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STABILITY OF MAXIMUM PRESERVING FUNCTIONAL EQUATIONS ON BANACH LATTICES

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Abstract. Analogous functional equations are studied by substituting in the Cauchy (linear) functional equation the addition with the maximum operation, to be called *maximum preserving functional equations*. The Hyers-Ulam stability problem thus is posed. We propose some sufficient conditions to solve it.

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1. INTRODUCTION

The most famous functional equation by Cauchy and known as *linear functional equation* reads:

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

We would point out that equation (1.1) has been investigated for many spaces and in various perspectives such as its stability which has been intensively considered in the literature. The stability problem was first posed by M. Ulam (see [31]) in the terms: "Give conditions in order for a linear mapping near an approximately linear mapping to exist."

Given two Banach algebras E and $E^{'}$ a transformation $f : E \to E^{'}$ is called δ-linear if

$$\|f(x+y) - f(x) - f(y)\| < \delta, \tag{1.2}$$

for all $x, y \in E$. The stability problem (of equation (1.1)) can be stated as follows. Does there exist for each $\varepsilon \in (0, 1)$ some $\delta > 0$ such that to each δ-linear transformation $f : E \to E'$ there corresponds a linear transformation $l : E \to E'$ satisfying the inequality $\|f(x) - l(x)\| < \varepsilon$ for all $x \in E$? This question was answered in the affirmative by Hyers [15]. Ever since various problems of stability on various spaces have come to light. We shall just list few of them: the stability of linear functional equation [4, 6], quadratic and cubic functional equations [8, 20, 23, 25, 27, 30], Jensen and Cauchy-Jensen functional equations [18], pexiderial functional equation [24, 29],

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non-Archimedian functional equation [17], functional differential equation [21], derivations and linear functions [5,22], entropy equation [9–12], functional inequalities [7, 13, 14, 16, 19, 26, 28].

Since the Cauchy linear functional equation (1.1) is an addition preserving equation, its Hyers-Ulam stability problem can be extended to the so-called *maximum preserving functional equation*, where the addition operation in (1.1) is replaced with the maximum operation on a given Banach lattice. The object of the present article is thus to formulate and solve some stability problems for this equation.

The motivation of studying the stability of maximum preserving functional equations lays in the optimal average (see [1–3], say) which, as a cone-related functional, is semi-homogeneous, maximum preserving and continuous from below (cf. [1]).

2. The main results

We would like to point out the similitude between the present communication and the result in [4].

If $\mathcal{B}$ is a Banach lattice, then $\mathcal{B}^+$ will stand for its positive cone, i.e. $\mathcal{B}^+ = \{x \in \mathcal{B} : x \geq 0\} = \{|x| : x \in \mathcal{B}\}$.

Given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$ we say that a functional $F : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if $F\left(\mathcal{X}^+\right) = \{F(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$.

Some Properties. Let be given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$ and, a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$.

P1. **Maximum Preserving Functional Equation:** $F(|x| \vee |y|) = F(|x|) \vee F(|y|)$ for all members $x, y \in \mathcal{X}$.

P2. **Semi-homogeneity:** $F(\alpha |x|) = \alpha F(|x|)$ for all $x \in \mathcal{X}$ and every number $\alpha \in [0, \infty)$.

P3. **Continuity From Below On The Positive Cone:** The identity $\lim_{n\to\infty} F(x_n) = F\left(\lim_{n\to\infty} x_n\right)$ holds for every increasing sequence $(x_n) \subset \mathcal{X}^+$.

P4. For any increasing sequence $(x_k) \subset \mathcal{X}^+$ the inequality hereafter holds

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{F(2^n x_k)}{2^n} \leq \lim_{k\to\infty} \lim_{n\to\infty} \frac{F(2^n x_k)}{2^n}, \tag{2.1}$$

provided that the limits exist.

We should note that the maximum preserving functional equation is known as a *join homomorphism* in Lattice Theory.

Remark 1. Given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$ let a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ satisfy property **P1**. Then the following statements are valid.

(1) $F(|x \vee y|) \leq F(|x|) \vee F(|y|)$ for all members $x, y \in \mathcal{X}$.
(2) The semi-homogeneity implies that $F(0) = 0$.

(3) F is an increasing operator, in the sense that if $x, y \in \mathcal{X}$ are such that $|x| \leq |y|$, then $F(|x|) \leq F(|y|)$.

Theorem 1. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F(\tau|x| \vee \eta|y|) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \leq \vartheta \tag{2.2}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality*

$$\frac{\|T(|x|) - F(|x|)\|}{\|x\|^{\alpha}} \leq \frac{2\vartheta}{2 - 2^{\alpha}} \tag{2.3}$$

for every $x \in \mathcal{X}$.

*Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

Each of the following theorem is a variation of the above result.

Theorem 2. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ with $p(0) = 0$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F(\tau|x| \vee \eta|y|) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) + p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \leq \vartheta \tag{2.4}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality (2.3) is valid for every $x \in \mathcal{X}$. Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

Theorem 3. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F\left(\frac{\tau p(\tau)|x| \vee \eta p(\eta)|y|}{p(\tau) \vee p(\eta)} \right) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \leq \vartheta \tag{2.5}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality (2.3) is valid for every $x \in \mathcal{X}$. Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

Theorem 4. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ with $p(0) = 0$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F\left(\frac{\tau p(\tau)|x| \vee \eta p(\eta)|y|}{p(\tau)+p(\eta)}\right) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau)+p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \le \vartheta \tag{2.6}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality (2.3) is valid for every $x \in \mathcal{X}$. Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

We point out that the proof of Theorem 1 can be suitably adapted to show the validity of Theorems 2, 3, 4.

It is worth to ask the question: *Under what conditions inequalities (2.2) and (2.4)-(2.6) hold true*? The answer is formulated in the following results without proof, because of their easiness (in fact, only the triangle inequality of the norm is needed to check their validity).

Lemma 1. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ and define the functional $F_c : \mathcal{X} \to \mathcal{Y}$ by $F_c(x) = F(x) \wedge c$, where $c \in \mathcal{Y}^{+}$. Let $\alpha \in (0, 1)$ be some number, $\mathcal{Y}_0 \subset \mathcal{Y}^{+}$ be some non-empty subset and consider the following 4 quantities:*

$$\sup_{c \in \mathcal{Y}_0} \sup_{\tau, \eta \in [0,\infty)} \sup_{x, y \in \mathcal{X}} \frac{\| F_c(\tau |x| \vee \eta |y|) - F(\tau |x| \vee \eta |y|) \|}{\|x\|^{\alpha} + \|y\|^{\alpha}},$$

$$\sup_{c \in \mathcal{Y}_0} \sup_{\tau, \eta \in [0,\infty)} \sup_{x, y \in \mathcal{X}} \frac{\left\| \frac{\tau p(\tau) F_c(|x|) \vee \eta p(\eta) F_c(|y|)}{p(\tau) \vee p(\eta)} - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}},$$

$$\sup_{c \in \mathcal{Y}_0} \sup_{\tau, \eta \in [0,\infty)} \sup_{x, y \in \mathcal{X}} \frac{\left\| F_c(\tau |x| \vee \eta |y|) - \frac{\tau p(\tau) F_c(|x|) \vee \eta p(\eta) F_c(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}}$$

and

$$\sup_{\tau, \eta \in [0,\infty)} \sup_{x, y \in \mathcal{X}} \frac{\left\| F(\tau |x| \vee \eta |y|) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

Lemma 2. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ and define the functional $F_c : \mathcal{X} \to \mathcal{Y}$ by $F_c(x) = F(x) \wedge c$, where $c \in \mathcal{Y}^{+}$. Let $\alpha \in (0, 1)$*

be some number and $\mathcal{Y}_0 \subset \mathcal{Y}^+$ *some non-empty subset and consider the following 4 quantities:*

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\|F_c(\tau|x|\vee\eta|y|)-F(\tau|x|\vee\eta|y|)\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F_c(\tau|x|\vee\eta|y|)-\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F(\tau|x|\vee\eta|y|)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

Lemma 3. *Let be given a continuous function* $p:[0,\infty)\to(0,\infty)$ *and, two Banach lattices* $\mathcal{X}$ *and* $\mathcal{Y}$*. Consider a cone-related functional* $F:\mathcal{X}\to\mathcal{Y}$ *and define the functional* $F_c:\mathcal{X}\to\mathcal{Y}$ *by* $F_c(x)=F(x)\wedge c$*, where* $c\in\mathcal{Y}^+$*. Let* $\alpha\in(0,1)$ *be some number and* $\mathcal{Y}_0\subset\mathcal{Y}^+$ *some non-empty subset and consider the following 4 quantities:*

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)-F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)\vee p(\eta)}-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)-\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}}$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

Lemma 4. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ and define the functional $F_c : \mathcal{X} \to \mathcal{Y}$ by $F_c(x) = F(x) \wedge c$, where $c \in \mathcal{Y}^+$. Let $\alpha \in (0,1)$ be some number and $\mathcal{Y}_0 \subset \mathcal{Y}^+$ some non-empty subset and consider the following 4 quantities:*

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\| F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right) - F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right)\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\| \frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)} - \frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\| F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right) - \frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\| F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right) - \frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

3. Proof of Theorem 1

Proof. We first show by induction that for any fixed $x \in \mathcal{X}$,

$$\frac{\left\| \frac{F(2^n|x|)}{2^n} - F(|x|)\right\|}{\|x\|^\alpha} \le \vartheta \sum_{j=0}^{n-1} 2^{j(\alpha-1)} \tag{3.1}$$

whenever $n \in \mathbb{N}$. In fact, for $n = 1$ the statement is obvious by choosing $\tau = \eta = 2$ and $x = y$ in inequality (2.2). Suppose the statement is true for $n = k$. Let us prove it for $n = k + 1$. In fact, let $2x$ replace x and $n = k$ in inequality (3.1) and observe that

$$\frac{\left\| \frac{F(2^k 2|x|)}{2^k} - F(2|x|)\right\|}{\|2x\|^\alpha} \le \vartheta \sum_{l=0}^{k-1} 2^{l(\alpha-1)}.$$

Hence

$$\frac{\left\| \frac{F(2^{(k+1)}|x|)}{2^{(k+1)}} - \frac{1}{2}F(2|x|)\right\|}{\|x\|^\alpha} \le \vartheta 2^{(\alpha-1)} \sum_{l=0}^{k-1} 2^{l(\alpha-1)} \le \vartheta \sum_{l=1}^{k} 2^{l(\alpha-1)}.$$

The triangle inequality yields

$$\left\| \frac{F\left(2^{(k+1)}|x|\right)}{2^{k+1}} - F(|x|) \right\| \leq \left\| \frac{F\left(2^{(k+1)}|x|\right)}{2^{k+1}} - \frac{F(2|x|)}{2} \right\| + \left\| \frac{F(2|x|)}{2} - F(|x|) \right\| \leq$$

$$\leq \left(\vartheta \sum_{j=1}^{k} 2^{j(\alpha-1)} + \vartheta \right) \|x\|^{\alpha} = \vartheta \|x\|^{\alpha} \sum_{j=0}^{k} 2^{j(\alpha-1)}.$$

We have just shown the validity of inequality (3.1) for every $n \in \mathbb{N}$. Since the geometric series $\sum_{l=0}^{\infty} 2^{l(\alpha-1)} = \frac{2}{2-2^{\alpha}}$, $(0 \leq \alpha < 1)$, we obtain that

$$\frac{\left\| \frac{F(2^n|x|)}{2^n} - F(|x|) \right\|}{\|x\|^{\alpha}} \leq \frac{2\vartheta}{2-2^{\alpha}}, \ 0 \leq \alpha < 1. \tag{3.2}$$

Next, note that for all $m > n > 0$ and making the change of variable $y = 2^n x$ we have

$$\left\|2^{-m} F\left(2^m |x|\right) - 2^{-n} F\left(2^n |x|\right)\right\| = 2^{-n} \left\|2^{-m+n} F\left(2^m |x|\right) - F\left(2^n |x|\right)\right\| =$$

$$= 2^{-n} \left\|2^{-m+n} F\left(2^{m-n} |y|\right) - F(|y|)\right\| \leq 2^{-n} \frac{2\vartheta}{2-2^{\alpha}} \|y\|^{\alpha} =$$

$$= 2^{-n(1-\alpha)} \frac{2\vartheta}{2-2^{\alpha}} \|x\|^{\alpha}.$$

Consequently, passing to the limit yields,

$$\lim_{n\to\infty} \left\|2^{-m} F\left(2^m |x|\right) - 2^{-n} F\left(2^n |x|\right)\right\| = 0.$$

Since $\mathcal{Y}$ is a Banach space, we can thus conclude that the sequence $\left(\frac{F(2^n|x|)}{2^n}\right) \subset \mathcal{Y}^+$ converges in the $\mathcal{Y}$-norm. Now, define the mapping $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(|x|) := \lim_{n\to\infty} \frac{F(2^n |x|)}{2^n} \tag{3.3}$$

Clearly, T is a cone-related operator. Let us show that T is maximum preserving. In fact, letting $\tau = \eta = 2^n$ in (2.2) yields

$$\left\| F\left(2^n (|x| \vee |y|)\right) - 2^n (F(|x|) \vee F(|y|)) \right\| \leq \vartheta \left(\|x\|^{\alpha} + \|y\|^{\alpha}\right).$$

Substituting x with $2^n x$ and y with $2^n y$ in this last inequality one can get

$$\left\| F\left(4^n (|x| \vee |y|)\right) - 2^n \left(F\left(2^n |x|\right) \vee F\left(2^n |y|\right)\right) \right\| \leq 2^{n\alpha} \vartheta \left(\|x\|^{\alpha} + \|y\|^{\alpha}\right)$$

which yields

$$4^{-n} \left\| F\left(4^n (|x| \vee |y|)\right) - 2^n \left(F\left(2^n |x|\right) \vee F\left(2^n |y|\right)\right) \right\| =$$

$$= \left\|4^{-n} F\left(4^n (|x| \vee |y|)\right) - 2^{-n} \left(F\left(2^n |x|\right) \vee F\left(2^n |y|\right)\right)\right\| \leq 2^{n(\alpha-2)} \vartheta \left(\|x\|^{\alpha} + \|y\|^{\alpha}\right).$$

Consequently, passing to the limit yields

$$\|T(|x|\vee|y|)-T(|x|)\vee T(|y|)\|=0,$$

$x,y\in\mathcal{X}$ or equivalently

$$T(|x|\vee|y|)=T(|x|)\vee T(|y|),$$

$x,y\in\mathcal{X}$, because

$$\lim_{n\to\infty}4^{-n}F\left(4^{n}|z|\right)=\lim_{m\to\infty}2^{-m}F\left(2^{m}|z|\right),\ z\in\mathcal{X}.$$

Next, we show the validity of the identity $T(\alpha|x|)=\alpha T(|x|)$ for all $x\in\mathcal{X}$ and every number $\alpha\in[0,\infty)$. In fact, in inequality (2.2) choose $\eta=\tau$, $y=0$ and substitute $2^{n}\tau$ for τ to observe via Remark 1/(2) that

$$\left\|F\left(2^{n}\tau|x|\right)-2^{n}\tau F(|x|)\right\|\leq\vartheta\|x\|^{\alpha}.$$

For all $x\in\mathcal{X}$ and every number $\alpha\in[0,\infty)$. This inequality can be transformed as

$$\left\|F\left(4^{n}\tau|x|\right)-2^{n}\tau F\left(2^{n}|x|\right)\right\|\leq\vartheta 2^{n\alpha}\|x\|^{\alpha}$$

if we replace x with $2^{n}x$. Consequently,

$$\left\|\frac{F(4^{n}\tau|x|)}{4^{n}}-\tau\frac{F(2^{n}|x|)}{2^{n}}\right\|\leq\vartheta 2^{-n(2-\alpha)}\|x\|^{\alpha}.$$

Hence on the one hand,

$$\lim_{n\to\infty}\frac{F(4^{n}\tau|x|)}{4^{n}}=\tau\lim_{n\to\infty}\frac{F(2^{n}|x|)}{2^{n}}=\tau T(|x|)$$

and on the other hand by changing the variable $z=\tau x$

$$\lim_{n\to\infty}\frac{F(4^{n}\tau|x|)}{4^{n}}=\lim_{n\to\infty}\frac{F(4^{n}|z|)}{4^{n}}=T(|z|)=T(\tau|x|).$$

Therefore, the semi-homogeneity holds true. In the next step taking the limit in (3.2) yields (2.3). Further, let us show the unicity. In fact, assume the existence of another such cone-related functional G such that $\mathcal{S}:=\{x\in\mathcal{X}:G(|x|)\neq T(|x|)\}\neq\varnothing$. Then (2.3) implies the existence of some $\vartheta_0>0$ and $\beta\in[0,1)$ such that for each $x\in\mathcal{S}$,

$$\|G(|x|)-T(|x|)\|\leq\vartheta_0\|x\|^{\beta}.\tag{3.4}$$

One can easily deduce from the semi-homogeneity of G and T that $kx\in\mathcal{S}$ for every $k\in\mathbb{N}$ whenever $x\in\mathcal{S}$. Taking into account (3.4) one can easily deduce by the triangle inequality and the semi-homogeneity that

$$\|G(|x|)-T(|x|)\|=n^{-1}\|G(|nx|)-T(|nx|)\|\leq n^{\alpha-1}\vartheta\|x\|^{\alpha}+n^{\beta-1}\vartheta_0\|x\|^{\beta}$$

which would imply in the limit that $\|G(|x|)-T(|x|)\|=0$, or equivalently $G(|x|)=T(|x|)$. This would mean that $x\in\mathcal{S}$ implies $x\notin\mathcal{S}$, or equivalently $\mathcal{S}\cap\mathcal{S}=\varnothing$, i.e. $\mathcal{S}=\varnothing$, a contradiction, indeed. Finally, let us prove the moreover-part. In fact, assuming that F satisfies property **P4**, pick arbitrarily an increasing sequence

$(|x_k|) \subset \mathcal{X}^+$ with limit $|x| \in \mathcal{X}^+$. Then by (3.3), the monotonicity of T and the continuity from below of F we have

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n} = \lim_{k\to\infty} T\left(|x_k|\right) \le T\left(|x|\right) = \lim_{n\to\infty}\frac{F\left(2^n|x|\right)}{2^n} = \\ = \lim_{n\to\infty}\lim_{k\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n}.$$

Thus

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n} \le T\left(|x|\right) = \lim_{n\to\infty}\lim_{k\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n}. \tag{3.5}$$

By the conjunction of both inequalities (2.1) and (3.5), it follows that operator T is continuous from below. To end the proof of the moreover-part we simply note that the reverse conditional is trivial. Therefore, we can conclude on the validity of the argument. □

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