz_0, Fz_0)) < \psi(m)$, a contradiction. Hence $m = 0$ and $\psi(m) = 0$. By (4.2) $\psi(\delta(Jy_0, Gy_0)) = 0$ which implies $\{Jy_0\} = Gy_0$. Therefore $\{Ix_0\} = Fx_0 = \{Jy_0\} = Gy_0 = \{p\}$. Hence, x_0 is a strict coincidence point of I and F and y_0 is a strict coincidence point of J and G.

By Theorem 6, p is the unique point of strict coincidence of I and F and also p is the unique point of strict coincidence of J and G.

If (I, F) and (J, G) are sowc, then by Theorem 5, p is the unique common fixed point of I, J, F and G, which is a strict fixed point for F and G. $\square$

Remark 6.   (1) By Example 2 and Theorem 7 we obtain a generalization of Theorem 3.

(2) By Example 5 and Theorem 7 we obtain a generalization of Theorem 4.

If $\psi(t) = t$ by Theorem 7 we obtain

Theorem 8. *Let (X, d) be a compact metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ satisfying the following conditions:*

a) *$Fx \subset J(X)$ and $Gx \subset I(X)$, $\forall x \in X$,*

b) *the functions I and F are continuous,*

c) *$\phi(\delta(Fx, Gy), d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)) \leq 0$, for all $x, y \in X$ and $\phi \in \mathcal{F}_c$. Then:*

d) *F and I have a strict coincidence point,*

e) *G and J have a strict coincidence point.*

Moreover, if the pairs (I, F) and (J, G) are strict owc, then I, J, F and G have an unique common fixed point which is a strict fixed point for F and G.

Remark 7. If I, J, F and G are self mappings of (X, d) then by Theorem 7 we obtain a generalization of Theorem 4.1 [7].

Corollary 1. *Let (X, d) be a compact metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ satisfying the following conditions:*

a) *$F(X) \subset J(X)$ and $G(X) \subset I(X)$, $\forall x \in X$,*

b) *the functions I and F are continuous,*

c) *$\phi(\delta(Fx, Gy)) \leq$*
   $ad(Ix, Jy) + b[\delta(Ix, Fx) + \delta(Jy, Gy)] + c[D(Ix, Gy) \cdot D(Jy, Fx)]^{1/2}$, for all $x, y \in X$, where $a > 0$, $b, c \geq 0$, $a + 2b < 1$ and $a + c < 1$. Then

d) *F and I have a strict coincidence point,*

e) *G and J have a strict coincidence point.*

Moreover, if the pairs (I, F) and (J, G) are strict owc, then I, J, F and G have an unique common fixed point which is a strict fixed point for F and G.

Proof. The proof follows by Theorem 8 and Example 2.   □

Example 6. Let $X = [0, 1]$ endowed with the Euclidean metric d. We define

$$Fx = \left\{\frac{1}{2}\right\}, x \in [0, 1] \qquad\qquad Gx = \begin{cases} \frac{1}{2}, x \in \left[0, \frac{1}{2}\right] \\ (\frac{1}{4}, \frac{1}{2}], x \in (\frac{1}{2}, 1] \end{cases}$$

$$Ix = \begin{cases} \frac{2x+1}{4}, x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, x \in (\frac{1}{2}, 1] \end{cases} \qquad\qquad Jx = \begin{cases} 1-x, x \in \left[0, \frac{1}{2}\right] \\ 0, x \in (\frac{1}{2}, 1] \end{cases}$$

Then we have

$$FX = \left\{\frac{1}{2}\right\}, \qquad GX = \left[\frac{1}{4}, \frac{1}{2}\right], \qquad IX = \left[\frac{1}{4}, \frac{1}{2}\right], \qquad JX = \{0\} \cup \left[\frac{1}{2}, 1\right].$$

Hence $F(X) \subset J(X), G(X) \subset I(X)$.
I and F are continuous.

$$J\left(\frac{1}{2}\right) = G\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\}, \qquad\qquad I\left(\frac{1}{2}\right) = F\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\},$$

$$IF\left(\frac{1}{2}\right) = I\left(\frac{1}{2}\right) = \frac{1}{2}, \qquad\qquad FI\left(\frac{1}{2}\right) = F\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\},$$

$$JG\left(\frac{1}{2}\right) = J\left(\frac{1}{2}\right) = \frac{1}{2}, \qquad\qquad GJ\left(\frac{1}{2}\right) = G\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\}.$$

Hence, (I, F) and (J, G) are strict owc.
If $x \in [0, 1]$ and $y \in \left[0, \frac{1}{2}\right]$ then $\delta(Fx, Gy) = 0$.
If $y \in (\frac{1}{2}, 1]$, then $\delta(Fx, Gy) = \delta\left(\{\frac{1}{2}\}, (\frac{1}{4}, \frac{1}{2}]\right) = \frac{1}{4}$ and $d(Ix, Jy) = d\left(\{\frac{1}{2}\}, 0\right) = \frac{1}{2}$.

Hence the condition $c)$ of Corollary 1 is satisfied for $a > \frac{1}{2}, a + 2b < 1, a + c < 1$.

Hence I, J, F and G have an unique common fixed point $x = \frac{1}{2}$, which is a strict fixed point for F and G.

5. ALTERING DISTANCE AND FIXED POINTS FOR HYBRID PAIRS SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE

Theorem 9. *Let (X, d) be a compact metric space, $I, J : (X, d) \to (X, d)$ and $F, G : X \to B(X)$ satisfying the following conditions:*

1) *$Fx \subset J(X)$ and $Gx \subset I(X)$, $\forall x \in X$,*
2) *the functions I and F are continuous,*

3) $\phi \left(\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(Ix,Jy)} h(t)dt, \int_0^{\delta(Ix,Fx)} h(t)dt, \right.$

    $\left. \int_0^{\delta(Jy,Gy)} h(t)dt, \int_0^{D(Ix,Gy)} h(t)dt, \int_0^{D(Jy,Fx)} h(t)dt \right) \le 0,$

    for all $x,y \in X$, where $\phi \in \mathcal{F}_c$ and $h(t)$ is as in Theorem 1. Then:

4) *F and I have a strict coincidence point,*

5) *G and J have a strict coincidence point.*

Moreover, if the pairs (I,F) and (J,G) are strict owc, then I,J,F and G have an unique common fixed point which is a strict fixed point for F and G.

Proof. As in Lemma 3 we have

$$\int_0^{\delta(Fx,Gy)} h(t)dt = \psi(\delta(Fx,Gy)), \qquad \int_0^{d(Ix,Jy)} h(t)dt = \psi(d(Ix,Jy)),$$

$$\int_0^{\delta(Ix,Fx)} h(t)dt = \psi(\delta(Ix,Fx)), \qquad \int_0^{\delta(Jy,Gy)} h(t)dt = \psi(\delta(Jy,Gy)),$$

$$\int_0^{D(Ix,Gy)} h(t)dt = \psi(D(Ix,Gy)), \qquad \int_0^{D(Jy,Fx)} h(t)dt = \psi(D(Jy,Fx)).$$

Then by 3) we obtain

$$\phi(\psi(\delta(Fx,Gy)), \psi(d(Ix,Jy)), \psi(\delta(Ix,Fx)),$$
$$\psi(\delta(Jy,Gy)), \psi(D(Ix,Gy)), D(Jy,Fx)) \le 0.$$

By Lemma 3 $\psi(t)$ is a weakly altering distance. Hence the conditions of Theorem 7 are satisfied and the conclusion of Theorem 9 follows from Theorem 7. $\qquad \square$

Remark 8. If $h(t) = 1$, by Theorem 9 we obtain Theorem 8.

By Theorem 9 and Example 2 - 5 we obtain particular results for mappings satisfying implicit relations in compact metric space. For example, by Theorem 9 and Example 2 we obtain

Corollary 2. *Let (X,d) be a compact metric space, $I,J : (X,d) \to (X,d)$, $F,G : X \to B(X)$ satisfying conditions (1) and (2) of Theorem 9 and*

$$\int_0^{\delta(Fx,Gy)} h(t)dt \le a \int_0^{d(Ix,Jy)} h(t)dt + b[\int_0^{\delta(Ix,Fx)} h(t)dt +$$

$$\int_0^{\delta(Jy,Gy)} h(t)dt] + c(\int_0^{D(Ix,Gy)} h(t)dt \int_0^{D(Jy,Fx)} h(t)dt)^{1/2} \le 0,$$

for all $x,y \in X$, where $h(t)$ is as in Theorem 1. Then:

a) *F and I have a strict coincidence point,*

b) *G and J have a strict coincidence point.*

Moreover, if the pairs (I,F) and (J,G) are strict owc, then I,J,F and G have an unique common fixed point which is a strict fixed point for F and G.

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