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ON POLYNOMIAL APPROXIMATIONS TO SOLUTIONS OF IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract. In this paper the possibility to present by a polynomial an independent variable for the approximate solutions of the systems of implicit ordinary differential equations under multi-point boundary conditions is substantiated.

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1. Introduction

There is a large number of methods which mathematicians elaborated for studying boundary value problems (BVPs). In the papers [1], [2] the numerical-analytic method based upon successive approximations was introduced. The polynomial version of this method in which the successive approximations are polynomials was proposed in [1] and then developed in [3], [4] for three- and multi-point boundary conditions. In this paper the issue of existence and approximate construction of the solutions of multi-point boundary conditions for the systems of implicit ordinary differential equations of the first order are studied by using polynomial approximations.

2. Construction of successive polynomial approximations

Let us consider a system of implicit equations

$$\frac{dx}{dt} = f(t, x, \frac{dx}{dt}), \tag{2.1}$$

with a multi-point boundary conditions

$$A_0 x(0) + \sum_{k=1}^{q} A_k x(t_k) + A_{q+1} x(T) = d, \tag{2.2}$$

where $x, d \in R^n$, $\quad f : [0, T] \times D_1 \times D_2 \to R^n$, D_1, D_2 are closed bounded domains in R^n, $0 = t_0 < t_1 < \ldots < t_q < t_{q+1} = T$, $\; A_k \quad (k = 0, 1, \ldots, q+1)$ - are $n \times n$ matrices so that $\det\left[\sum_{k=1}^{q=1} A_k t_k\right] \neq 0$.

First of all we will introduce some notations [1].

It is known that for $f(t) \in C\,[0,T]$ there is a unique polynomial $P_m^0(t)$ among all the polynomials $P_m(t)$ with no more than m degree which is the best approximation for $f(t)$:

$$E_m(f) \equiv \left\| f(t) - P_m^0(t) \right\| = \inf_{P_m(t)} \left\| f(t) - P_m(t) \right\|.$$

Let us set in the interval $[0,T]$ the nodes

$$\tau_i = \frac{T}{2}\left(\cos\frac{(2i-1)\,\pi}{2(p+1)} + 1\right), \qquad i = 1, 2, \ldots, p+1, \tag{2.3}$$

which are obtained by the substitution $\tau = \frac{T}{2}\,(\tau' + 1)$ from the corresponding zeroes $\tau_i' \in [-1,1]$ of the Chebyshev polynomials

$$T_{p+1}(t) = \cos\left((p+1)\arccos t\right).$$

For arbitrary continuous function $x_r(t)$ by $f^p\left(t, x_r(t), y_r(t)\right)$ we denote the Lagrange interpolation polynomial with p degree and with respect to the points (2.3):

$$f^p\left(t, x_r(t,x_0), y_r(t,x_0)\right) = \left(f_1^p\left(t, x_r(t,x_0), y_r(t,x_0)\right), \ldots, f_n^p\left(t, x_r(t,x_0), y_r(t,x_0)\right)\right),$$

where $y_r(t) := \frac{dx_r(t)}{dt}$, $f_j^p\left(t, x_r(t,x_0), y_r(t,x_0)\right) = a_{0j}^r + a_{1j}^r t + \ldots + a_{pj}^r$, $j = 1, 2, \ldots, n$, $f_j^p\left(\tau_i, x_r(\tau_i), y_r(\tau_i)\right) = f_j(\tau_i, x_r(\tau_i), y_r(\tau_i)$, $i = 1, 2, \ldots, p+1$.

Let us denote by

$$\overline{\mathcal{L}}\left(f, x, y, t, x_0\right) = f\left(t, x(t,x_0), y(t,x_0)\right) - \frac{1}{T}\int_0^T f\left(s, x(s,x_0), y(s,x_0)\right) ds,$$

$$\mathcal{L}\left(f, x, y, t, x_0\right) = \int_0^t \left(f\left(\tau, x(\tau,x_0), y(\tau,x_0)\right) - \frac{1}{T}\int_0^T f\left(s, x(s,x_0), y(s,x_0)\right) ds \right) d\tau.$$

We assume that the following conditions hold for the BVP (2.1), (2.2):

a) the vector-function $f(t,x,y)$ is continuous in $\Omega = [0,T] \times D_1 \times D_2$ (and therefore it is bounded by some vector M) and Lipschitzian in x and y, i.e.,

$$|f(t,x,y)| \leq M, \; |f(t,x,y) - f(t,\overline{x},\overline{y})| \leq K_1\,|x - \overline{x})| + K_2\,|y - \overline{y})|, \tag{2.4}$$

where M and $n \times n$ matrices K_1, K_2 have non-negative components. The absolute value sign and the inequalities we understand component-wise;

b) domains D_1 and D_2 satisfy the conditions

$$D_{\beta_1} := \{x \in R^n \mid B\left(x, \beta_1(x)\right) \subset D_1\} \neq \varnothing, \quad B\left(0, \beta_2(x)\right) \subset D_2,$$

where $B\left(x,\rho(x)\right)$ is the ball of radius $\rho(x)$ with center x and

$$\beta_1(x) = \left(\frac{T}{2}E+G\right)\cdot\left(M'+L_p\right)+T\left|d(x)\right|, \quad G=T\cdot\sum_{k=1}^{q}\left|HA_k\right|\cdot\alpha_1\left(t_k\right),$$

$$\beta_2(x) = 2\left(M+L_p\right)+\frac{1}{T}G\left(M'+L_p\right)+\left|d(x)\right|, \quad H=\left[\sum_{k=1}^{q+1}A_kt_k\right]^{-1},$$

$$d(x)=H\cdot\left(d-\sum_{k=0}^{q+1}A_kx\right), \quad \alpha_1(t)=2t\left(1-\frac{t}{T}\right),$$

$$M'=\frac{1}{2}\left[\max_{(t,x,y)\in\Omega}f(t,x,y)-\min_{(t,x,y)\in\Omega}f(t,x,y)\right],$$

$$L_p=(5+\lg p)\,\max_r E_p\left(f\left(t,x_r^{p+1}\left(t,x_0\right),y_r^p\left(t,x_0\right)\right)\right)=$$

$$=(5+\lg p)\cdot\left(\max_r E_p\left(f_1\left(t,x_r^{p+1}\left(t,x_0\right),y_r^p\left(t,x_0\right)\right)\right),\dots\right.$$

$$\left.\dots,\max_r E_p\left(f_n\left(t,x_r^{p+1}\left(t,x_0\right),y_r^p\left(t,x_0\right)\right)\right)\right);$$

c) the eigenvalues $\lambda_j(Q)$ of the matrix $Q=K_1\left(\frac{T}{2}E+G\right)+K_2\left(2E+\frac{1}{T}G\right)$ satisfy the inequalities

$$\left|\lambda_j(q)\right|<1, \quad j=1,\dots,n. \tag{2.5}$$

Let us introduce the sequence of polynomials with $p+1$ degree

$$\begin{aligned} x_m^{p+1}(t,x_0)&=x_0+\mathcal{L}\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t,x_0\right)+tHd(x_0)-\\ &-tH\sum_{k=1}^{q}A_k\mathcal{L}\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t_k,x_0\right),\ x_0^{p+1}\left(t,x_0\right)=x_0,\ m=1,2,\dots \end{aligned} \tag{2.6}$$

Their derivatives look as follows:

$$\begin{aligned} y_m^p(t,x_0)&=\overline{\mathcal{L}}\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t,x_0\right)+Hd(x_0)-\\ &-H\sum_{k=1}^{q}A_k\mathcal{L}\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t_k,x_0\right),\ y_0^p\left(t,x_0\right)=0,\ m=1,2,\dots \end{aligned} \tag{2.7}$$

Here the above index means that this expression is a polynomials of a correspondent degree. It is easy to see that all the members of the sequence (2.6) satisfy the boundary condition (2.2) for arbitrary $x_0\in D_{\beta_1}$.
The next theorem establishes the convergence of the sequence (2.6) and the properties of the limit functions.

Theorem 1. *Let BVP (2.1), (2.2) satisfy the conditions a)-c). Then:*

(1) *the sequences (2.6) and (2.7) converge to the functions $x^*(t,x_0)$ and $y^*(t,x_0)$, respectively, as $m\to\infty$, uniformly in $(t,x_0)\in[0,T]\times D_{\beta_1}$:*

$$x^*(t,x_0)=\lim_{m\to\infty}x_m^{p+1}(t,x_0), \quad y^*(t,x_0)=\lim_{m\to\infty}y_m^p(t,x_0),$$

where $y^*(t,x_0) = \frac{dx^*(t,x_0)}{dt}$;

(2) *the limit function* $x^*(t,x_0)$ *satisfies the "perturbed" BVP*

$$\begin{cases} \frac{dx}{dt} = f\left(t,x,\frac{dx}{dt}\right) + \Delta(x_0), \\ A_0x(0) + \sum\limits_{k=1}^{q} A_k x(t_k) + A_{q+1}x(T) = d, \end{cases} \tag{2.8}$$

where

$$\Delta(x_0) = -\frac{1}{T}\int\limits_0^T f^p\left(s, x^*\left(s,x_0\right), y^*\left(s,x_0\right)\right)ds + Hd(x_0) - \\ -H\sum_{k=1}^{q} A_k \mathcal{L}\left(f^p, x^*, y^*, t_k, x_0\right), \tag{2.9}$$

with the initial value $x^*\left(0,x_0\right) = x_0$;

(3) *the following error estimations hold:*

$$\left|x^*\left(t,x_0\right) - x_m^{p+1}\left(t,x_0\right)\right| \le \left(\alpha_1(t)E + G\right)\cdot W_{m-1}^p, \tag{2.10}$$

$$\left|y^*\left(t,x_0\right) - y_m^{p}\left(t,x_0\right)\right| \le \left(2E + \frac{1}{T}G\right)\cdot W_{m-1}^p, \tag{2.11}$$

where

$$W_{m-1}^p = \left[\sum_{i=0}^{m-1} Q^i\right]\cdot L_p + Q^{m-1}(E-Q)^{-1}\cdot \\ \cdot\left[K_1\left\{\left(\tfrac{T}{2}E + G\right)M' + T\left|d(x_0)\right|\right\} + K_2\left\{2M + \tfrac{1}{T}GM' + \left|d(x_0)\right|\right\}\right].$$

Proof. In addition to (2.6), (2.7) let us introduce the sequence of functions.

$$x_m(t,x_0) = x_0 + \mathcal{L}\left(f, x_{m-1}, y_{m-1}, t, x_0\right) + tHd(x_0) - \\ -tH\sum_{k=1}^{q} A_k\mathcal{L}\left(f, x_{m-1}, y_{m-1}, t_k, x_0\right), \quad x_0\left(t,x_0\right) = x_0, \quad m = 1,2,\ldots, \tag{2.12}$$

$$y_m(t,x_0) := \frac{dx_m(t,x_0)}{dt} = \overline{\mathcal{L}}\left(f, x_{m-1}, y_{m-1}, t, x_0\right) + Hd(x_0) - \\ -H\sum_{k=1}^{q} A_k\mathcal{L}\left(f, x_{m-1}, y_{m-1}, t_k, x_0\right), \quad y_0\left(t,x_0\right) = 0, \quad m = 1,2,\ldots \tag{2.13}$$

Also we introduce some notations:

$$x_m := x_m(t,x_0),\ x_m^{p+1} := x_m^{p+1}(t,x_0),\ r_{m+1}(t,x_0) := \left|x_{m+1}(t,x_0) - x_m(t,x_0)\right|,$$

$$y_m := y_m(t,x_0),\ y_m^{p} := y_m^{p}(t,x_0),\ \widehat{r}_{m+1}(t,x_0) := \left|y_{m+1}(t,x_0) - y_m(t,x_0)\right|.$$

We note [1] that

$$\left|f^p(t,x_m^{p+1},y_m^p) - f(t,x_m^{p+1},y_m^p)\right| \le L_p, \tag{2.14}$$

and making use of (2.4) we get

$$\left|f^p(t,x_m^{p+1},y_m^p) - f(t,x_m,y_m)\right| \le \left|f^p(t,x_m^{p+1},y_m^p) - f(t,x_m^{p+1},y_m^p)\right| + \\ + \left|f(t,x_m^{p+1},y_m^p) - f(t,x_m,y_m)\right| \le L_p + K_1\left|x_m^{p+1} - x_m\right| + K_2\left|y_m^p - y_m\right|. \tag{2.15}$$

Using Lemma 3 of [5] we have that

$$|\mathcal{L}\left(f,x,y,t,x_0\right)| \le \alpha_1(t)M^{'} \le \frac{T}{2}M^{'}, \tag{2.16}$$

$$\left|TH\sum_{k=1}^{q}A_k\mathcal{L}\left(f,x,y,t_k,x_0\right)\right| \le GM^{'}, \tag{2.17}$$

$$\left|\mathcal{L}\left(f^p,x_m^{p+1},y_m^p,t,x_0\right)-\mathcal{L}\left(f,x_m^{p+1},y_m^p,t,x_0\right)\right| \le \alpha_1(t)L_p, \tag{2.18}$$

$$\left|TH\sum_{k=1}^{q}A_k\left[\mathcal{L}\left(f^p,x_m^{p+1},y_m^p,t_k,x_0\right)-\mathcal{L}\left(f,x_m^{p+1},y_m^p,t_k,x_0\right)\right]\right| \le GL_p. \tag{2.19}$$

We have to show that (2.6) is a Cauchy sequence in the space of continuous vector functions. To begin with, we establish for arbitrary $(t,x_0)\in[0,T]\times D_{\beta_1}$, and $m=0,1,2,\dots$ that $x_m^{p+1}(t,x_0)\in D_1$ and $y_m^p(t,x_0)\in D_2$ by using (2.16)-(2.19):

$$\left|x_1^{p+1}-x_0\right| \le \left|\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t,x_0\right)\right| + \left|TH\sum_{k=1}^{q}A_k\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t_k,x_0\right)\right| +$$

$$+T\left|d(x_0)\right| \le \left|\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t,x_0\right)-\mathcal{L}\left(f,x_0,y_0,t,x_0\right)\right| + \left|\mathcal{L}\left(f,x_0,y_0,t,x_0\right)\right| +$$

$$+T\left|d(x_0)\right| + \left|TH\sum_{k=1}^{q}A_k\left[\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t_k,x_0\right)-\mathcal{L}\left(f,x_0,y_0,t_k,x_0\right)\right]\right| +$$

$$+\left|TH\sum_{k=1}^{q}A_k\mathcal{L}\left(f,x_0,y_0,t_k,x_0\right)\right| \le \left(\alpha_1(t)E+G\right)\left(L_p+M'\right)+T\left|d(x_0)\right| \le \beta_1(x_0),$$

$$+T\left|d(x_0)\right| \le \left|\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t,x_0\right)-\mathcal{L}\left(f,x_0,y_0,t,x_0\right)\right| + \left|\mathcal{L}\left(f,x_0,y_0,t,x_0\right)\right| +$$

$$+T\left|d(x_0)\right| + \left|TH\sum_{k=1}^{q}A_k\left[\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t_k,x_0\right)-\mathcal{L}\left(f,x_0,y_0,t_k,x_0\right)\right]\right| +$$

$$+\left|TH\sum_{k=1}^{q}A_k\mathcal{L}\left(f,x_0,y_0,t_k,x_0\right)\right| \le \left(\alpha_1(t)E+G\right)\left(L_p+M'\right)+T\left|d(x_0)\right| \le \beta_1(x_0),$$

$$|y_1^p| \le \left|\overline{\mathcal{L}}\left(f^p,x_0^{p+1},y_0^p,t,x_0\right)\right| + |d(x_0)| + \left|H\sum_{k=1}^{q}A_k\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t_k,x_0\right)\right| \le$$

$$\le \left|\overline{\mathcal{L}}\left(f^p,x_0^{p+1},y_0^p,t,x_0\right)-\overline{\mathcal{L}}\left(f,x_0,y_0,t,x_0\right)\right| + \left|\overline{\mathcal{L}}\left(f,x_0,y_0,t,x_0\right)\right| + |d(x_0)| +$$

$$+\left|H\sum_{k=1}^{q}A_k\left[\mathcal{L}\left(f^p,x_0^{p+1},y_0^p,t_k,x_0\right)-\mathcal{L}\left(f,x_0,y_0,t_k,x_0\right)\right]\right| +$$

$$+\left|H\sum_{k=1}^{q}A_k\mathcal{L}\left(f,x_0,y_0,t_k,x_0\right)\right| \le 2\left(M+L_p\right)+G\left(M'+L_p\right)+|d(x_0)| \le \beta_2(x_0).$$

It follows that $x_1^{p+1}(t,x_0)\in D_1$, $y_1^p(t,x_0)\in D_2$. By induction in a similar way we can establish that

$$\left|x_m^{p+1}-x_0\right| \le \beta_1(x_0), \quad |y_m^p| \le \beta_2(x_0).$$

Now we consider the differences $x_m - x_m^{p+1}$ and $y_m - y_m^p$. For $m = 1$ we have

$$\begin{gathered}\left|x_1 - x_1^{p+1}\right| \le \left|\mathcal{L}\left(f, x_0, y_0, t, x_0\right) - \mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right)\right| + \\ + \left|TH \sum_{k=1}^{q} A_k \left[\mathcal{L}\left(f, x_0, y_0, t_k, x_0\right) - \mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right)\right]\right| \le \\ \le \left(\alpha_1(t)E + G\right) L_p,\end{gathered} \tag{2.20}$$

$$\begin{gathered}\left|y_1 - y_1^p\right| \le \left|\overline{\mathcal{L}}\left(f, x_0, y_0, t, x_0\right) - \overline{\mathcal{L}}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right)\right| + \\ + \left|H \sum_{k=1}^{q} A_k \left[\mathcal{L}\left(f, x_0, y_0, t_k, x_0\right) - \mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right)\right]\right| \le \\ \le \left(2E + \tfrac{1}{T}G\right) L_p.\end{gathered} \tag{2.21}$$

Using (2.14)-(2.21) and Lemma 4 of [5] we get

$$\begin{gathered}\left|x_2 - x_2^{p+1}\right| \le \left|\mathcal{L}\left(f, x_1, y_1, t, x_0\right) - \mathcal{L}\left(f^p, x_1^{p+1}, y_1^p, t, x_0\right)\right| + \\ + \left|TH \sum_{k=1}^{q} A_k \left[\mathcal{L}\left(f, x_1, y_1, t_k, x_0\right) - \mathcal{L}\left(f^p, x_1^{p+1}, y_1^p, t_k, x_0\right)\right]\right| \le \\ \le \left[\alpha_1(t)E + K_1\left(\alpha_2(t)E + \alpha_1(t)G\right) + \alpha_1(t)K_2\left(2E + \tfrac{1}{T}G\right)\right] L_p + \\ + \left|TH \sum_{k=1}^{q} A_k \left[\alpha_1(t_k)E + K_1\left(\alpha_2(t_k)E + \alpha_1(t_k)G\right) + \alpha_1(t_k)K_2\left(2E + \tfrac{1}{T}G\right)\right] L_p\right| \le \\ \le \left(\alpha_1(t)E + G\right)\left[E + K_1\left(\tfrac{T}{3}E + G\right) + K_2\left(2E + \tfrac{1}{T}G\right)\right] L_p \le \\ \le \left(\alpha_1(t)E + G\right)\left[E + Q\right] L_p,\end{gathered}$$

$$\begin{gathered}\left|y_2 - y_2^p\right| \le \left|\overline{\mathcal{L}}\left(f, x_1, y_1, t, x_0\right) - \overline{\mathcal{L}}\left(f^p, x_1^{p+1}, y_1^p, t, x_0\right)\right| + \\ + \left|H \sum_{k=1}^{q} A_k \left[\mathcal{L}\left(f, x_1, y_1, t_k, x_0\right) - \mathcal{L}\left(f^p, x_1^{p+1}, y_1^p, t_k, x_0\right)\right]\right| \le \\ \le 2 \max_{t\in[0,T]} \left|f(t, x_1, y_1) - f^p(t, x_1^{p+1}, y_1^p)\right| + \\ + \left|H \sum_{k=1}^{q} A_k \left|\mathcal{L}_1\left(f, x_1, y_1, t_k, x_0\right) - \mathcal{L}_1\left(f^p, x_1^{p+1}, y_1^p, t_k, x_0\right)\right|\right| \le \\ \le \left(2E + \tfrac{1}{T}G\right)\left[E + Q\right] L_p.\end{gathered}$$

We can obtain by induction that

$$\left|x_m(t, x_0) - x_m^{p+1}(t, x_0)\right| \le \left(\alpha_1(t)E + G\right)\left[\sum_{i=1}^{m-1} Q^i\right] L_p, \tag{2.22}$$

$$\left|y_m(t, x_0) - y_m^p(t, x_0)\right| \le \left(2E + \frac{1}{T}G\right)\left[\sum_{i=1}^{m-1} Q^i\right] L_p. \tag{2.23}$$

Now we have to estimate $r_{m+1}(t,x_0)$ and $\widehat{r}_{m+1}(t,x_0)$ for every $m = 0,1,2,\dots$ by using Lemmas 3 and 4 of [5]:

$$r_1(t,x_0) \le |\mathcal{L}(f,x_0,y_0,t,x_0)| + T\,|d(x_0)| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\mathcal{L}(f,x_0,y_0,t_k,x_0)\right| \le \left(\tfrac{T}{2}E+G\right)M' + T\,|d(x_0)| \equiv \gamma_1(x_0),$$

$$\widehat{r}_1(t,x_0) \le \left|\overline{\mathcal{L}}(f,x_0,y_0,t,x_0)\right| + |d(x_0)| + \left|H\sum_{k=1}^{q} A_k\mathcal{L}(f,x_0,y_0,t_k,x_0)\right| \le$$

$$\le 2M + |d(x_0)| + \tfrac{1}{T}GM' \equiv \gamma_2(x_0),$$

$$r_2(t,x_0) \le |\mathcal{L}(f,x_1,y_1,t,x_0) - \mathcal{L}(f,x_0,y_0,t,x_0)| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}(f,x_1,y_1,t_k,x_0) - \mathcal{L}(f,x_0,y_0,t_k,x_0)\right]\right| \le$$

$$\le \left(1-\tfrac{t}{T}\right)\int_0^t \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau + \tfrac{t}{T}\int_t^T \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau +$$

$$+\left|TH\sum_{k=1}^{q} A_k\left[\left(1-\tfrac{t_k}{T}\right)\int_0^{t_k}\left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau +\right.\right.$$

$$\left.\left.+\tfrac{t_k}{T}\int_{t_k}^T \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau\right]\right| \le (\alpha_1(t)E+G)\cdot\left[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)\right],$$

$$\widehat{r}_2(t,x_0) \le \left|\overline{\mathcal{L}}(f,x_1,y_1,t,x_0) - \overline{\mathcal{L}}(f,x_0,y_0,t,x_0)\right| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}(f,x_1,y_1,t_k,x_0) - \mathcal{L}(f,x_0,y_0,t_k,x_0)\right]\right| \le$$

$$\le 2\max_{t\in[0,T]}\left|K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right| +$$

$$+\left|H\sum_{k=1}^{q} A_k\left[\left(1-\tfrac{t_k}{T}\right)\int_0^{t_k}\left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau +\right.\right.$$

$$\left.\left.+\tfrac{t_k}{T}\int_{t_k}^T \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau\right]\right| \le \left(2E+\tfrac{1}{T}G\right)\cdot\left[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)\right].$$

Similarly,

$$r_3(t,x_0) \le \Big\{ \left(1-\tfrac{t}{T}\right) \int_0^t \left[K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+ \tfrac{t}{T} \int_t^T \left[K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+ \Big| TH \sum_{k=1}^{q} A_k \Big[\left(1-\tfrac{t_k}{T}\right) \int_0^{t_k} \left[K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+ \tfrac{t_k}{T} \int_{t_k}^T \left[K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right] d\tau \Big] \Big| \Big\} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right] \le$$

$$\le \left(\alpha_1(t)E+G\right) \cdot Q \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right],$$

$$\widehat{r}_3(t,x_0) \le \Big\{ 2 \max_{t\in[0,T]} \left| K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right| +$$

$$+ \Big| H \sum_{k=1}^{q} A_k \Big[\left(1-\tfrac{t_k}{T}\right) \int_0^{t_k} \left[K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+ \tfrac{t_k}{T} \int_{t_k}^T \left[K_1 \left(\alpha_1(\tau)E+G\right) + K_2 \left(2E+\tfrac{1}{T}G\right)\right] d\tau \Big] \Big| \Big\} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right] \le$$

$$\le \left(2E+\tfrac{1}{T}G\right) \cdot Q \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right].$$

We can show by induction that for arbitrary $m = 0, 1, 2, \ldots$

$$r_{m+1}(t,x_0) \le \left(\alpha_1(t)E+G\right) \cdot Q^{m-1} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right], \tag{2.24}$$

$$\widehat{r}_{m+1}(t,x_0) \le \left(2E+\frac{1}{T}G\right) \cdot Q^{m-1} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right]. \tag{2.25}$$

From (2.24) and assumption c) we obtain the inequality

$$\begin{aligned} &\left|x_{m+j}(t,x_0) - x_m(t,x_0)\right| \le \sum_{i=0}^{j-1} \left|x_{m+i+1}(t,x_0) - x_{m+i}(t,x_0)\right| \le \\ &\le \sum_{i=0}^{j-1} r_{m+i+1}(t,x_0) \le \sum_{i=0}^{j-1} \left(\alpha_1(t)E+G\right) Q^{m+i-1} \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right] \le \\ &\le \left(\alpha_1(t)E+G\right) \cdot Q^{m-1} \left(E-Q\right)^{-1} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right]. \end{aligned} \tag{2.26}$$

For the derivatives $y_m(t,x_0)$ from (2.25) in a similar way we have:

$$\begin{aligned} &\left|y_{m+j}(t,x_0) - y_m(t,x_0)\right| \le \\ &\le \left(2E+\tfrac{1}{T}G\right) Q^{m-1} \left(E-Q\right)^{-1} \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right]. \end{aligned} \tag{2.27}$$

It follows that (2.12) and (2.13) are uniformly convergent sequences:

$$\lim_{m\to\infty} x_m(t,x_0) = x^*(t,x_0), \ \lim_{m\to\infty} y_m(t,x_0) = y^*(t,x_0).$$

Taking the limit as $j \to \infty$ in (2.26) and (2.27) we get the error estimates

$$|x^*(t,x_0) - x_m(t,x_0)| \le (\alpha_1(t)E + G)\cdot Q^{m-1}(E-Q)^{-1}\cdot[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)],$$

$$|y^*(t,x_0) - y_m(t,x_0)| \le \left(2E + \tfrac{1}{T}G\right)\cdot Q^{m-1}(E-Q)^{-1}\cdot[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)].$$

Combining the last two inequalities with (2.22) and (2.23), we get the error estimates (2.10) and (2.11). Passing to the limit as $m \to \infty$ in (2.6) we obtain that $x^*(t,x_0)$ satisfies the integral equation

$$x(t) = x_0 + \mathcal{L}(f,x,y,t,x_0) + tHd(x_0) - tH\sum_{k=1}^{q} A_k\mathcal{L}(f,x,y,t_k,x_0).$$

While differentiating it, we get that $x^*(t,x_0)$ is a solution of the perturbed BVP (2.8)-(2.9). □

The following statement gives necessary and sufficient conditions for the existence of a solution of the BVP (2.1)-(2.2).

Theorem 2. *Under the conditions of Theorem 1, the limit function $x^*(t,x_0^*)$ is a solution of the BVP (2.1)-(2.2) if and only if x_0^* verifies the determining equation*

$$\begin{aligned}\Delta(x_0) = -\tfrac{1}{T}\int_0^T f(s,x^*(s,x_0),y^*(s,x_0))\,ds + Hd(x_0) + \\ +H\sum_{k=1}^{q} A_k\mathcal{L}(f,x^*,y^*,t_k,x_0) = 0.\end{aligned} \tag{2.28}$$

Proof. The proof can be carried out in the same way as for the corresponding statements from [2] (Theorem 2.3). □

3. Sufficient existence conditions

Consider the m-th approximation to the determining equation (2.28)

$$\begin{aligned}\Delta_m^p(x_0) = -\tfrac{1}{T}\int_0^T f^p\left(s,x_m^{p+1}(s,x_0),y_m^p(s,x_0)\right)ds + Hd(x_0) + \\ +H\sum_{k=1}^{q} A_k\mathcal{L}\left(f^p,x_m^{p+1},y_m^p,t_k,x_0\right) = 0.\end{aligned} \tag{3.1}$$

Theorem 3. *Suppose that the conditions of Theorem 1 hold. Furthermore, assume that*

d) *there exists a closed, convex subset $D' = D_1' \times D_2' \subset D_1 \times D_2$ so that for arbitrary m and fixed p the approximate determining equation (3.1) has only one solution $x_0 = x_{0m}^p$ with non-zero topological index;*

e) *on the boundary ∂D of the subset D the inequality*

$$\inf_{x_0 \in \partial D} |\Delta_m^p(x_0)| > \left(E + \frac{1}{T} G\right) W_m^p$$

holds.

Then there exists a solution $x = x^*(t)$ to the BVP (2.1)-2.2) with the initial value $x^*(0) = x_0^*$, where $x_0^* \in D_1'$.

Proof. Similarly to (2.15) and making use of (2.10) and (2.11), we get

$$\left|f(t, x^*, y^*) - f^p(t, x_m^{p+1}, y_m^p)\right| \leq \left[K_1 (\alpha_1(t) E + G) + K_2 \left(2E + \frac{1}{T} G\right)\right] W_{m-1}^p + L_p.$$

For the deviation of the exact and approximate determining functions we have that

$$|\Delta(x_0) - \Delta_m^p(x_0)| \leq \tfrac{1}{T} \int_0^T |f^p(s, x^*(s, x_0), y^*(s, x_0)) -$$

$$-f^p\left(s, x_m^{p+1}(s, x_0), y_m^p(s, x_0)\right)\Big| + H \sum_{k=1}^{q} A_k \left|\mathcal{L}(f^p, x^*, y^*, t_k, x_0) -\right.$$

$$\left.-\mathcal{L}\left(f^p, x_m^{p+1}, y_m^p, t_k, x_0\right)\right| \leq \left(E + \tfrac{1}{T} G\right)\left(Q W_{m-1}^p + L_p\right) \leq \left(E + \tfrac{1}{T} G\right) W_m^p.$$

Similarly to Theorem 3.1 of [2], one can prove that the vector fields $\Delta(x_0)$ and $\Delta_m^p(x_0)$ are homotopic, which completes the proof of Theorem 3. □

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