Uniform approximation by means of some piecewise linear functions

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UNIFORM APPROXIMATION BY MEANS OF SOME PIECEWISE LINEAR FUNCTIONS

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[Received: September 21, 2001]

Abstract. In the present paper we construct new piecewise linear functions using a special partition of the interval \([-1, 1]\). This construction leads to the definition of some new linear operators and we shall obtain global estimates for the remainder in approximating continuous functions by these operators using the second order modulus of smoothness of Ditzian - Totik.

Mathematical Subject Classification: 41A10, 41A35

Keywords: piecewise linear function, Ditzian - Totik modulus of smoothness

1. Introduction

Let \((\delta_k)_{k=-n}^n\) be a sequence such that \(\delta_k = \delta_{-k} \ (k = 1, 2, \ldots, n)\) and \(1/n^2 = \delta_n < \delta_{n-1} < \ldots < \delta_1 < \delta_0 < c_0/n\). Furthermore, let \((x_k)_{k=-n}^n : -1 = x_{-n} < x_{-n+1} < \ldots < x_{n-1} < x_n = 1\) be a partition of the interval \([-1, 1]\) with the properties:

\((i)\) \(c_1 \delta_k \leq x_{k+1} - x_k \leq c_2 \delta_k \ (k = -n, \ldots, n - 1)\)

\((ii)\) for any \(u \in [x_k, x_{k+1}] \ (k = -n + 1, \ldots, n - 2)\) we have \(x_{k+1} - x_k \leq c \cdot \sqrt{1 - u^2/n} \).

Here and throughout \(c_0, c_1, c_2\) and \(c\) denote absolute constants and the value of \(c\) may vary with each occurrence, even on the same line.

The existence of \((\delta_k)_{k=-n}^n\) and \((x_k)_{k=-n}^n\) is guaranteed by the constructive proof of DeVore and Yu given in [1, p. 326 and p. 329]. This proof establishes a pointwise estimate of the Timan - Teljakovski type for monotone polynomial approximation [1, p. 324, Theorem 1]. The idea of DeVore and Yu was successfully applied by Leviatan [3, p. 3, Theorem 1'] to give a global estimate on monotone approximation. Both proofs are based on a two - stage approximation. At first the function \(f \in C[-1, 1]\) is approximated by a piecewise linear function \(S_nf\) which interpolates \(f\) at the points \(x_k \ (k = -n, -n + 1, \ldots, n)\). By Newton’s formula we have

\[(S_nf)(x) - f(x) = (x - x_k) (x_{k+1} - x) \ [x_k, x, x_{k+1} ; f],\]
Theorem 1. If \( f \in C[-1, 1] \), then \( \|f - U_n f\| \leq c \omega_n^2(f, n^{-1}) \).

**Proof.** By (i) we get \( n^{-\gamma} \varphi(x_{n+1}) \leq n^{-\gamma} \leq c_1 \delta_n \leq x_{n+1} - x_n \). Hence \( x_n = x_{-n+1} - n^{-\gamma} \varphi(x_{n+1}) \leq x_{n+1} - x_n \). Again, by (i) and by the properties of \( (\delta_k)_{k=-n}^{n} \) we have \( n^{-\gamma} \varphi(x_k) \leq n^{-\gamma} \leq c_1 \delta_n \leq c_1 \delta_{k} \leq x_{k+1} - x_k \ (k = -n + 1, \ldots, n - 1) \). Then
Now, in view of Popoviciu’s representation theorem [5] we have

\[ (S_n f)(x) = \frac{x-x_n}{x_n-x} \cdot f(x_n) \]

\[ + \sum_{k=-n+1}^{n-1} \frac{x_{k+1}-x_k}{2} \cdot [x_{k-1}, x_k, x_{k+1} ; |t-x|] f(x_k) \]

\[ + \frac{x-x_{n-1}}{2(x_n-x_{n-1})} \cdot f(x_n). \]

Hence

\[ (U_n f)(x) - (S_n f)(x) = \]

\[ \frac{x-x_n}{x_n-x} \cdot \left\{ f(x_{n+1} + n^{-\gamma} \varphi(x_{n+1})) \right\} - 2f(x_{n+1}) + f(x_{n+1} - n^{-\gamma} \varphi(x_{n+1})) \] (2.1)

for \( x_n \leq x \leq x_{n+1} \);

\[ (U_n f)(x) - (S_n f)(x) = \]

\[ \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \left\{ f(x_k + n^{-\gamma} \varphi(x_k)) \right\} - 2f(x_k) + f(x_k - n^{-\gamma} \varphi(x_k)) \] +

\[ \frac{x-x_k}{x_{k+1}-x_k} \cdot \left\{ f(x_{k+1} + n^{-\gamma} \varphi(x_{k+1})) \right\} - 2f(x_{k+1}) + f(x_{k+1} - n^{-\gamma} \varphi(x_{k+1})) \] (2.2)

for \( x_k \leq x \leq x_{k+1} \) \( (k = -n+1, \ldots, n-1) \);

\[ (U_n f)(x) - (S_n f)(x) = \]

\[ \frac{x_n-x}{x_n-x_{n-1}} \cdot \left\{ f(x_{n-1} + n^{-\gamma} \varphi(x_{n-1})) \right\} - 2f(x_{n-1}) + f(x_{n-1} - n^{-\gamma} \varphi(x_{n-1})) \] (2.3)

for \( x_{n-1} \leq x \leq x_n \).

On the other hand, by definition of the Ditzian-Totik modulus of smoothness we obtain from (2), (3), (4) and \( \gamma \geq 1 \) the estimates

\[ |(U_n f)(x) - (S_n f)(x)| = \]

\[ \frac{x-x_n}{x_n-x_{n+1}} \cdot |\Delta^2_{n^{-\gamma} \varphi(x_{n+1})} f(x_{n+1})| \leq \| \Delta^2_{n^{-\gamma} \varphi(x)} f(x) \| \leq \omega^2(f, n^{-1}), \]
where \( x_{-n} \leq x \leq x_{-n+1} \):

\[
|(U_n f)(x) - (S_n f)(x)| \leq \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot |\Delta^2_{n-\varphi(x_k)} f(x_k)| + \frac{x - x_k}{x_{k+1} - x_k} \cdot |\Delta^2_{n-\varphi(x_{k+1})} f(x_{k+1})| \leq \|\Delta^2_{n-\varphi(x)} f(x)\| \leq \omega^2(f, n^{-1}),
\]

where \( x_k \leq x \leq x_{k+1} \):

\[
|(U_n f)(x) - (S_n f)(x)| = \frac{x_n - x}{x_n - x_{n-1}} \cdot |\Delta^2_{n-\varphi(x_{n-1})} f(x_{n-1})| \leq \|\Delta^2_{n-\varphi(x)} f(x)\| \leq \omega^2(f, n^{-1}),
\]

where \( x_{n-1} \leq x \leq x_n \). Hence \( \|U_n f - S_n f\| \leq \omega^2(f, n^{-1}) \). Using (1) we obtain the assertion of the theorem.

Our next piecewise linear function is the following:

\[
(V_n f)(x) = \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \int_{x_k}^{x_{k+1}} f(u) \, du + \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \int_{x_{k+1}}^{x_{k+1} - c_1 \delta_k} f(u) \, du,
\]

if \( x_k < x < x_{k+1} \) (\( k = -n, \ldots, n - 1 \)) and \( (V_n f)(x) = f(x_k) \), if \( x = x_k \) (\( k = -n, \ldots, n \)).

Then \( V_n : C[-1, 1] \rightarrow L_\infty[-1, 1] \) is a linear, positive operator such that \( V_n f \) interpolates the function \( f \) at the points \( x_k \) (\( k = -n, \ldots, n \)). The main difference between \( V_n \) and \( S_n \) is that \( V_n f \) is not necessarily continuous at \( x_k \) (\( k = -n + 1, \ldots, n - 1 \)). Moreover, \( V_n e_0 = e_0 \) and \( V_n e_1 = e_1 + O(n^{-1}) \) (here \( e_0(x) = 1 \) and \( e_1(x) = x \) for \( x \in [-1, 1] \)). Indeed, the first statement is obvious and for the second one we have for \( x_k < x < x_{k+1} \) (\( k = -n, \ldots, n - 1 \)):

\[
(V_n e_1)(x) = \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left( \frac{1}{2} (2x_k c_1 \delta_k + c_1^2 \delta_k^2) \right) + \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left( \frac{1}{2} (2x_{k+1} c_1 \delta_k - c_1^2 \delta_k^2) \right)
\]

Hence, by properties of \( (\delta_k^\gamma)_{k=-n} \) we obtain

\[
|(V_n e_1)(x) - e_1(x)| \leq \frac{1}{2} c_1 \delta_k \leq \frac{c_0 c_1}{2n}, \quad n \geq 1.
\]
Furthermore, we denote the set of all algebraic polynomials of degree at most \( n \) by \( \Pi_n \) and the best uniform approximation on \([-1, 1]\) by \( E_n(f) = \inf\{\|f - p\| : p \in \Pi_n\} \). Our next results are the following:

**Theorem 2.** Let \( f \in C[-1, 1] \) and \( V_n \) \((n \in N, n \geq 2)\) be defined as above. Then
\[
\|f - V_n f\| \leq c \left\{ \omega_2(f, n^{-1}) + n^{-1} \int_{-1}^{1} \frac{\omega_2^2(f, t)}{t^3} \, dt + n^{-1} E_0(f) \right\}.
\]

**Corollary 1.** Let \( f \in C[-1, 1], n \in N, n \geq 2 \). Then
\[
\|f - V_n f\| \leq c \left\{ n^{-1} \left\| g' \right\| + n^{-2} \left\| \varphi^2 g'' \right\| \right\}.
\]

To prove our statements we need a lemma:

**Lemma 1.** For \( g \in C^2[-1, 1], n \geq 2 \) we have
\[
\|g - V_n g\| \leq c \left\{ n^{-1} \left\| g' \right\| + n^{-2} \left\| \varphi^2 g'' \right\| \right\}.
\]

**Proof.** If \( x_k < x < x_{k+1} \) \((k = -n + 1, \ldots, n - 2)\), then

\[
(V_n g)(x) - g(x) =
\]

\[
= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \left\{ \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} [g(u) - g(x_k)] \, du + g(x_k) - g(x) \right\}
\]

\[
+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \left\{ \frac{1}{c_1 \delta_k} \int_{x_k + c_1 \delta_k}^{x_{k+1}} [g(u) - g(x_{k+1})] \, du + g(x_{k+1}) - g(x) \right\}.
\]

Simple computations show that
\[
\int_{\alpha}^{\beta} [g(u) - g(\alpha)] \, du = \int_{\alpha}^{\beta} (\beta - u) \, g'(u) \, du
\]
and
\[
\int_{\alpha}^{\beta} [g(\beta) - g(u)] \, du = \int_{\alpha}^{\beta} (u - \alpha) \, g'(u) \, du.
\]
Hence

\[(V_n g)(x) - g(x) = \]

\[
\frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \left\{ \frac{1}{c_1 \delta_k} \int_{x_k}^{x_{k+1} + c_1 \delta_k} \left( x_k + c_1 \delta_k - u \right) g'(u) \, du - \int_{x_k}^{x} g'(u) \, du \right\}
\]

\[
+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \left\{ \frac{1}{c_1 \delta_k} \right\}
\]

\[
\cdot \left\{ \int_{x_k}^{x_{k+1} + c_1 \delta_k} \left( x_k + c_1 \delta_k - u \right) g'(u) \, du + \int_{x}^{x_{k+1} - c_1 \delta_k} g'(u) \, du \right\}
\]

\[
= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k}
\]

\[
\cdot \left\{ \int_{x_k}^{x_{k+1} + c_1 \delta_k} \left( x_k + c_1 \delta_k - u \right) g'(u) \, du - \int_{x_k}^{x} c_1 \delta_k g'(u) \, du \right\}
\]

\[
+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \cdot \left\{ \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} \left( x_k + c_1 \delta_k - u \right) g'(u) \, du + \int_{x}^{x_{k+1}} \left( x_k + c_1 \delta_k - u \right) g'(u) \, du \right\}
\]

By partial integration we obtain

\[(V_n g)(x) - g(x) = \]

\[
\frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ -\frac{1}{2} \left( x_k - x \right)^2 g'(x) + \int_{x_k}^{x} \left( x_k - u \right)^2 g''(u) \, du + \int_{x}^{x_{k+1} - c_1 \delta_k} \left( x_k + c_1 \delta_k - u \right)^2 g''(u) \, du \right\} + \]

\[
+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ \frac{1}{2} \int_{x_k}^{x_{k+1} + c_1 \delta_k} \left( x_k + c_1 \delta_k - u \right)^2 g''(u) \, du \right\} + \]
\[
+ \frac{1}{2} \int_{x}^{x + c_1 \delta_k} (x_k + c_1 \delta_k - u)^2 g''(u) \, du \bigg) + \\
+ \frac{x - x_k}{x_k + 1 - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ -\frac{1}{2} (x_{k+1} - c_1 \delta_k - x)^2 g'(x) + \\
+ \frac{1}{2} \int_{x + 1 - c_1 \delta_k}^{x} (x_{k+1} - c_1 \delta_k - u)^2 g''(u) \, du \bigg) + \\
+ \frac{1}{2} (x_{k+1} - x)^2 g'(x) + \frac{1}{2} \int_{x}^{x_{k+1}} (x_{k+1} - u)^2 g''(u) \, du \bigg) .
\]

(2.4)

On the other hand, we get for \( x_k \leq u \leq x_{k+1} \) by (i) and (ii)

\[
\left| -\frac{1}{2} (x_k - x)^2 + \frac{1}{2} (x_k + c_1 \delta_k - x)^2 \right| = \\
= |c_1 \delta_k (x_k - x) + \frac{1}{2} (c_1 \delta_k)^2| = c_1 \delta_k |(x_k - x) + \frac{1}{2} c_1 \delta_k| \\
\leq c_1 \delta_k (x_k - x + \frac{1}{2} c_1 \delta_k) \leq \frac{3}{2} c_1 \delta_k (x_{k+1} - x_k) \\
\leq \frac{3}{2} c_1 \delta_k \cdot \frac{\sqrt{1 - u^2}}{n} \leq c \cdot \frac{c_1 \delta_k}{n},
\]

we have for \( x_k \leq u \leq x < x_{k+1} \), by (ii)

\[
(x_k - u)^2 \leq (x_{k+1} - x_k)^2 \leq c \frac{1 - u^2}{n^2}
\]

(2.6)

and for \( x_k < x \leq u \leq x_k + c_1 \delta_k \leq x_{k+1} \) or \( x_k \leq x_k + c_1 \delta_k \leq u \leq x < x_{k+1} \) we have in view of (i) and (ii)

\[
(x_k + c_1 \delta_k - u)^2 \leq 2 (x_k - u)^2 + 2 (c_1 \delta_k)^2 \leq 4 (x_{k+1} - x_k)^2 \leq c \frac{1 - u^2}{n^2} .
\]

(2.7)

Using the same arguments we obtain

\[
\left| -\frac{1}{2} (x_{k+1} - c_1 \delta_k - x)^2 + \frac{1}{2} (x_{k+1} - x)^2 \right| \leq \frac{c_1 \delta_k}{n}
\]

(2.8)

for \( x_k \leq u \leq x_{k+1} \);

\[
(x_{k+1} - c_1 \delta_k - u)^2 \leq c \frac{1 - u^2}{n^2}
\]

(2.9)

for \( x_k \leq x_{k+1} - c_1 \delta_k \leq u \leq x_{k+1} \) or \( x_k < x \leq u \leq x_{k+1} - c_1 \delta_k \leq x_{k+1} \) and

\[
(x_{k+1} - u)^2 \leq c \frac{1 - u^2}{n^2}
\]

(2.10)

for \( x_k < x \leq u \leq x_{k+1} \). Then (5) – (11) and (ii) imply
\[
| (V_ng)(x) - g(x) | \leq \\
\leq \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ \left| -\frac{1}{2} (x_k - x)^2 + \frac{1}{2} (x_k + c_1 \delta_k - x)^2 \right| \right. \\
\cdot | g'(x) | + \left. \frac{1}{2} \int_{x_k}^x (u - x_k)^2 | g''(u) | \, du \right\} + \\
+ \frac{1}{2} \int_{x_k}^{x_{k+1}} (x_{k+1} - u)^2 | g''(u) | \, du \\
\leq \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ \left| -\frac{1}{2} (x_{k+1} - c_1 \delta_k - x)^2 + \frac{1}{2} (x_{k+1} - x)^2 \right| \right. \\
\cdot | g'(x) | + \left. \frac{1}{2} \int_{x_k}^{x_{k+1}} (x_{k+1} - c_1 \delta_k - u)^2 | g''(u) | \, du \right\} + \\
+ \frac{1}{2} \int_{x_k}^{x_{k+1}} (x_{k+1} - u)^2 | g''(u) | \, du \\
\leq \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ \left| c \cdot \frac{c_1 \delta_k}{n} \cdot \| g' \| \right| + \\
\cdot | g'(x) | + \frac{1}{2} \int_{x_k}^{x_{k+1}} c \cdot \frac{1 - u^2}{n^2} | g''(u) | \, du \right\} + \\
+ \frac{1}{2} \int_{x_k}^{x_{k+1}} c \cdot \frac{1 - u^2}{n^2} | g''(u) | \, du \\
+ \frac{1}{2} \int_{x_k}^{x_{k+1}} c \cdot \frac{1 - u^2}{n^2} | g''(u) | \, du \right\}.
\] (2.11)

But
\[
\int_{x_k}^{x_{k+1}} + \int_{x}^{x_k + c_1 \delta_k} = \int_{x_k}^{x_{k+1}} + \int_{x}^{x_k + c_1 \delta_k} = \int_{x_k}^{x_{k+1}}
\]
if \( x_k < x \leq x_k + c_1 \delta_k \) or
\[
\int_{x_k}^{x_{k+1}} + \int_{x}^{x_k + c_1 \delta_k} = \int_{x_k}^{x_{k+1}} + \int_{x_k + c_1 \delta_k} \leq \int_{x_k}^{x_{k+1}} + \int_{x_k}^{x_{k+1}} = 2 \int_{x_k}^{x_{k+1}}
\]
if \( x_k + c_1 \delta_k \leq x < x_{k+1} \) and
\[
\int_{x_k + c_1 \delta_k}^{x_{k+1}} + \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} = \int_{x_k + c_1 \delta_k}^{x_{k+1}} + \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} = \int_{x_k + c_1 \delta_k}^{x_{k+1}}
\]
if \( x_{k+1} - c_1 \delta_k \leq x < x_{k+1} \) or

\[
\left| \int_{x_{k+1} - c_1 \delta_k}^{x} - \int_{x}^{x_{k+1}} \right| + \int_{x}^{x_{k+1}} = \int_{x}^{x_{k+1} - c_1 \delta_k} + \int_{x}^{x_{k+1}} \leq \\
\leq \int_{x}^{x_{k+1}} + \int_{x}^{x_{k+1}} = 2 \int_{x}^{x_{k+1}}
\]

if \( x_k < x \leq x_{k+1} - c_1 \delta_k \). So we obtain in view of (12) that

\[
| (V_n g)(x) - g(x) | \leq \\
\leq \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ c \cdot \frac{c_1 \delta_k}{n} \| g' \| + \\
+ c \cdot \| \varphi^2 g'' \| \cdot \max \left( c_1 \delta_k ; x - x_k \right) \right\} + \\
+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ c \cdot \frac{c_1 \delta_k}{n} \| g' \| + \\
+ c \cdot \frac{\| \varphi^2 g'' \|}{n^2} \cdot \max \left( c_1 \delta_k ; x_{k+1} - x \right) \right\} \leq c \left\{ n^{-1} \| g' \| + n^{-2} \| \varphi^2 g'' \| \cdot \max \left( 1 ; \frac{x_{k+1} - x_k}{c_1 \delta_k} \right) \right\} .
\]

By (i) we have \( x_{k+1} - x_k \leq c_2 \delta_k \). Therefore

\[
| (V_n g)(x) - g(x) | \leq c \left\{ n^{-1} \| g' \| + n^{-2} \| \varphi^2 g'' \| \right\} . \tag{2.12}
\]

If \( x_{n-1} < x < x_{n+1} \) then

\[
(V_n g)(x) - g(x) = \\
= \frac{x_{n+1} - x}{x_{n+1} - x_{n-1}} \cdot \left\{ \frac{1}{c_1 \delta_{n-1}} \cdot \\
\cdot \int_{x_{n-1}}^{x_{n-1} + c_1 \delta_{n-1}} [g(u) - g(x_{n-1})] \ du + g(x_{n-1}) - g(x) \right\} + \\
+ \frac{x - x_{n-1}}{x_{n+1} - x_{n-1}} \cdot \left\{ \frac{1}{c_1 \delta_{n-1}} \cdot \\
\cdot \int_{x_{n-1}}^{x_{n+1}} [g(u) - g(x_{n+1})] \ du + g(x_{n+1}) - g(x) \right\}
\]
\[
\begin{align*}
\quad &= \frac{x_{n+1} - x}{x_{n+1} - x_n} \cdot \left\{ \frac{1}{c_1 \delta_n} \right\} \\
&\quad \cdot \int_{x_n}^{x_{n+1} + c_1 \delta_n} [g(u) - g(x_n)] \, du - \int_{x_n}^{x} g'(u) \, du \\
&+ \frac{x - x_n}{x_{n+1} - x_n} \cdot \left\{ \frac{1}{c_1 \delta_n} \right\} \\
&\quad \cdot \int_{x_n}^{x_{n+1}} [g(u) - g(x_{n+1})] \, du + \int_{x_n}^{x} g'(u) \, du \\
&= \frac{x_{n+1} - x}{x_{n+1} - x_n} \cdot \left\{ \frac{1}{c_1 \delta_n} \right\} \\
&\quad \cdot \int_{x_n}^{x_{n+1} + c_1 \delta_n} \left[ \int_{x_n}^{u} g'(v) \, dv \right] \, du - \int_{x_n}^{x} g'(u) \, du \\
&+ \frac{x - x_n}{x_{n+1} - x_n} \cdot \left\{ \frac{1}{c_1 \delta_n} \right\} \\
&\quad \cdot \int_{x_n}^{x_{n+1}} \left[ \int_{x_n}^{x_{n+1}} g'(v) \, dv \right] \, du + \int_{x}^{x_{n+1}} g'(u) \, du \\
&\leq \frac{x_{n+1} - x}{x_{n+1} - x_n} \cdot \left\{ \frac{1}{c_1 \delta_n} \right\} \\
&\quad \cdot \int_{x_n}^{x_{n+1} + c_1 \delta_n} (u - x_n) \, du + (x - x_n) \cdot \|g'\| \\
&+ \frac{x - x_n}{x_{n+1} - x_n} \cdot \left\{ \frac{1}{c_1 \delta_n} \right\} \\
&\quad \cdot \int_{x_n}^{x_{n+1}} (x_{n+1} - u) \, du + (x_{n+1} - x) \cdot \|g'\|.
\end{align*}
\]

Hence

\[
|V_n g(x) - g(x)| \leq
\]
In view of (i) we have \( x_n \leq u \leq x_n + c_1 \delta_n \leq x_{n+1}, \ x_n \leq x_{n+1} - c_1 \delta_n \leq u \leq x_{n+1} \) and \( x_n \leq x \leq x_{n+1} \), respectively. So

\[
\|(V_n g)(x) - g(x)\| \leq \frac{x_n - x}{x_{n+1} - x_n} \left\{ \frac{1}{c_1 \delta_n} \right\} \int_{x_n}^{x_{n+1} + c_1 \delta_n} (x_{n+1} - x_n) \, du + (x_{n+1} - x_n) \right\} \|g'\|
\]

\[
+ \frac{x - x_n}{x_{n+1} - x_n} \left\{ \frac{1}{c_1 \delta_n} \right\} \int_{x_n}^{x_{n+1} - c_1 \delta_n} (x_{n+1} - x_n) \, du + (x_{n+1} - x_n) \right\} \|g'\|
\]

\[
\leq 2 \left( x_{n+1} - x_n \right) \|g'\|.
\]

Again, (i) implies \( x_{n+1} - x_n \leq c_2 \delta_n \). This means that

\[
\|(V_n g)(x) - g(x)\| \leq 2 c_2 \delta_n \|g'\| \leq \frac{c}{n^2} \|g'\|. \tag{2.13}
\]

Analogously

\[
\|(V_n g)(x) - g(x)\| \leq \frac{c}{n^2} \|g'\| \tag{2.14}
\]

for \( x_{n-1} < x < x_n \). In conclusion (13), (14) and (15) imply

\[
\|g - V_n g\| \leq c \left\{ n^{-1} \|g'\| + n^{-2} \|\varphi'^2 g''\| \right\},
\]

which completes the proof. \( \square \)

**Proof of Theorem 2.** We have

\[
\|(V_n f)(x)\| \leq \frac{x_k - x}{x_{k+1} - x_k} \left\{ \frac{1}{c_1 \delta_k} \right\} \int_{x_k}^{x_{k+1} + c_1 \delta_k} |f(u)| \, du
\]

\[
+ \frac{x - x_k}{x_{k+1} - x_k} \left\{ \frac{1}{c_1 \delta_k} \right\} \int_{x_k}^{x_{k+1} - c_1 \delta_k} |f(u)| \, du \leq \|f\|
\]

for \( x_k < x < x_{k+1} \) \( (k = -n, \ldots, n - 1) \). Thus

\[
\|V_n f\| \leq \|f\|. \tag{2.15}
\]

On the other hand, let us denote by \( p_n \in \Pi_n \) the best \( n \)th degree polynomial approximation to \( f \). Then we know for \( f \in C[-1, 1] \) (see [2, p. 79, Theorem 7.2.1]) that

\[
E_n(f) = \|f - p_n\| \leq c \omega_2^2(f, n^{-1}). \tag{2.16}
\]

Moreover, we have the following Bernstein type inequality [2, p. 84, Theorem 7.3.1] for the best approximation polynomial

\[
\|\varphi'^2 p_n\| \leq n^2 \omega_3^2(f, n^{-1}). \tag{2.17}
\]
Then, using (16), Lemma 1, (17), (18) and the proof of [4, pp. 83 - 84, Theorem 3.2] we obtain the conclusion of our theorem. □

**Proof of Corollary 1.** In view of [4, p. 86, Remark 3.4], the proof is a direct consequence of the fact that we can drop the first term on the right hand side of the estimate given in Theorem 2 because of

\[
\int_{\frac{1}{2}}^{1} \frac{\omega^2(f,t)}{t^3} \, dt \geq \frac{\omega^2(f,n^{-1})}{n^2} \cdot \int_{\frac{1}{2}}^{1} \frac{dt}{t^3} \geq c \, n^2 \, \omega^2(f,n^{-1}), \quad n > 2.
\]

□

**Acknowledgement.** This work was done with the financial support of *Domus Hungarica Scientiarum et Artium*.

**REFERENCES**


