Convergence of the solution of an impulsive differential equation with piecewise constant argument

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CONVERGENCE OF THE SOLUTION OF AN IMPULSIVE DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. We show the existence of the unique solution of impulsive differential equation

$$\begin{cases} x'(t) = a(t)(x(t) - x(\lfloor t-1 \rfloor)) + f(t), \ t \neq n \in \mathbb{Z}^+ = \{1,2,\ldots\}, \ t \geq 0, \\ \Delta x(t) = c_t x(t) + d_t, \ t = n \in \mathbb{Z}^+, \end{cases}$$

with the initial conditions

$$x(-1) = x_{-1}, \ x(0) = x_0,$$

where $\lfloor . \rfloor$ denotes the floor integer function. Moreover, we obtain sufficient conditions for the asymptotic constancy of this equation and we compute, as $t \to \infty$, the limits of the solutions of the impulsive equation with $c_n = 0$ in terms of the initial conditions, a special solution of the corresponding adjoint equation and a solution of the corresponding difference equation.

1. INTRODUCTION

The theory of differential equations with piecewise constant arguments *(DEP-CA)* of the type

$$x'(t) = f(t, x(t), x(h(t)))$$

was initiated in [14, 40] where $h(t) = \lfloor t \rfloor$, $\lfloor t-n \rfloor$, $\lfloor t+n \rfloor$, *etc.* and $\lfloor . \rfloor$ denotes the floor integer function. These types of equations have been intensively investigated for twenty five years. Systems described by *DEPCA* exist in a large area such as biomedicine, chemistry, physics and mechanical engineering. Busenberg and Cooke [13] first established a mathematical model with a piecewise constant argument for analyzing vertically transmitted diseases. Examples in practice include machinery driven by servo units, charged particles moving in a piecewise constantly varying electric field and elastic systems impelled by a Geneva wheel.

DEPCA are also closely related to difference and differential equations. So, they describe hybrid dynamical systems and combine the properties of both differential

and difference equations. The oscillation, periodicity and some asymptotic properties of various differential equations with piecewise constant arguments were methodically demonstrated in [1–5, 26–28, 35, 36, 38, 42]. Also, Wiener's book [43] is a distinguished source in this area.

Impulsive differential equations are a basic tool to study the dynamics of processes that are subjected to abrupt changes in their states. Theory of impulsive differential equations has been motivated by a number of applied problems such as control theory [24, 25], population dynamics [37], chemotherapeutic treatment in medicine [31] and some physics problems [32]. A significant development has been made in the mathematical theory of impulsive differential equations in the last two decades; see the monographs [7, 39].

But, there are only a few papers on impulsive differential equations with piecewise constant arguments (*IDEPCA*) [9, 29, 33, 44]. In [33], Li and Shen considered the problem

$$y'(t) = f(t, y\lfloor t - k \rfloor), \ t \neq n, \ t \in J,$$
$$\Delta y(n^+) = I_n(y(n)), \ n = 1, 2, \ldots, p, \ y(0) = y(T).$$

Using the method of upper and lower solutions, they proved that it has at least one solution. In [44], Wiener and Lakshmikantham established the existence and uniqueness of solutions of the initial value problem

$$x'(t) = f(x(t), x(g(t))), \ x(0) = x_0,$$

and they also studied the cases of oscillation and stability, where f is a continuous function and $g : [0, \infty) \to [0, \infty)$, $g(t) \leq t$, is a step function. In [9] and [29], some qualitative aspects of advanced and delay *IDEPCA* are investigated. In [34], the authors investigated the impulsive stabilization of certain delay differential equations with piecewise constant argument by using Lyapunov function and analysis methods. They showed that some nonimpulsive systems can be stabilized by imposition of impulsive controls.

Lately, the problem of the asymptotic constancy of solutions was studied for some functional differential equations [6, 8, 11, 12, 15–23, 41] and as well the same problem has been considered for some impulsive delay differential equations [10, 30]. So, due to the practical reasons and the papers mentioned above one can be motivated to deal with the problem of asymptotic constancy of solutions of an impulsive differential equation with piecewise constant arguments.

In this paper, we consider the first order nonhomogeneous linear impulsive differential equation with piecewise constant argument

$$x'(t) = a(t)(x(t) - x(\lfloor t - 1 \rfloor)) + f(t), \ t \neq n \in \mathbb{Z}^+, \ t \geq 0, \qquad (1.1)$$

$$\Delta x(t) = c_t x(t) + d_t, \ t = n \in \mathbb{Z}^+, \qquad (1.2)$$

with the initial conditions

$$x(-1) = x_{-1}, \ x(0) = x_0, \tag{1.3}$$

where $a(t)$ and $f(t)$ are continuous real valued functions on $[0, \infty)$, $c_n \in \mathbb{R} \setminus \{1\}$, $d_n \in \mathbb{R}$, $n \in \mathbb{Z}^+$, $\mathbb{Z}^+ = \{1, 2, \ldots\}$, x_0, $x_{-1} \in \mathbb{R}$, $\Delta x(n) = x(n^+) - x(n^-)$, $x(n^+) = \lim\limits_{t \to n^+} x(t)$, $x(n^-) = \lim\limits_{t \to n^-} x(t)$ and $\lfloor . \rfloor$ denotes the floor integer function.

The main purpose of this work is to obtain sufficient conditions for asymptotic constancy of the solution $x(t)$ of $(1.1) - (1.3)$ and also, as $t \to \infty$, to compute the limit of the solution of the impulsive differential equation with piecewise constant argument

$$x'(t) = a(t)(x(t) - x(\lfloor t - 1 \rfloor)) + f(t), \ t \neq n \in \mathbb{Z}^+, \ t \geq 0, \tag{1.4}$$

$$\Delta x(t) = d_t, \ t = n \in \mathbb{Z}^+, \tag{1.5}$$

in terms of initial conditions, the solution of an integral equation and the solution of a corresponding difference equation. To the best of author's knowledge, this problem has not been studied yet.

2. Existence of Solutions

Definition 1. A function $x(t)$ defined on $[0, \infty)$ is said to be a solution of $(1.1) - (1.3)$ if it satisfies the following conditions:

(d_1) $x : [0, \infty) \to \mathbb{R}$ is continuous with the possible exception of the points $t \in \mathbb{Z}^+$,

(d_2) $x(t)$ is right continuous and has left-hand limits at the points $t \in \mathbb{Z}^+$,

(d_3) $x'(t)$ exists for every $t \in [0, \infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$ where one-sided derivatives exist,

(d_4) $x(t)$ satisfies (1.1) for any $t \in (0, \infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$,

(d_5) $x(t)$ satisfies (1.2) for every $t = n \in \mathbb{Z}^+$,

(d_6) $x(-1) = x_{-1}, \ x(0) = x_0$.

Theorem 1. *The initial value problem* $(1.1) - (1.3)$ *has a unique solution* $x(t)$ *on* $[0, \infty)$

$$x(t) = \exp\left(\int_{\lfloor t \rfloor}^{t} a(u)\,du\right) z_{\lfloor t \rfloor} + \left(1 - \exp\left(\int_{\lfloor t \rfloor}^{t} a(u)\,du\right)\right) z_{\lfloor t \rfloor - 1}$$

$$+ \int_{\lfloor t \rfloor}^{t} \exp\left(\int_{s}^{t} a(u)\,du\right) f(s)\,ds \tag{2.1}$$

where $\lfloor . \rfloor$ denotes the floor integer function, $z_{\lfloor t \rfloor} = x(\lfloor t \rfloor)$ for $t \in [0, \infty)$ and it is a solution of the corresponding difference equation

$$
\begin{aligned}
z_{\lfloor t \rfloor+1} = & \left((1-c_{\lfloor t \rfloor+1})^{-1} \exp \left(\int_{\lfloor t \rfloor}^{\lfloor t \rfloor+1} a(u)\,du \right) \right) z_{\lfloor t \rfloor} \\
& + (1-c_{\lfloor t \rfloor+1})^{-1} \left(1 - \exp \left(\int_{\lfloor t \rfloor}^{\lfloor t \rfloor+1} a(u)\,du \right) \right) z_{\lfloor t \rfloor-1} \\
& + (1-c_{\lfloor t \rfloor+1})^{-1} \left(\int_{\lfloor t \rfloor}^{\lfloor t \rfloor+1} \exp \left(\int_{s}^{\lfloor t \rfloor+1} a(u)\,du \right) f(s)\,ds + d_{\lfloor t \rfloor+1} \right). \quad (2.2)
\end{aligned}
$$

Proof. Let $x(t)$ be a solution of (1.1) on $n \leq t < n+1$. Then, Eq.(1.1) reduces to linear ordinary differential equation

$$
x'(t) = a(t)(x(t) - x(n-1)) + f(t).
$$

When we solve this equation, we have

$$
\begin{aligned}
x(t) = & \exp \left(\int_{n}^{t} a(u)\,du \right) x(n) + \left(1 - \exp \left(\int_{n}^{t} a(u)\,du \right) \right) x(n-1) \\
& + \int_{n}^{t} \exp \left(\int_{s}^{t} a(u)\,du \right) f(s)\,ds, \; n \leq t < n+1. \quad (2.3)
\end{aligned}
$$

If we denote the solution $x(t)$ defined by (2.3) as $x_n(t)$, then $x_{n+1}(t)$ implies the solution of Eq.(1.1) on the interval $n+1 \leq t < n+2$:

$$
\begin{aligned}
x_{n+1}(t) = & \exp \left(\int_{n+1}^{t} a(u)\,du \right) x(n+1) + \left(1 - \exp \left(\int_{n+1}^{t} a(u)\,du \right) \right) x(n) \\
& + \int_{n+1}^{t} \exp \left(\int_{s}^{t} a(u)\,du \right) f(s)\,ds, \; n+1 \leq t < n+2. \quad (2.4)
\end{aligned}
$$

Using impulse conditions (1.2) for $t = n+1$, we have

$$
\Delta x(n+1) = c_{n+1} x(n+1) + d_{n+1},
$$

that is,

$$
x(n+1)^{+} - x(n+1)^{-} = c_{n+1} x(n+1) + d_{n+1}.
$$

Since $x(t)$ is right continuous, the previous equality reduces to

$$x_n\left((n+1)^-\right) = (1-c_{n+1})\,x_{n+1}(n+1) - d_{n+1}.$$

Considering this equality together with (2.3) and (2.4), we obtain the second order nonhomogeneous difference equation

$$\begin{aligned}
x(n+1) = {} &(1-c_{n+1})^{-1} \exp\left(\int_n^{n+1} a(u)\,du\right) x(n) \\
&+ (1-c_{n+1})^{-1}\left(1 - \exp\left(\int_n^{n+1} a(u)\,du\right)\right) x(n-1) \\
&+ (1-c_{n+1})^{-1}\left(\int_n^{n+1} \exp\left(\int_s^{n+1} a(u)\,du\right) f(s)\,ds + d_{n+1}\right).
\end{aligned} \tag{2.5}$$

Let us define $x(\lfloor t \rfloor) = z_{\lfloor t \rfloor}$ for $t \in [0, \infty)$. So, $x(n) = z_n$ for $n = 0, 1, 2, \ldots$, and Eq.(2.5) can be rewritten as

$$\begin{aligned}
z_{n+1} = {} &(1-c_{n+1})^{-1} \exp\left(\int_n^{n+1} a(u)\,du\right) z_n \\
&+ (1-c_{n+1})^{-1}\left(1 - \exp\left(\int_n^{n+1} a(u)\,du\right)\right) z_{n-1} \\
&+ (1-c_{n+1})^{-1}\left(\int_n^{n+1} \exp\left(\int_s^{n+1} a(u)\,du\right) f(s)\,ds + d_{n+1}\right)
\end{aligned} \tag{2.6}$$

which implies Eq.(2.2) for $t \in [0, \infty)$.

Taking into account the initial conditions $x(-1) = x_{-1} = z_{-1}$ and $x(0) = x_0 = z_0$, the solution of difference equation (2.6) is obtained uniquely. So, the unique solution of $(1.1) - (1.3)$ can be written as (2.1) on $[0, \infty)$. □

In this while, we note the following statements:

a) Along this paper we will assume that all solutions of the difference equation (2.6) are bounded, that is, for any solution z_n of (2.6) there is a real positive constant L such that

$$|z_n| \le L, \ n \in \mathbb{Z}^+. \tag{2.7}$$

b) A straightforward verification shows that the solution of the initial value problem $(1.1) - (1.3)$ satisfies the following integral equation

$$
x(t) = \begin{cases} x_{-1}, \ t = -1, \\ x_0 + \int\limits_0^t a(s)x(s)\,ds - \int\limits_0^t a(s)x(\lfloor s-1 \rfloor)\,ds + \int\limits_0^t f(s)\,ds + \sum\limits_{i=1}^{\lfloor t \rfloor} c_i z_i + \sum\limits_{i=1}^{\lfloor t \rfloor} d_i, \\ \hspace{6cm} t \geq 0, \end{cases}
$$

(2.8)

where $z_i = x(i)$, $i = 1, 2, \ldots, \lfloor t \rfloor$.

3. MAIN RESULTS

This section contains the statements of our main results.

Theorem 2. *Let $a(t)$ and $f(t)$ be continuous functions on the interval $[0, \infty)$, $c : \mathbb{Z}^+ \longrightarrow \setminus \{1\}$, $d : \mathbb{Z}^+ \longrightarrow$ and $n \in \mathbb{Z}^+$.*
If

(i) $\int\limits_0^\infty |a(s)|\,ds \leq K_1 < \infty$,

(ii) $\int\limits_0^\infty |f(s)|\,ds \leq K_2 < \infty$,

(iii) $\prod\limits_{i=1}^\infty (1 + |c_i|) \leq L_1 < \infty$,

(iv) $\sum\limits_{i=1}^\infty |d_i| \leq L_2 < \infty$,

then the solution $x(t)$ of $(1.1) - (1.3)$ tends to a constant as $t \to \infty$, where K_j, L_j, $j = 1, 2$, are real positive constants.

Theorem 3. *Suppose that all assumptions of Theorem 2, except (iii), are satisfied. Let $x(t)$ be the solution of $(1.4) - (1.5)$ and $\lim\limits_{t \to \infty} x(t) = l(x_0, x_{-1})$.*
If

$$
\int\limits_t^{\lfloor t+1 \rfloor} |a(s)|\,ds \leq \rho < 1,
$$

(3.1)

then

$$
l(x_0, x_{-1}) = \left(1 + \int\limits_0^1 y(s)a(s)\,ds \right) x_0 + \left(\int\limits_0^1 y(s)a(s)\,ds \right) x_{-1}
$$

$$
+ \int\limits_0^\infty y(s)f(s)\,ds + \sum\limits_{i=1}^\infty d_i + \sum\limits_{i=1}^\infty (z_i - z_{i-1}) \int\limits_i^{i+1} y(s)a(s)\,ds \quad (3.2)
$$

where y is a solution of the integral equation

$$y(t) = 1 + \int\limits_{t}^{\lfloor t+1 \rfloor} y(s) a(s) ds, \ t \geq 0, \tag{3.3}$$

and z_i is a solution of the difference equation (2.6) *for $n = i$.*

4. PROOFS

In this section the proofs of the main results are given.

4.1. *Proof of Theorem* 2.

For the proof of Theorem 2 we consider the following well known lemma [39]:

Lemma 1. *Let a nonnegative piecewise continuous function $u(t)$ satisfy for $t \geq t_0$ the inequality*

$$u(t) \leq \alpha + \int\limits_{t_0}^{t} v(s) u(s) ds + \sum_{t_0 \leq \tau_i < t} \beta_i u(\tau_i)$$

where $\alpha \geq 0$, $\beta_i \geq 0$, $v(s) > 0$, τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq \alpha \prod_{t_0 \leq \tau_i < t} (1 + \beta_i) \exp\left(\int\limits_{t_0}^{t} v(s) ds\right).$$

Now we can give the proof of Theorem 2:

Proof of Theorem 2. Let $x(t)$ be the solution of $(1.1) - (1.3)$. Then, from (2.8),

$$|x(t)| \leq |x_0| + \int\limits_{0}^{t} |a(s)| |x(s)| ds + \int\limits_{0}^{t} |a(s)| |x(\lfloor s-1 \rfloor)| ds$$

$$+ \int\limits_{0}^{t} |f(s)| ds + \sum_{i=1}^{\lfloor t \rfloor} |c_i| |z_i| + \sum_{i=1}^{\lfloor t \rfloor} |d_i|.$$

Considering $x(\lfloor s-1 \rfloor) = z_{\lfloor s-1 \rfloor}$, $s \in [0, \infty)$, and the boundedness of z_n, $n \in \mathbb{Z}^+$, we have

$$|x(t)| \leq |x_0| + \int\limits_{0}^{t} |a(s)| |x(s)| ds + L \int\limits_{0}^{\infty} |a(s)| ds$$

$$+ \int_0^\infty |f(s)|\, ds + \sum_{i=1}^{\lfloor t \rfloor} |c_i|\,|z_i| + \sum_{i=1}^\infty |d_i|.$$

By using (i), (ii), (iv) and (2.7), we obtain

$$|x(t)| \leq \alpha + \int_0^t |a(s)|\,|x(s)|\, ds + \sum_{i=1}^{\lfloor t \rfloor} |c_i|\,|z_i|$$

where $\alpha = |x_0| + LK_1 + K_2 + L_2$. Applying Lemma 1,

$$|x(t)| \leq \alpha \prod_{i=0}^{\lfloor t \rfloor} (1 + |c_i|) \exp\left(\int_0^t |a(s)|\, ds \right)$$

$$\leq \alpha \prod_{i=0}^\infty (1 + |c_i|) \exp\left(\int_0^\infty |a(s)|\, ds \right).$$

Hence, by (i) and (iii)

$$|x(t)| \leq \alpha L_1 e^{K_1} = M, \ t \geq 0, \tag{4.1}$$

where M is a positive real constant.

On the other hand,

$$|x(t) - x(s)| \leq \int_s^t |a(u)|\,|x(u)|\, du + \int_s^t |a(u)|\,|x(\lfloor u - 1 \rfloor)|\, du$$

$$+ \int_s^t |f(u)|\, du + \sum_{i=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} |c_i|\,|z_i| + \sum_{i=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} |d_i| \tag{4.2}$$

for $0 \leq s < t < \infty$.

From (2.7) and (4.1), we obtain

$$|x(t) - x(s)| \leq (M + L) \int_s^\infty |a(u)|\, du + \int_s^\infty |f(u)|\, du + L \sum_{i=\lfloor s \rfloor + 1}^\infty |c_i| + \sum_{i=\lfloor s \rfloor + 1}^\infty |d_i|.$$

In this while, we note that

$$\sum_{i=1}^\infty |c_i| < \infty \tag{4.3}$$

since (iii) is valid. So, by using (4.3) and the conditions (i), (ii) and (iv), it is easy to verify that

$$\lim_{s \to \infty} |x(t) - x(s)| = 0.$$

Thus, by the Cauchy convergence criterion, $\lim\limits_{t\to\infty} x(t) \in \mathbb{R}$.   □

4.2. *Proof of Theorem 3.*

For the proof of Theorem 3, it is necessary to prove the following theorem and lemmas.

Theorem 4. *Suppose $a(t)$ is continuous and (3.1) is satisfied. Then, there is a unique bounded function $y \in PRC([0,\infty),\mathbb{R})$ such that (3.3) holds.*

Proof. Denote the set of piecewise right continuous functions by $PRC([0,\infty),\mathbb{R})$, that is, $\varphi \in PRC$ means that $\varphi : [0,\infty) \to \mathbb{R}$ is continuous for $t \in [0,\infty)$, $t \neq n$, $n = 1,2,\ldots$, and is continuous from the right for $t = n$, $n = 0,1,2,\ldots$.
Now, let us take the space

$$B = \left\{ y \in PRC([0,\infty),\mathbb{R}) : |y|_B \leq \lambda, \ \lambda \geq \frac{1}{1-\rho} \right\}.$$

B is a Banach space with the norm

$$|y|_B = \sup_{t\geq 0} |y(t)|, \ y \in B.$$

For $y \in B$ and $t \geq 0$, define

$$Ty(t) = 1 + \int\limits_{t}^{\lfloor t+1 \rfloor} y(s)\,a(s)\,ds.$$

It can be easily shown that for every integer point n, $n = 0,1,2,\ldots$,

$$Ty(n^+) = \lim_{t\to n^+} Ty(t) = Ty(n)$$

and

$$Ty(n^-) = \lim_{t\to n^-} Ty(t) = 1.$$

As well as, for $t_* \in (n, n+1)$, $n = 0,1,2,\ldots$,

$$Ty(t_*^+) = Ty(t_*^-) = Ty(t_*).$$

So, $Ty \in PRC([0,\infty),\mathbb{R})$.
Also, from (3.1), it follows that

$$|Ty|_B \leq 1 + \rho|y|_B \leq \lambda.$$

Hence T maps B into itself. On the other hand, for $y, z \in B$

$$|Ty - Tz|_B \leq \rho|y - z|_B.$$

Since $\rho < 1$, $T : B \to B$ is a contraction. Therefore, the unique solution $y \in B$ of $Ty = y$ is the unique piecewise right continuous and bounded solution of (3.3).   □

Lemma 2. *Let a (t) is continuous and* (3.1) *is true. Then, the solution y of the integral equation* (3.3) *satisfies the following integral equation*

$$\begin{cases} y'(t) = -y(t)a(t), \; t \neq n, \\ \Delta y(n) = \int\limits_{n}^{n+1} y(s)a(s)\,ds, \; n \in \mathbb{Z}^{+}, \end{cases} \tag{4.4}$$

where $\Delta y(n) = y(n^{+}) - y(n^{-})$, $y(n^{+}) = \lim\limits_{t \to n^{+}} y(t)$ and $y(n^{-}) = \lim\limits_{t \to n^{-}} y(t)$.

Proof. Taking the derivative of (3.3) for $t \in (n, n+1)$, $n \in \mathbb{Z}^{+}$, we obtain

$$y'(t) = -y(t)a(t).$$

On the other hand,

$$\begin{aligned} \Delta y(n) &= y(n^{+}) - y(n^{-}) \\ &= 1 + \int\limits_{n}^{n+1} y(s)a(s)\,ds - 1 \\ &= \int\limits_{n}^{n+1} y(s)a(s)\,ds. \end{aligned}$$

So, the proof of Lemma 2 is complete.   □

Now, let us denote the function

$$C(t) = y(t)x(t) - \int\limits_{t}^{\lfloor t+1 \rfloor} y(s)a(s)x(\lfloor s-1 \rfloor)\,ds, \; t \geq 0, \tag{4.5}$$

where y is the solution of the integral equation (3.3) and x is the solution of (1.4) − (1.5).

Lemma 3. *Suppose that $a(t)$ and $f(t)$ are continuous functions on $[0, \infty)$ and* (3.1) *is hold. Then,*

$$C(t) = C(0) + \int\limits_{0}^{t} y(s)f(s)\,ds + \sum_{i=1}^{\lfloor t \rfloor} d_{i} + \sum_{i=1}^{\lfloor t \rfloor} (z_{i} - z_{i-1}) \int\limits_{i}^{i+1} y(s)a(s)\,ds \tag{4.6}$$

where y is a solution of (3.3) *and z_{i} is a solution of the difference equation* (2.6) *for $n = i$.*

Proof. To obtain (4.6), it is enough to show that $C(t)$, defined by (4.6), satisfies

$$\begin{cases} C'(t) = y(t)f(t), \; t \neq n, \; t \geq 0, \\ \Delta C(n) = d_{n} + (z_{n} - z_{n-1}) \int\limits_{n}^{n+1} y(s)a(s)\,ds, \; n \in \mathbb{Z}^{+}. \end{cases} \tag{4.7}$$

For $t \in (n, n+1)$, (4.5) can be written as

$$C(t) = y(t)x(t) - x(n-1) \int_t^{n+1} y(s)a(s)\,ds. \tag{4.8}$$

By taking derivative of both sides of (4.8), we have

$$C'(t) = y'(t)x(t) + y(t)x'(t) + x(n-1)y(t)a(t).$$

Considering (1.4) and (4.4), we get

$$C'(t) = -y(t)a(t)x(t) + y(t)\{a(t)x(t) - a(t)x(n-1) + f(t)\} + x(n-1)y(t)a(t)$$
$$= y(t)f(t).$$

Also, from (4.8),

$$\Delta C(n) = C(n^+) - C(n^-)$$

$$= y(n)x(n) - x(n-1) \int_n^{n+1} y(s)a(s)\,ds - y(n^-)x(n^-). \tag{4.9}$$

Substituting

$$y(n^-) = y(n) - \int_n^{n+1} y(s)a(s)\,ds$$

and

$$x(n^-) = x(n) - d_n$$

into (4.9), we obtain

$$\Delta C(n) = d_n + (z_n - z_{n-1}) \int_n^{n+1} y(s)a(s)\,ds$$

where $z_n = x(n)$, $n \in \mathbb{Z}^+$. So, we obtain (4.7). Integrating both sides of (4.7), we get (4.6). $\qquad\square$

Now, we can prove Theorem 3:

Proof of Theorem 3. Let $x(t)$ be the solution of $(1.4) - (1.5)$. For the proof, it is sufficient to show that

$$\lim_{t \to \infty} x(t) = C(0) + \int_0^\infty y(s)f(s)\,ds + \sum_{i=1}^\infty d_i + \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s)a(s)\,ds \tag{4.10}$$

where C is defined by (4.5) and z_i is a solution of the difference equation (2.6) for $n = i$. From (4.6), we have for $t \geq 0$,

$$x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^\infty d_i - \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds$$

$$= x(t) - \left(C(0) + \int_0^t y(s) f(s) ds + \sum_{i=1}^{\lfloor t \rfloor} d_i + \sum_{i=1}^{\lfloor t \rfloor} (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \right)$$

$$- \int_t^\infty y(s) f(s) ds - \sum_{i=\lfloor t \rfloor+1}^\infty d_i - \sum_{i=\lfloor t \rfloor+1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds$$

$$= x(t) - C(t) - \int_t^\infty y(s) f(s) ds - \sum_{i=\lfloor t \rfloor+1}^\infty d_i - \sum_{i=\lfloor t \rfloor+1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds.$$

Using (4.5), it follows for $t \geq 0$,

$$x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^\infty d_i - \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds$$

$$= x(t) - y(t) x(t) + \int_t^{\lfloor t+1 \rfloor} y(s) a(s) x(\lfloor s-1 \rfloor) ds - \int_t^\infty y(s) f(s) ds$$

$$- \sum_{i=\lfloor t \rfloor+1}^\infty d_i - \sum_{i=\lfloor t \rfloor+1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds. \tag{4.11}$$

On the other hand, multiplying (3.3) by $x(t)$ we obtain

$$x(t) = y(t) x(t) - \int_t^{\lfloor t+1 \rfloor} y(s) a(s) x(t) ds$$

for $t \geq 0$. Substituting the last expression into (4.11), we find

$$x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^\infty d_i - \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds$$

$$= \int_t^{\lfloor t+1 \rfloor} y(s) a(s) (x(\lfloor s-1 \rfloor) - x(t)) ds - \int_t^\infty y(s) f(s) ds$$

$$- \sum_{i=\lfloor t \rfloor +1}^{\infty} d_i - \sum_{i=\lfloor t \rfloor +1}^{\infty} (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds. \tag{4.12}$$

From (4.12), together with (2.7), (4.1) and the boundedness of $y(t)$ on $[0, \infty)$, we get

$$\left| x(t) - C(0) - \int_0^{\infty} y(s) f(s) ds - \sum_{i=1}^{\infty} d_i - \sum_{i=1}^{\infty} (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \right|$$

$$\leq |y|_B (L + M) \int_t^{\lfloor t+1 \rfloor} |a(s)| ds + |y|_B \int_t^{\infty} f(s) ds$$

$$+ \sum_{i=\lfloor t \rfloor +1}^{\infty} |d_i| + 2L |y|_B \sum_{i=\lfloor t \rfloor +1}^{\infty} \int_i^{i+1} |a(s)| ds$$

where $|y|_B = \sup\limits_{t \geq 0} |y(t)|$. Thus, it follows that (4.10) is correct. Taking into account (4.5), it is easily verified that the limit relation (4.10) is reduced to (3.2). So, the proof is completed. $\qquad \square$

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