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Oana Mleçniţe

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EXISTENCE AND HYERS-ULAM STABILITY RESULTS FOR A COINCIDENCE PROBLEM WITH APPLICATIONS

OANA MLEŞNIŢE

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Abstract. The purpose of the work is to present some Hyers-Ulam stability results for the coincidence point problem associated to a single-valued operator problem. As an application, a Hyers-Ulam stability theorem for a initial value problem associated to a differential equation is given.

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1. INTRODUCTION

Let (X,d) be a metric space and $f : X \to X$ an operator. We denote by

$$Fix(f) := \{x \in X \mid f(x) = x\},$$

the fixed point set of the operator f. By definition, f is called a weakly Picard operator if the sequence $(f^n(x))_{n\in\mathbb{N}}$, of successive approximations converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f. For example, self Caristi type operators and self graphic contractions on complete metric spaces are examples of weakly Picard operators (see [3], [4]).

If f is weakly Picard operator, then we define the operator $f^\infty : X \to X$ defined by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$. It is clear that $f^\infty(X) = Fix(f)$. Moreover, f^∞ is a set retraction of X to $Fix(f)$.

If f is weakly Picard operator and $Fix(f) = \{x^*\}$, then by definition f is a Picard operator. In this case f^∞ is the constant operator, $f^\infty(x) = x^*$, for all $x \in X$. Self Banach contractions, Kannan contractions and Ciric-Reich-Rus contractions on complete metric spaces are nice examples of Picard operators (see [3], [4]).

The following concepts are important in our consideration, see [4].

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Definition 1. Let $f : X \to X$ be a weakly Picard operator and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function which is continuous in 0 and $\psi(0) = 0$. By definition the operator f is ψ-weakly Picard operator if

$$d(x, f^{\infty}(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In particular, if $\psi(t) = c \cdot t$ with $c > 0$ then we say that f is c-weakly Picard operator.

For some examples of weakly Picard operators and ψ-weakly Picard operators see [1].

Example 1. Let (X,d) be a complete metric space and $f : X \to X$ an operator with closed graphic. We suppose that f is a graphic α-contraction, i.e.,

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X.$$

Then f is a c-weakly Picard operator, with $c = \dfrac{1}{1-\alpha}$.

Definition 2. Let (x,d) be a metric space. A function $f : X \to X$ is a φ-contraction if φ is increasing and $(\varphi^n(t)) \to 0$, $n \to \infty$ for all $t \geq 0$ and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

Theorem 1 ([3]). *Let (X,d) be a complete metric space and $f : X \to X$ a φ-contraction. Then f is a Picard operator.*

Definition 3. Let (X,d) be a metric space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{1.1}$$

is said to be generalized Hyers-Ulam stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ that is increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{1.2}$$

there exists a solution x^* of the equation (1.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If $\psi(t) = ct$, for each $t \in \mathbb{R}_+$ (for some $c > 0$), then the equation (1.1) is said to be Hyers-Ulam stable.

Theorem 2 (see [4]). *Let (X,d) be a metric space. If $f : X \to X$ is a ψ-weakly Picard operator, then the fixed point equation (1.1) is generalized Hyers-Ulam stable. In particular, if f is c-weakly Picard operator, then the equation (1.1) is Hyers-Ulam stable.*

Proof. Let $\varepsilon > 0$ and y^* a solution of (1.2). Since f is ψ-weakly Picard operator, we have that

$$d(x, f^{\infty}(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

If we take $x := y^*$ and $x^* := f^{\infty}(y)$, we have that $d(y^*, x^*) \leq \psi(\varepsilon)$. So, the fixed point equation (1.1) is generalized Hyers-Ulam stable. □

Let (X,d) and (Y,ρ) be two metric spaces and $f,g : X \to Y$ two operators. Let us consider the following coincidence point problem

$$f(x) = g(x) \tag{1.3}$$

Definition 4 ([4]). Let (X,d) and (Y,ρ) be two metric spaces and $f,g : X \to Y$ be two operators. The coincidence problem (1.3) is called generalized Hyers-Ulam stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ that is increasing, continuous in 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution u^* of the inequality

$$\rho(f(u), g(u)) \leq \varepsilon \tag{1.4}$$

there exists a solution x^* of (1.3) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$ then the coincidence point (1.3) is said to be Hyers-Ulam stable.

2. Hyers-Ulam stability for coincidence equations

Our main abstract result is an existence and Hyers-Ulam stability result for the coincidence point problem.

Theorem 3. *Let $A \neq \varnothing$ be an arbitrary set and let (M,d) be a metric space. Let $S,T : A \to M$ such that $S(A) \subset T(A)$ and $(T(A),d)$ is a complete subspace of M. Suppose that there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with φ increasing and $(\varphi^n(t)) \to 0$, $n \to \infty$, for all $t \in \mathbb{R}_+$ such that*

$$d(Sx, Sy) \leq \varphi(d(Tx, Ty)), \text{ for all } x, y \in A.$$

Then:

a) $C(S,T) \neq \varnothing$;

b) If additionally, we suppose that the function $\beta(t) := t - \varphi(t)$ is increasing and bijective and there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that

$$d(y, S(T^{-1}(y))) \leq \psi(d(Ty, Sy)), \text{ for all } y \in T(A), \tag{2.1}$$

then the coincidence point problem (1.3) is $(\beta^{-1} \circ \psi)-$ generalized Hyers-Ulam stable.

Proof. a) The proof is organized in several steps. Let $f := S \circ T^{-1}$. We prove:
i) f is a singlevalued operator on $T(A)$;
Let $y_1, y_2 \in f(x)$. We get $y_1 \in S(T^{-1}(x))$ and $y_2 \in S(T^{-1}(x))$. So exists $u_1, u_2 \in T^{-1}(x)$ such that $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Because $u_1, u_2 \in T^{-1}(x)$ we have $T(u_1) = x$ and $T(u_2) = x$. Then we have:

$$d(y_1, y_2) = d(Su_1, Su_2) \le \varphi(d(Tu_1, Tu_2)) = \varphi(0).$$

Taking into account that φ is increasing and $(\varphi^n(t)) \to 0$, $n \to \infty$, for all $t \in \mathbb{R}_+$ we deduce that $\varphi(0) = 0$.
We get that $d(y_1, y_2) = 0$. So $y_1 = y_2$ and thus $f(x)$ is a singleton.

ii) $f : T(A) \to T(A)$;
Let $x \in T(A)$. Then exists $a \in A$ such that $x = T(a)$. So we have $a \in T^{-1}(x)$ $\Longrightarrow S(a) \subseteq S(T^{-1}(x)) \Longrightarrow S(a) \subseteq f(x)$. Since f is a siglevalued operator we get $S(a) = f(x) \Longrightarrow f(x) = S(a) \subseteq S(A) \subseteq T(A)$.

iii) $f : T(A) \to T(A)$ is a φ-contraction;
Let $x_1, x_2 \in T(A)$ and $u_1, u_2 \in A$ such that $u_1 \in T^{-1}(x_1)$ and $u_2 \in T^{-1}(x_2)$. Then we have:

$$d(f(x_1), f(x_2)) = d(S(T^{-1}(x_1)), S(T^{-1}(x_2))) = d(Su_1, Su_2) \le$$
$$\le \varphi(d(Tu_1, Tu_2)) = \varphi(d(x_1, x_2)).$$

So f is self φ-contraction on the complete metric space $(T(A), d)$.

iv) We can apply now Theorem 1 for f and we deduce that f is a Picard operator. So we get that there exists a unique $y^* \in T(A)$ such that

$$y^* = f(y^*) = S(T^{-1}(y^*)).$$

Let $x^* = T^{-1}(y^*)$. Then $y^* = T(x^*)$ and so we get $y^* = S(x^*)$. Hence we conclude

$$S(x^*) = T(x^*) = y^*.$$

b) We prove that the coincidence point problem is generalized Hyers-Ulam stable.
Let $\varepsilon > 0$ and $v^* \in X$ be such that $d(T(v^*), S(v^*)) \le \varepsilon$.
If we take into account of (2.1), we have

$$d(v^*, f(v^*)) = d(v^*, S(T^{-1}(v^*))) \le \psi(d(S(v^*), T(v^*))) \le \psi(\varepsilon).$$

So we get

$$d(v^*, y^*) = d(v^*, f(y^*))$$
$$\le d(f(v^*), v^*) + d(f(v^*), f(y^*)) \le \psi(\varepsilon) + \varphi(d(v^*, y^*)).$$

Then

$$\beta((d(v^*, y^*))) \le \psi(\varepsilon)$$

Hence we get that

$$d(v^*, y^*) \leq (\beta^{-1} \circ \psi)(\varepsilon).$$

Thus, the coincidence point problem (1.3) is $(\beta^{-1} \circ \psi)$ - generalized Hyers-Ulam stable. □

Remark 1. Our theorem is an extension of Goebel's Theorem (see [2]), which can be obtained from our result by taking $\varphi(t) = kt$ for each $t \in \mathbb{R}_+$ (for some $k \in [0,1)$).

We will present now an application of Theorem 3.

Theorem 4. *Consider the differential equation*

$$x' = f(t,x) \tag{2.2}$$

with the initial condition

$$x(0) = \xi. \tag{2.3}$$

Suppose that the function f is defined in the half-plane $t \geq 0$, $-\infty < x < +\infty$ and satisfies following conditions:

- *i) $f(t,x)$ is a continuous function of x for almost all $t \geq 0$;*
- *ii) $f(t,x)$ is a measurable function of t for all $x \in \mathbb{R}$;*
- *iii) Lipschitz inequality, i.e.*

  $$|f(t,x) - f(t,y)| \leq L(t)|x-y|,$$

  where L is locally integrable function on the interval $(0,\infty)$;
- *iv) $\int_0^t f(\tau,0)d\tau = O(e^{\int_0^t L(\tau)d\tau})$;*
- *v) $f(t,\gamma u) \geq \gamma f(t,u)$ for all $\gamma \geq 1, t > 0, u \in \mathbb{R}$*

Then, the differential equation (2.2) has, for every $\xi \in \mathbb{R}$, a unique solution and the equation (2.2) is Hyers-Ulam stable.

Proof. Let us consider the set

$$A = \{x \in C[0,\infty) : x(t) = O(e^{\int_0^t L(\tau)d\tau})\}.$$

We define the operators $S, T : A \to B$ by

$$(Sx)(t) = \left\{ \int_0^t f(\tau, x(\tau))d\tau + \xi \right\} e^{-p\int_0^t L(\tau)d\tau},$$

$$(Tx)(t) = x(t)e^{-p\int_0^t L(\tau)d\tau},$$

where B is a Banach space of bounded continuous functions on $[0,\infty)$ with the norm $\|x\| = \sup\limits_{(0,\infty)} |x(t)|$ and $p > 1$. By simple calculation we have

$$|(Sx)(t) - (Sy)(t)| \leq \frac{1}{p}\|Tx - Ty\|$$

and further $\|Sx - Sy\| \le \frac{1}{p}\|Tx - Ty\|$. But $S(A) \subset T(A)$ and $T(A)$ is a complete subspace of B. By Theorem 3 there exists $\bar{x} \in A$ such that $S(\bar{x}) = T(\bar{x})$. From this we have

$$\bar{x}(t) = \int_0^t f(\tau, \bar{x}(\tau))d\tau + \xi.$$

Since T is a single-valued operator, $\bar{x}$ is unique. Then the differential equation (2.2) has for every $\xi \in \mathbb{R}$ a unique solution with the initial condition $\bar{x}(0) = \xi$.

Next we prove that the equation (2.2) is Hyers-Ulam stable.

We have $(T^{-1}y)(t) = y(t) \cdot e^{p\int_0^t L(\tau)d\tau}$. We prove that $d(y, S(T^{-1}(y))) \le \alpha d(Ty, Sy)$, for all $y \in T(A)$. We obtain that

$$S(T^{-1}(y))(t) = \left\{ \int_0^t f(\tau, (T^{-1}y)(\tau))d\tau + \xi \right\} \cdot e^{-p\int_0^t L(\tau)d\tau} =$$

$$= e^{-p\int_0^t L(\tau)d\tau} \left\{ \int_0^t f(\tau, y(\tau)e^{p\int_0^t L(\tau)d\tau})d\tau + \xi \right\}.$$

By calculations we get

$$
\begin{aligned}
& |y(t) - S(T^{-1}(y))(t)| \\
&= \left| y(t) - e^{-p\int_0^t L(\tau)d\tau} \left\{ \int_0^t f(\tau, y(\tau)e^{p\int_0^t L(\tau)d\tau})d\tau + \xi \right\} \right| \le \\
&\le \left| y(t) - e^{-p\int_0^t L(\tau)d\tau} \left\{ e^{p\int_0^t L(\tau)d\tau} \int_0^t f(\tau, y(\tau))d\tau + \xi \right\} \right| = \\
&= \left| y(t) - \int_0^t f(\tau, y(\tau))d\tau - e^{-p\int_0^t L(\tau)d\tau}\xi \right| = \\
&= \Bigg| -e^{p\int_0^t L(\tau)d\tau} \Bigg\{ -y(t)e^{-p\int_0^t L(\tau)d\tau} + \\
&\quad e^{-p\int_0^t L(\tau)d\tau} \int_0^t f(\tau, y(\tau))d\tau + e^{-2p\int_0^t L(\tau)d\tau}\xi \Bigg\} \Bigg| \le \\
&\le e^{p\int_0^t L(\tau)d\tau} \Bigg| -y(t)e^{-p\int_0^t L(\tau)d\tau} + \\
&\quad e^{-p\int_0^t L(\tau)d\tau} \int_0^t f(\tau, y(\tau))d\tau + e^{-p\int_0^t L(\tau)d\tau}\xi \Bigg| = \\
&= e^{p\int_0^t L(\tau)d\tau} |(Sy)(t) - (Ty)(t)|.
\end{aligned}
$$

Since, all the condition of Theorem 3 hold, then the equation (2.2) is Hyers-Ulam stable. □

REFERENCES

[1] A. Chis-Novac, R. Precup, and I. A. Rus, "Data dependence of fixed points for non-self generalized contractions," *Fixed Point Theory*, vol. 10, no. 1, pp. 73–87, 2009.
[2] K. Goebel, "A coincidence theorem," *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.*, vol. 16, pp. 733–735, 1968.
[3] I. A. Rus, *Generalized contractions and applications*. Cluj-Napoca: Cluj University Press, 2001.
[4] I. A. Rus, "Ulam stability of ordinary differential equations," *Stud. Univ. Babeş-Bolyai, Math.*, vol. 54, no. 4, pp. 125–133, 2009.

Author's address

Oana Mleşniţe
Department of Mathematics, Babeş-Bolyai University Cluj-Napoca,, Kogălniceanu Street No.1, 400084, Cluj-Napoca, Romania.
E-mail address: oana.mlesnite@math.ubbcluj.ro