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A note on \ast complete intersections generated by linear forms

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A NOTE ON *COMPLETE INTERSECTIONS GENERATED BY LINEAR FORMS

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Abstract. Let R be a commutative noetherian graded ring. In $R[Y_1, \dots, Y_m]$ we consider the linear forms $a_i = \sum_{j=1}^m a_{ji} Y_j$, $1 \leq i \leq n$, with a_{ji} homogeneous elements of R . We state a necessary and sufficient condition, in terms of *grades of the determinantal ideals of the matrix $A = (a_{ji})$, for the ideal (a_1, \dots, a_n) to be a *complete intersection of *grade n in $R[Y_1, \dots, Y_m]$.

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INTRODUCTION

Let R be a commutative noetherian ring with unit. In [1] the author considered relation ideals of symmetric algebras of finitely generated R -modules and stated a condition to be a complete intersection. If R is a graded ring, similar properties can be established and studied. Definitions and results are inspired by the results known in the not graded case, however it is interesting to establish them in the graduate case. In fact, for the graded algebras good properties can be determined, as in the local case ([2–4]).

Let E be a finitely generated R -module and $Sym_R(E) = \bigoplus_{t=0}^{\infty} Sym_t(E)$ be its symmetric algebra. It is known that the symmetric algebra $Sym_R(E)$ has a presentation $R[Y_1, \dots, Y_m]/J$, where m is the number of the generators of E and J is the relation ideal generated by the linear forms $a_i = \sum_{j=1}^m a_{ji} Y_j$, $1 \leq i \leq n$ in the variables Y_1, \dots, Y_m . Theoretical properties of $Sym_R(E)$ (such as integrity, regularity, to be Cohen-Macaulay, Gorenstein) were studied by various authors. For a finitely generated module E this topic developed and culminated in the study of approximation complex of E . For best results the projective resolution of E can be examined, when it is finite. In particular, Avramov studied theoretic properties of $Sym_R(E)$ in terms of a condition on Fitting ideals of E , when the projective dimension of E is ≤ 1 , and stated a condition for the presentation ideal of $Sym_R(E)$ to be a complete intersection.

The aim of this note is to investigate such properties in the graduate case.

The paper is organized as follows. In Section 1 some preliminary notions and properties on graded rings are introduced. Let R be a graded ring and I be an arbitrary ideal of R . I^* is the graded ideal generated by all homogeneous elements of I and it is the largest graded ideal contained in I . In particular, for each prime ideal $\wp \subset R$, the graded ideal \wp^* is a prime ideal too. In [2] there are theorems that link theoretic properties of \wp and \wp^* by localizations of finitely generated graded R -modules, that is $\dim E_\wp = \dim E_{\wp^*} + 1$ and $\text{depth} E_\wp = \text{depth} E_{\wp^*} + 1$, for all not graded prime ideal \wp of R . Now, we generalize the classic definitions for graded ideals. We define * grade of I the common length of the maximal homogeneous R -sequences in I^* . Then I is a * complete intersection if it can be generated by a number of homogeneous elements equal to its * grade. In Section 2 we consider homogeneous linear forms $a_i = \sum_{j=1}^m a_{ji} Y_j$, $1 \leq i \leq n$, of $R[Y_1, \dots, Y_m]$ with a_{ji} homogeneous elements of R . We state a necessary and sufficient condition, in terms of * grades of the determinantal ideals of the matrix $A = (a_{ji})$, such that (a_1, \dots, a_n) is a * complete intersection of * grade n in $R[Y_1, \dots, Y_m]$.

1. PRELIMINARY AND NOTATIONS

Let R be a commutative noetherian graded ring. In [2] there are some definitions related to the graded ideals of R and some properties on the dimension and the depth of some localizations of finitely generated graded R -modules.

Definition 1. Let $I \subset R$ be an ideal (not necessarily graded). I^* is the graded ideal of R generated by all the homogeneous elements of I (the homogeneous elements of I of degree j are $f \in I_j = I \cap R_j$).

The ideal I^* is the largest graded ideal contained in I . If I is homogeneous, then $I = I^*$.

Remark 1. Let $\wp \subset R$ be a prime ideal, then \wp^* is a prime ideal too ([2], 1.5.6).

Example 1. Let $I = (X_1^2 - 2, X_1 X_2 - 2) \subset R = K[X_1, X_2]$. We have:

$$I_0 = I \cap R_0 = (0)$$

$$I_1 = I \cap R_1 = (0)$$

$$I_2 = I \cap R_2 = \langle X_1^2 - X_1 X_2 \rangle,$$

in fact $f = X_1^2 - 2 - (X_1 X_2 - 2) = X_1^2 - X_1 X_2 \in I_2$. In general, for $i > 2$:

$$I_i = I \cap R_i = \langle g(X_1^2 - X_1 X_2) \rangle,$$

with $\deg(g) = i - 2$. It follows that $I^* = (X_1^2 - X_1 X_2) \subset I$.

We recall the following well-known property ([2]).

Proposition 1. Let R be a commutative noetherian graded domain and $I = (f) \subset R$ be a principal ideal. $I^* = (0) \iff f$ is not a homogeneous element.

The previous proposition is not true if R is not a domain as the following example shows.

Example 2. Let $R =_8 [X]$ be the graded ring with standard graduation and I be the ideal of R generated by the not homogeneous element $f = X - 4$. In I there are homogeneous elements, hence $I^* \neq (0)$. In fact, let $a = 2X \in R_1$ be a homogeneous element of R , we have $af = 2X^2 \in I$ that is a homogeneous element of degree two. Hence $2X^2 \in I^*$.

Remark 2. Let R be a graded ring that is not a domain and $\wp = (f)$ be a prime ideal of R generated by a not homogeneous element. Since \wp^* is a prime ideal, it follows that $\wp^* \neq (0)$ because (0) is not prime if R is not a domain. Hence in this case the condition \Leftarrow of the Proposition 1 is not true.

In [2] the following definitions related to the graded ideals of R are given.

Definition 2. Let R be a graded ring. A graded ideal m of R is called $*$ maximal if every graded ideal that properly contains m equals R .

Definition 3. A graded ring R is a $*$ local ring if it has a unique $*$ maximal ideal.

Example 3. Let R be a graded ring and \wp be a prime ideal of R . Let S be the set of the homogeneous elements of R that don't lie in \wp . S is a multiplicatively closed set. Denote $R_{(\wp)}$ the homogeneous localization of R . $R_{(\wp)}$ is a graded ring and the graduation is defined setting $(R_{(\wp)})_i = \{\frac{x}{a} \in R_{(\wp)} \mid \deg(\frac{x}{a}) = i\}$, where $\deg(\frac{x}{a}) = \deg(x) - \deg(a)$, with x and a homogeneous elements. Then $R_{(\wp)}$ is a $*$ local ring with $*$ maximal ideal $\wp^* R_{(\wp)}$.

Now we introduce the following definitions.

Definition 4. Let R be a graded ring. A sequence x_1, \dots, x_n of homogeneous elements of R is called $*$ regular sequence on R (or a homogeneous R -sequence) if the following conditions are satisfied:

- 1) (x_1, \dots, x_n) is a proper homogeneous ideal;
- 2) x_{i+1} is a regular homogeneous element in $R/(x_1, \dots, x_i)$, for $i = 0, \dots, n-1$.

Definition 5. Let R be a graded ring and E be a finitely generated graded R -module. A sequence $\underline{x} = \{x_1, \dots, x_n\}$ of homogeneous elements of R is called E - $*$ regular sequence (or a homogeneous E -sequence) if the following conditions are satisfied:

- 1) $\underline{x}E \neq E$;
- 2) x_i is a nonzero-divisor in $E/(x_1, \dots, x_{i-1})E$, for $i = 1, \dots, n$.

Definition 6. Let R be a graded ring, E be a finitely generated graded R -module and I be an ideal of R such that $IE \neq E$. The common length of the maximal homogeneous E -sequences in I^* is called the $*$ grade of I on E , denoted by $*$ grade(I, E).

A special situation occurs for $*$ local rings:

Definition 7. Let (R, m) be a graded noetherian ring and E be a finitely generated graded R -module. The $*$ grade of m on E is called the $*$ depth of E , denoted by $*$ depth(E).

Let I be an ideal of a graded ring R . It is customary to set: $*$ grade(I) = $*$ grade(R/I) = $*$ grade(I, R). More precisely we give the following:

Definition 8. The $*$ grade of I is the common length of the maximal homogeneous R -sequences in I^* , denoted $*$ grade(I).

Definition 9. I is a $*$ complete intersection if it can be generated by a number of homogeneous elements equal to its $*$ grade.

2. $*$ COMPLETE INTERSECTIONS

Let R be a graded ring.

Definition 10. Let $A = (a_{ji})$ be a $m \times n$ matrix whose entries are homogeneous elements of R , for $i = 1, \dots, n, j = 1, \dots, m$. A is called a graded matrix on R .

Definition 11. Let $A = (a_{ji})$ be a graded $m \times n$ matrix on R for $i = 1, \dots, n, j = 1, \dots, m$. The ideal of R generated by the $t \times t$ minors of A , $1 \leq t \leq \min(m, n)$ is called t -determinantal ideal, denoted $I_t(A)$.

By definition $I_0(A) = R$ and $I_t(A) = 0$ for $t > \min(m, n)$.

Remark 3.

- 1) If the entries of the matrix A are homogeneous elements of the same degree, $I_t(A)$ is a graded ideal.
- 2) If the elements of every column (row) are homogeneous of the same degree, then the ideal $I_t(A)$ is graded.

Remark 4. If the matrix A is associated to a graded homomorphism φ of free graded modules, then $I_t(\varphi)$ denotes the ideal $I_t(A)$.

Theorem 1. *The homogeneous linear forms of $R[Y_1, \dots, Y_m]$*

$$a_i = \sum_{j=1}^m a_{ji} Y_j, \quad 1 \leq i \leq n,$$

generate a $$ complete intersection of $*$ grade n if and only if the determinantal ideals of $A = (a_{ji})$ satisfy the conditions*

$$*\text{grade}(I_k(A)) \geq n - k + 1, \quad 1 \leq k \leq n,$$

where A is a graded matrix whose entries are homogeneous elements of R of the same degree d_i for each column.

Proof. Set $S = R[Y_1, \dots, Y_m]$. Let $a_i = \sum_{j=1}^m a_{ji} Y_j \in S$.

Being $(a_1, \dots, a_n) \subset (Y_1, \dots, Y_m)$ and $\text{grade}(Y_1, \dots, Y_m) = m$, then we suppose $n \leq m$.

\Rightarrow) Let (a_1, \dots, a_n) be a *complete intersection of *grade n in $R[Y_1, \dots, Y_m]$ with a_i homogeneous linear forms in the variables Y_j and bidegree $(g_i, 1)$. We consider the Koszul complex on a_1, \dots, a_n that is a graded complex. By [1] (Proposition 1) the assert follows.

\Leftarrow) We use induction on n and we prove that a_1, \dots, a_n generate a *complete intersection in S of *grade n .

$n = 1$: we prove that a_1 is a nonzero-divisor if and only if $\text{*grade}(a_{11}, a_{21}, \dots, a_{m1}) \geq 1$. If $\text{*grade}(a_{11}, a_{21}, \dots, a_{m1}) \geq 1$, then some homogenous element a_{j1} is nonzero-divisor. It follows that $a_{j1} Y_j$ is a nonzero-divisor. Choose an element $f \in S$ with $f \neq 0$, one has

$$fa_1 = f(a_{11}Y_1 + a_{21}Y_2 + \dots + a_{m1}Y_m) \neq 0$$

because at least $fa_{j1}Y_j \neq 0$. Then a_1 is a nonzero-divisor. Conversely, suppose a_1 nonzero-divisor. For all nonzero element $f \in S$, one has $fa_1 \neq 0$. It follows

$$fa_1 = f(a_{11}Y_1 + a_{21}Y_2 + \dots + a_{m1}Y_m) \neq 0$$

hence $fa_{j1}Y_j \neq 0$ for all f and for any $j = 1, \dots, m$. Being Y_j nonzero-divisor one has that each a_{j1} is nonzero-divisor and $\text{*grade}(a_{11}, a_{21}, \dots, a_{m1}) \geq 1$.

Suppose by induction that the assert is true for $n - 1$ homogeneous elements. Let \wp be a graded prime ideal of S associated to a maximal homogeneous S -regular sequence in (a_1, \dots, a_n) and containing a_i 's. Hence:

$$\text{*grade}(\wp) = \text{*grade}(\wp S_\wp) = \text{*grade}(a_1, \dots, a_n) S_\wp.$$

It is sufficient to prove that $\text{*grade}(\wp)$ can not be smaller than n . In fact from $\text{*grade}(\wp) \geq n$ it follows that $\text{*grade}(\wp) = n$. Assume that $\text{*grade}(\wp) < n$. By hypothesis for per $k = 1$ one has $\text{*grade}(I_1(A)) \geq n$, then some $a_{ji} = a$ is not in \wp . Consider the ring $R_{(a)}$, that is the homogeneous localization of R in the set of powers of a , with $a \notin \wp$. $R_{(a)}$ is a *local ring, whose graduation is given by $(R_{(a)})_i = \{\frac{x}{y} \in R_{(a)} \mid \deg(\frac{x}{y}) = i\}$, where $\deg(\frac{x}{y}) = \deg(x) - \deg(y)$ with x and y homogeneous elements of R (Example 3). Consider the graded ideal $J = (a_1, \dots, a_n)_{(a)}$ of $R_{(a)}$ and set a system of generators consisting of homogenous linear forms with coefficient in $R_{(a)}$. By elementary transformations over $R_{(a)}$ of the matrix A , which involve a linear change of the indeterminates Y_1, \dots, Y_m to T_1, \dots, T_m , we can assume that J is generated by T_1 and b_2, \dots, b_m , with $b_i = \sum_{j=2}^m b_{ji} T_j \in R_{(a)}[T_2, \dots, T_m]$ and $b_{ji} \in R_{(a)}$. We obtain a graded matrix $B = (b_{ji})$ whose entries are homogenous elements of the same degree in each column. Hence

$$I_k(A)_{(a)} = I_{k-1}(B),$$

with $I_k(A)_{(a)}$ graded ideal of $R_{(a)}$ obtained by a homogeneous localization of the graded ideal $I_k(A)$. Then:

$$*\text{grade}(I_{k-1}(B)) = *\text{grade}(I_k(A)_{(a)}) \geq *\text{grade}(I_k(A)) \geq n - k + 1,$$

hence $*\text{grade}(I_{k-1}(B)) \geq (n-1) - (k-1) + 1$.

By induction hypothesis one concludes that b_2, \dots, b_m generate a $*$ complete intersection of $*$ grade $n-1$ in $R(a)[T_2, \dots, T_m]$. It is enough to prove that T_1, b_2, \dots, b_m generate a $*$ complete intersection of grade n in $R(a)[T_2, \dots, T_m]$. Being T_1 a nonzero-divisor in $R(a)[T_2, \dots, T_m]$, it follows that T_1, b_2, \dots, b_m generate a $*$ complete intersection. It follows $*\text{grade}(a_1, \dots, a_n)_{(a)} = n$, hence

$$*\text{grade}(a_1, \dots, a_n)S_\varphi \geq n.$$

Being

$$*\text{grade}(\varphi) = *\text{grade}(a_1, \dots, a_n)S_\varphi$$

one has $*\text{grade}(\varphi) \geq n$, that is a contradiction. \square

Now, let $R = K[X_1, \dots, X_q]$ be the polynomial ring over a field K with standard graduation. Let $E = Rf_1 + \dots + Rf_m$ be a finitely generated graded R -module, $d_j = \deg(f_j)$ for $j = 1, \dots, m$, and $\text{Sym}_R(E) = \bigoplus_{t \geq 0} \text{Sym}_t(E)$ be its symmetric algebra.

If E has free graded presentation:

$$\bigoplus_{i=1}^n R(-c_i) \xrightarrow{\varphi} \bigoplus_{j=1}^m R(-d_j) \xrightarrow{\psi} E \rightarrow 0,$$

with $\varphi = (a_{ji})$ graded matrix on R , then the kernel J of the surjective homomorphism $\text{Sym}(\psi)$ induced by ψ

$$\text{Sym}(\psi) : R[Y_1, \dots, Y_m] \rightarrow \text{Sym}_R(E) \rightarrow 0$$

is generated by linear forms in the variables Y_j

$$a_i = \sum_{j=1}^m a_{ji} Y_j,$$

for $i = 1, \dots, n$ and such that $\sum_{j=1}^m a_{ji} f_j = 0$, $1 \leq i \leq n$.

Hence the symmetric algebra $\text{Sym}_R(E)$ has a presentation $R[Y_1, \dots, Y_m]/J$. The ring $R[Y_1, \dots, Y_m] = K[X_1, \dots, X_q; Y_1, \dots, Y_m]$ is bigraded by $\text{bideg}(X_i) = (1, 0)$ for $i = 1, \dots, q$ and $\text{bideg}(Y_j) = (0, d_j)$ for $j = 1, \dots, m$.

Remark 5. If E is a free graded R -module of rank m , then its symmetric algebra is

$$\text{Sym}_R(E) \cong R[Y_1, \dots, Y_m],$$

with $E = \bigoplus_j R^m(-d_j)$.

By Theorem 1 it follows the following result for $\text{Sym}_R(E)$.

Corollary 1. *Let $E = Rf_1 + \cdots + Rf_m$ be a finitely generated graded module on $R = K[X_1, \dots, X_q]$, with $d_j = \deg(f_j)$ for $j = 1, \dots, m$. Let*

$$F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\psi} E \rightarrow 0,$$

be a free graded presentation of E with F_0 and F_1 free graded modules of rank n and m , respectively, and $\varphi = (a_{ji})$ be a graded matrix whose columns have homogenous elements of R of the same degree. The kernel of the canonical epimorphism

$$\text{Sym}(\psi) : R[Y_1, \dots, Y_m] \rightarrow \text{Sym}_R(E) \rightarrow 0$$

*is a *complete intersection of *grade n in $R[Y_1, \dots, Y_m]$ if and only if*

$$*\text{grade}(I_k(\varphi)) \geq n - k + 1, \quad 1 \leq k \leq n.$$

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