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UNIFORM APPROXIMATION BY MEANS OF SOME PIECEWISE LINEAR FUNCTIONS

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Abstract. In the present paper we construct new piecewise linear functions using a special partition of the interval $[-1, 1]$. This construction leads to the definition of some new linear operators and we shall obtain global estimates for the remainder in approximating continuous functions by these operators using the second order modulus of smoothness of Ditzian - Totik.

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1. **Introduction**

Let $(\delta_k)_{k=-n}^{n}$ be a sequence such that $\delta_k = \delta_{-k}$ $(k = 1, 2, \ldots, n)$ and $1/n^2 = \delta_n < \delta_{n-1} < \ldots < \delta_1 < \delta_0 < c_0/n$. Furthermore, let $(x_k)_{k=-n}^{n} : -1 = x_{-n} < x_{-n+1} < \ldots < x_{n-1} < x_n = 1$ be a partition of the interval $[-1, 1]$ with the properties:

- **:** (i) $c_1\ \delta_k \ \leq\ x_{k+1} - x_k \ \leq\ c_2\ \delta_k$ $(k = -n, \ldots, n-1)$
- **:** (ii) for any $u \in [x_k, x_{k+1}]$ $(k = -n+1, \ldots, n-2)$ we have $x_{k+1} - x_k \leq c \cdot \frac{\sqrt{1-u^2}}{n}$.

Here and throughout c_0, c_1, c_2 and c denote absolute constants and the value of c may vary with each occurrence, even on the same line.

The existence of $(\delta_k)_{k=-n}^{n}$ and $(x_k)_{k=-n}^{n}$ is guaranteed by the constructive proof of DeVore and Yu given in [1, p. 326 and p. 329]. This proof establishes a pointwise estimate of the Timan - Teljakovski type for monotone polynomial approximation [1, p. 324, Theorem 1]. The idea of DeVore and Yu was successfully applied by Leviatan [3, p. 3, Theorem 1'] to give a global estimate on monotone approximation. Both proofs are based on a two - stage approximation. At first the function $f \in C[-1, 1]$ is approximated by a piecewise linear function $S_n f$ which interpolates f at the points x_k $(k = -n, -n+1, \ldots, n)$. By Newton's formula we have

$$(S_n f)(x) - f(x) \ =\ (x - x_k)\ (x_{k+1} - x)\ [\ x_k,\ x,\ x_{k+1}\ ;\ f\ ],$$

$x_k \le x \le x_{k+1}$ $(k = -n, \dots, n-1)$, where the square brackes denote the divided difference of f at x_k, x, x_{k+1}. Using the proof of [3, p. 7, Theorem 7] we can deduce the following result:

$$\|f - S_n f\| \;\le\; c\, \omega_\varphi^2(f, n^{-1}), \tag{1.1}$$

where $\|\cdot\|$ is the sup - norm on $[-1,1]$, $\varphi(x) = \sqrt{1-x^2}$, $x \in [-1,1]$ and $\omega_\varphi^2(f,t) = \sup_{0<h\le t} \|\Delta^2_{h\varphi(x)} f(x)\|$ is the Ditzian - Totik modulus of smoothness [2], where

$$\Delta^2_{h\varphi(x)} f(x) = f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x)),$$

if $x \pm h\varphi(x) \in [-1,1]$ and $\Delta^2_{h\varphi(x)} f(x) = 0$, otherwise.

In this note we are interested in uniform approximation by piecewise linear functions different from $S_n f$.

2. Approximation by piecewise linear functions

Using a representation theorem given by Popoviciu for the operator S_n [5], we consider the following piecewise linear function:

$$\begin{aligned}
(U_n f)(x) = \\
&= \frac{x_{-n+1} - x + |x_{-n+1} - x|}{2(x_{-n+1} - x_{-n})} \cdot f(x_{-n}) \; + \\
&+ \sum_{k=-n+1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2} \cdot [\; x_{k-1},\ x_k,\ x_{k+1}\ ;\ |t-x|\;]_t \cdot \\
&\qquad \cdot \big\{ f(x_k + n^{-\gamma}\varphi(x_k)) - f(x_k) + f(x_k - n^{-\gamma}\varphi(x_k)) \big\} + \\
&+ \frac{x - x_{n-1} + |x - x_{n-1}|}{2(x_n - x_{n-1})} \cdot f(x_n),
\end{aligned}$$

where $f \in C[-1,1]$ and $\gamma \ge 1$ such that $c_1 \delta_{-n} = c_1 n^{-2} \ge n^{-\gamma}$. Here we denote by $[\; x_{k-1},\ x_k,\ x_{k+1}\ ;\ |t-x|]_t$ the fact that the divided difference is applied to the variable t.

The operator $U_n : C[-1,1] \to C[-1,1]$ is linear which preserves the linear functions. The function $U_n f$ interpolates f at the endpoints of the interval $[-1,1]$. Moreover, if $f \in C[-1,1]$ is a positive, convex function, then $U_n f$ is also positive. Indeed, for $k = -n+1, \dots, n-1$ we have $f(x_k + n^{-\gamma}\varphi(x_k)) - f(x_k) + f(x_k - n^{-\gamma}\varphi(x_k)) \ge f(x_k) \ge 0$. Hence, by definition of $U_n f$ we obtain $(U_n f)(x) \ge (S_n f)(x) \ge 0$, $x \in [-1,1]$ because $S_n : C[-1,1] \to C[-1,1]$ is a positive linear operator.

Our first result is the following:

Theorem 1. *If $f \in C[-1,1]$, then $\|f - U_n f\| \;\le\; c\, \omega_\varphi^2(f, n^{-1})$.*

Proof. By (i) we get $n^{-\gamma}\varphi(x_{-n+1}) \le n^{-\gamma} \le c_1 \delta_{-n} \le x_{-n+1} - x_{-n}$. Hence $x_{-n} \le x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1}) \le x_{-n+1}$. Again, by (i) and by the properties of $(\delta_k)_{k=-n}^n$ we have $n^{-\gamma}\varphi(x_k) \le n^{-\gamma} \le c_1\delta_{-n} \le c_1\delta_k \le x_{k+1} - x_k$ $(k = -n+1, \dots, n-1)$. Then

$x_k \le x_k + n^{-\gamma}\varphi(x_k) \le x_{k+1}$ $(k = -n+1, \ldots, n-1)$. With the same hypotheses we obtain $n^{-\gamma}\varphi(x_{k-1}) \le n^{-\gamma} \le c_1\delta_{-n} \le c_1\delta_{k-1} \le x_k - x_{k-1}$ $(k = -n+2, \ldots, n)$. Hence $x_{k-1} \le x_k - n^{-\gamma}\varphi(x_{k-1}) \le x_k$ $(k = -n+2, \ldots, n)$. This means that $U_n f$ is well - defined.

Now, in view of Popoviciu's representation theorem [5] we have

$$(S_n f)(x) = \frac{x_{-n+1} - x + |x_{-n+1} - x|}{2(x_{-n+1} - x_{-n})} \cdot f(x_{-n}) \; + \\ + \sum_{k=-n+1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2} \cdot [\, x_{k-1},\ x_k,\ x_{k+1}\ ;\ |t-x|\,]_t \cdot f(x_k) \; + \\ + \frac{x - x_{n-1} + |x - x_{n-1}|}{2(x_n - x_{n-1})} \cdot f(x_n).$$

Hence

$$(U_n f)(x) - (S_n f)(x) = \\ = \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \cdot \big\{ f(x_{-n+1} + n^{-\gamma}\varphi(x_{-n+1})) - \\ - 2f(x_{-n+1}) + f(x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1})) \big\} \tag{2.1}$$

for $x_{-n} \le x \le x_{-n+1}$;

$$(U_n f)(x) - (S_n f)(x) = \\ = \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \big\{ f(x_k + n^{-\gamma}\varphi(x_k)) - \\ - 2\, f(x_k) + f(x_k - n^{-\gamma}\varphi(x_k)) \big\} \; + \\ + \frac{x - x_k}{x_{k+1} - x_k} \cdot \big\{ f(x_{k+1} + n^{-\gamma}\varphi(x_{k+1})) - \\ - 2\, f(x_{k+1}) + f(x_{k+1} - n^{-\gamma}\varphi(x_{k+1})) \big\} \tag{2.2}$$

for $x_k \le x \le x_{k+1}$ $(k = -n+1, \ldots, n-1)$;

$$(U_n f)(x) - (S_n f)(x) = \\ = \frac{x_n - x}{x_n - x_{n-1}} \cdot \big\{ f(x_{n-1} + n^{-\gamma}\varphi(x_{n-1})) - \\ - 2\, f(x_{n-1}) + f(x_{n-1} - n^{-\gamma}\varphi(x_{n-1})) \big\} \tag{2.3}$$

for $x_{n-1} \le x \le x_n$.

On the other hand, by definition of the Ditzian-Totik modulus of smoothness we obtain from (2), (3), (4) and $\gamma \ge 1$ the estimates

$$|(U_n f)(x) - (S_n f)(x)| = \\ = \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_{-n+1})} f(x_{-n+1})| \\ \le \|\Delta^2_{n^{-\gamma}\varphi(x)} f(x)\| \le \omega^2_\varphi(f, n^{-1}),$$

where $x_{-n} \le x \le x_{-n+1}$;
$|(U_n f)(x) - (S_n f)(x)| \le$

$$\begin{aligned} &\le \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_k)} f(x_k)| + \\ &+ \frac{x-x_k}{x_{k+1}-x_k} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_{k+1})} f(x_{k+1})| \\ &\le \|\Delta^2_{n^{-\gamma}\varphi(x)} f(x)\| \le \omega^2_\varphi(f, n^{-1}), \end{aligned}$$

where $x_k \le x \le x_{k+1}$;

$$\begin{aligned} |(U_n f)(x) - (S_n f)(x)| &= \frac{x_n - x}{x_n - x_{n-1}} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_{n-1})} f(x_{n-1})| \\ &\le \|\Delta^2_{n^{-\gamma}\varphi(x)} f(x)\| \le \omega^2_\varphi(f, n^{-1}), \end{aligned}$$

where $x_{n-1} \le x \le x_n$. Hence $\|U_n f - S_n f\| \le \omega^2_\varphi(f, n^{-1})$. Using (1) we obtain the assertion of the theorem. □

Our next piecewise linear function is the following:

$$\begin{aligned} (V_n f)(x) &= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_k}^{x_k + c_1\delta_k} f(u)\, du + \\ &+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}} f(u)\, du, \end{aligned}$$

if $x_k < x < x_{k+1}$ $(k = -n, \ldots, n-1)$ and $(V_n f)(x) = f(x_k)$, if $x = x_k$ $(k = -n, \ldots, n)$.

Then $V_n : C[-1,1] \to L_\infty[-1,1]$ is a linear, positive operator such that $V_n f$ interpolates the function f at the points x_k $(k = -n, \ldots, n)$. The main difference between V_n and S_n is that $V_n f$ is not necessarily continuous at x_k $(k = -n+1, \ldots, n-1)$. Moreover, $V_n e_0 = e_0$ and $V_n e_1 = e_1 + O(n^{-1})$ (here $e_0(x) = 1$ and $e_1(x) = x$ for $x \in [-1,1]$). Indeed, the first statement is obvious and for the second one we have for $x_k < x < x_{k+1}$ $(k = -n, \ldots, n-1)$:

$$\begin{aligned} (V_n e_1)(x) &= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \frac{1}{2}\left(2x_k c_1\delta_k + c_1^2\delta_k^2\right) + \\ &+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \frac{1}{2}\left(2x_{k+1} c_1\delta_k - c_1^2\delta_k^2\right) \\ &= x + \frac{1}{2}\, c_1\delta_k \left(\frac{x_{k+1}-x}{x_{k+1}-x_k} - \frac{x-x_k}{x_{k+1}-x_k}\right). \end{aligned}$$

Hence, by properties of $(\delta_k)_{k=-n}^n$ we obtain

$$|(V_n e_1)(x) - e_1(x)| \le \frac{1}{2}\, c_1\delta_k \le \frac{c_0 c_1}{2n}, \qquad n \ge 1.$$

Furthermore, we denote the set of all algebraic polynomials of degree at most n by Π_n and the best uniform approximation on $[-1,1]$ by $E_n(f) = \inf\{\|f-p\| \ : \ p \in \Pi_n\}$. Our next results are the following:

Theorem 2. *Let $f \in C[-1,1]$ and V_n $(n \in N,\ n \geq 2)$ be defined as above. Then*

$$\|f - V_n f\| \ \leq$$

$$\leq \ c \left\{ \omega_\varphi^2(f, n^{-1}) \ + \ n^{-1} \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\omega_\varphi^2(f,t)}{t^3} \, dt \ + \ n^{-1} \ E_0(f) \right\}.$$

Corollary 1. *Let $f \in C[-1,1]$, $n \in N$, $n \geq 2$. Then*

$$\|f - V_n f\| \ \leq \ c \, n^{-1} \left\{ \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\omega_\varphi^2(f,t)}{t^3} \, dt \ + \ \|f\| \right\}.$$

To prove our statements we need a lemma:

Lemma 1. *For $g \in C^2[-1,1]$, $n \geq 2$ we have*

$$\|g - V_n g\| \ \leq \ c \ \left\{ n^{-1} \ \|g'\| \ + \ n^{-2} \ \|\varphi^2 g''\| \right\}.$$

Proof. If $x_k < x < x_{k+1}$ $(k = -n+1, \ldots, n-2)$, then

$$(V_n g)(x) - g(x) =$$

$$\begin{aligned}
&= \ \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \\
&\cdot \ \left\{ \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} [g(u) - g(x_k)] \ du + g(x_k) - g(x) \right\} \\
&+ \ \frac{x - x_k}{x_{k+1} - x_k} \cdot \\
&\cdot \ \left\{ \frac{1}{c_1 \delta_k} \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} [g(u) - g(x_{k+1})] \ du + g(x_{k+1}) - g(x) \right\}.
\end{aligned}$$

Simple computations show that

$$\int_\alpha^\beta [g(u) - g(\alpha)] \ du \ = \ \int_\alpha^\beta (\beta - u) \ g'(u) \ du$$

and

$$\int_\alpha^\beta [g(\beta) - g(u)] \ du \ = \ \int_\alpha^\beta (u - \alpha) \ g'(u) \ du.$$

Hence

$$
\begin{aligned}
(V_n g)(x) - g(x) &= \\
&= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \\
&\cdot \left\{ \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} (x_k + c_1 \delta_k - u)\ g'(u)\ du - \int_{x_k}^{x} g'(u)\ du \right\} \\
&+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \left\{ \frac{1}{c_1 \delta_k} \cdot \right. \\
&\cdot \left. \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} (x_{k+1} - c_1 \delta_k - u)\ g'(u)\ du + \int_{x}^{x_{k+1}} g'(u)\ du \right\} \\
&= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \cdot \\
&\cdot \left\{ \int_{x_k}^{x_k + c_1 \delta_k} (x_k + c_1 \delta_k - u)\ g'(u)\ du - \int_{x_k}^{x} c_1 \delta_k g'(u)\ du \right\} \\
&+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \cdot \left\{ \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} (x_{k+1} - c_1 \delta_k - u)\ g'(u)\ du + \right. \\
&+ \left. \int_{x}^{x_{k+1}} c_1 \delta_k g'(u)\ du \right\} \\
&= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \\
&\cdot \left\{ \int_{x_k}^{x} (x_k - u) g'(u)\ du + \int_{x}^{x_k + c_1 \delta_k} (x_k + c_1 \delta_k - u) g'(u)\ du \right\} \\
&+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \cdot \left\{ \int_{x_{k+1} - c_1 \delta_k}^{x} (x_{k+1} - c_1 \delta_k - u) g'(u)\ du + \right. \\
&+ \left. \int_{x}^{x_{k+1}} (x_{k+1} - u) g'(u)\ du \right\}.
\end{aligned}
$$

By partial integration we obtain

$$
\begin{aligned}
(V_n g)(x) - g(x) &= \\
&= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1 \delta_k} \left\{ -\frac{1}{2} (x_k - x)^2\ g'(x) + \right. \\
&+ \frac{1}{2} \int_{x_k}^{x} (x_k - u)^2 g''(u)\ du + \frac{1}{2} (x_k + c_1 \delta_k - x)^2\ g'(x) + \\
&+ \left. \frac{1}{2} \int_{x}^{x_k + c_1 \delta_k} (x_k + c_1 \delta_k - u)^2 g''(u)\ du \right\} +
\end{aligned}
$$

$$+ \quad \frac{1}{2} \int_{x}^{x_k+c_1\delta_k} (x_k + c_1\delta_k - u)^2 g''(u)\, du \Big\} \; +$$
$$+ \quad \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1\delta_k} \; \Big\{ -\frac{1}{2}\, (x_{k+1} - c_1\delta_k - x)^2 g'(x) \; +$$
$$+ \quad \frac{1}{2} \int_{x_{k+1}-c_1\delta_k}^{x} (x_{k+1} - c_1\delta_k - u)^2 g''(u)\, du \; +$$
$$+ \quad \frac{1}{2}\, (x_{k+1} - x)^2 g'(x) + \frac{1}{2} \int_{x}^{x_{k+1}} (x_{k+1} - u)^2 g''(u)\, du \Big\} . \qquad (2.4)$$

On the other hand, we get for $x_k \le u \le x_{k+1}$ by (i) and (ii)

$$\left| -\tfrac{1}{2}\, (x_k - x)^2 + \tfrac{1}{2}\, (x_k + c_1\delta_k - x)^2 \right| =$$
$$= \quad |c_1\delta_k(x_k - x) + \frac{1}{2}\, (c_1\delta_k)^2| = c_1\delta_k |(x_k - x) + \frac{1}{2}\, c_1\delta_k|$$
$$\le \quad c_1\delta_k(x - x_k + \frac{1}{2}\, c_1\delta_k) \; \le \; \frac{3}{2}\, c_1\delta_k \; (x_{k+1} - x_k)$$
$$\le \quad \frac{3}{2}\, c_1\delta_k \cdot c\, \frac{\sqrt{1-u^2}}{n} \; \le \; c \; \cdot \frac{c_1\delta_k}{n}; \qquad (2.5)$$

we have for $x_k \le u \le x < x_{k+1}$, by (ii)

$$(x_k - u)^2 \le (x_{k+1} - x_k)^2 \le c\, \frac{1-u^2}{n^2} \qquad (2.6)$$

and for $x_k < x \le u \le x_k + c_1\delta_k \le x_{k+1}$ or $x_k \le x_k + c_1\delta_k \le u \le x < x_{k+1}$ we have in view of (i) and (ii)

$$(x_k + c_1\delta_k - u)^2 \le 2\, (x_k - u)^2 + 2\, (c_1\delta_k)^2 \le 4\, (x_{k+1} - x_k)^2 \le c\frac{1-u^2}{n^2} \, . \qquad (2.7)$$

Using the same arguments we obtain

$$\left| -\frac{1}{2}\, (x_{k+1} - c_1\delta_k - x)^2 + \frac{1}{2}\, (x_{k+1} - x)^2 \right| \; \le \; c\frac{c_1\delta_k}{n} \qquad (2.8)$$

for $x_k \le u \le x_{k+1}$;

$$(x_{k+1} - c_1\delta_k - u)^2 \le c\, \frac{1-u^2}{n^2} \qquad (2.9)$$

for $x_k \le x_{k+1} - c_1\delta_k \le u \le x < x_{k+1}$ or $x_k < x \le u \le x_{k+1} - c_1\delta_k \le x_{k+1}$ and

$$(x_{k+1} - u)^2 \le c\frac{1-u^2}{n^2} \qquad (2.10)$$

for $x_k < x \le u \le x_{k+1}$. Then (5) – (11) and (ii) imply

$$
\begin{aligned}
&|(V_n g)(x) - g(x)| \le \\
&\le \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ |-\frac{1}{2}(x_k - x)^2 + \frac{1}{2}(x_k + c_1\delta_k - x)^2| \cdot \right.\\
&\cdot \quad |g'(x)| + \frac{1}{2} \int_{x_k}^{x} (u-x_k)^2 |g''(u)|\, du \; + \\
&+ \left. \frac{1}{2} \left| \int_{x}^{x_k + c_1\delta_k} (x_k + c_1\delta_k - u)^2 |g''(u)|\, du \right| \right\} + \\
&+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ |-\frac{1}{2}(x_{k+1} - c_1\delta_k - x)^2 + \frac{1}{2}(x_{k+1} - x)^2| \cdot \right.\\
&\cdot \quad |g'(x)| + \frac{1}{2} \left| \int_{x_{k+1}-c_1\delta_k}^{x} (x_{k+1} - c_1\delta_k - u)^2 |g''(u)|\, du \right| + \\
&+ \left. \frac{1}{2} \int_{x}^{x_{k+1}} (x_{k+1} - u)^2 |g''(u)|\, du \right\} \\
&\le \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ c \cdot \frac{c_1\delta_k}{n} \cdot \|g'\| + \right.\\
&+ \frac{1}{2} \int_{x_k}^{x} c \cdot \frac{1-u^2}{n^2} \cdot |g''(u)|\, du \; + \\
&+ \left. \frac{1}{2} \left| \int_{x}^{x_k + c_1\delta_k} c \cdot \frac{1-u^2}{n^2} |g''(u)|\, du \right| \right\} + \\
&+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ c \cdot \frac{c_1\delta_k}{n} \|g'\| + \right.\\
&+ \frac{1}{2} \left| \int_{x_{k+1}-c_1\delta_k}^{x} c \cdot \frac{1-u^2}{n^2} |g''(u)|\, du \right| + \\
&+ \left. \frac{1}{2} \int_{x}^{x_{k+1}} c \cdot \frac{1-u^2}{n^2} |g''(u)|\, du \right\}. \qquad (2.11)
\end{aligned}
$$

But

$$\int_{x_k}^{x} + \left| \int_{x}^{x_k+c_1\delta_k} \right| = \int_{x_k}^{x} + \int_{x}^{x_k+c_1\delta_k} = \int_{x_k}^{x_k+c_1\delta_k}$$

if $x_k < x \le x_k + c_1\delta_k$ or

$$\int_{x_k}^{x} + \left| \int_{x}^{x_k+c_1\delta_k} \right| = \int_{x_k}^{x} + \int_{x_k+c_1\delta_k}^{x} \le \int_{x_k}^{x} + \int_{x_k}^{x} = 2 \int_{x_k}^{x}$$

if $x_k + c_1\delta_k \le x < x_{k+1}$ and

$$\left| \int_{x_{k+1}-c_1\delta_k}^{x} \right| + \int_{x}^{x_{k+1}} = \int_{x_{k+1}-c_1\delta_k}^{x} + \int_{x}^{x_{k+1}} = \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}}$$

if $x_{k+1} - c_1\delta_k \le x < x_{k+1}$ or

$$\left| \int_{x_{k+1}-c_1\delta_k}^{x} \right| + \int_x^{x_{k+1}} = \int_x^{x_{k+1}-c_1\delta_k} + \int_x^{x_{k+1}} \le$$
$$\le \int_x^{x_{k+1}} + \int_x^{x_{k+1}} = 2 \int_x^{x_{k+1}}$$

if $x_k < x \le x_{k+1} - c_1\delta_k$. So we obtain in view of (12) that

$|(V_n g)(x) - g(x)| \le$

$$\le \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ c \cdot \frac{c_1\delta_k}{n} \|g'\| + \right.$$
$$\left. + c \cdot \frac{\|\varphi^2 g''\|}{n^2} \cdot \max\left(c_1\delta_k; x - x_k\right) \right\}$$
$$+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ c \cdot \frac{c_1\delta_k}{n} \|g'\| + \right.$$
$$\left. + c \cdot \frac{\|\varphi^2 g''\|}{n^2} \cdot \max(c_1\delta_k; x_{k+1} - x) \right\}$$
$$\le c \left\{ n^{-1} \|g'\| + n^{-2} \|\varphi^2 g''\| \cdot \max\left(1; \frac{x_{k+1}-x_k}{c_1\delta_k}\right) \right\}.$$

By (i) we have $x_{k+1} - x_k \le c_2\delta_k$. Therefore

$$|(V_n g)(x) - g(x)| \le c \left\{ n^{-1}\|g'\| + n^{-2}\|\varphi^2 g''\| \right\}. \tag{2.12}$$

If $x_{-n} < x < x_{-n+1}$ then

$$(V_n g)(x) - g(x) =$$
$$= \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right.$$
$$\left. \cdot \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} [g(u) - g(x_{-n})] \, du + g(x_{-n}) - g(x) \right\}$$
$$+ \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right.$$
$$\left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} [g(u) - g(x_{-n+1})] \, du + g(x_{-n+1}) - g(x) \right\}$$

$$\begin{aligned}
&= \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} [g(u)-g(x_{-n})] \ du - \int_{x_{-n}}^{x} g'(u) \ du \right\} \\
&+ \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} [g(u)-g(x_{-n+1})] \ du + \int_{x}^{x_{-n+1}} g'(u) \ du \right\} \\
&= \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} \left[\int_{x_{-n}}^{u} g'(v) \ dv \right] \ du - \int_{x_{-n}}^{x} g'(u) \ du \right\} \\
&+ \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ -\frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} \left[\int_{u}^{x_{-n+1}} g'(v) \ dv \right] \ du + \int_{x}^{x_{-n+1}} g'(u) \ du \right\}.
\end{aligned}$$

Hence

$|(V_ng)(x)-g(x)| \leq$

$$\begin{aligned}
&\leq \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} \left[\int_{x_{-n}}^{u} |g'(v)| \ dv \right] \ du + \int_{x_{-n}}^{x} |g'(u)| \ du \right\} \\
&+ \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} \left[\int_{u}^{x_{-n+1}} |g'(v)| \ dv \right] \ du + \int_{x}^{x_{-n+1}} |g'(u)| \ du \right\} \\
&\leq \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} (u-x_{-n}) \ du + (x-x_{-n}) \right\} \ \|g'\| \\
&+ \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
&\cdot \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} (x_{-n+1}-u) \ du + (x_{-n+1}-x) \right\} \ \|g'\|.
\end{aligned}$$

In view of (i) we have $x_{-n} \le u \le x_{-n} + c_1\delta_{-n} \le x_{-n+1}$, $x_{-n} \le x_{-n+1} - c_1\delta_{-n} \le u \le x_{-n+1}$ and $x_{-n} \le x \le x_{-n+1}$, respectively. So

$$\begin{aligned}
|(V_n g)(x) - g(x)| \le& \\
\le\ & \frac{x_{-n+1} - x}{x_{-n+1} - x_{-n}} \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot\ & \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} (x_{-n+1} - x_{-n})\, du + (x_{-n+1} - x_{-n}) \right\} \|g'\| \\
+\ & \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot\ & \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} (x_{-n+1} - x_{-n})\, du + (x_{-n+1} - x_{-n}) \right\} \|g'\| \\
\le\ & 2\, (x_{-n+1} - x_{-n})\, \|g'\|.
\end{aligned}$$

Again, (i) implies $x_{-n+1} - x_{-n} \le c_2\delta_{-n}$. This means that

$$|(V_n g)(x) - g(x)| \ \le\ 2\ c_2\ \delta_{-n}\ \|g'\| \ \le\ \frac{c}{n^2}\ \|g'\|. \tag{2.13}$$

Analogously

$$|(V_n g)(x) - g(x)| \ \le\ \frac{c}{n^2}\ \|g'\| \tag{2.14}$$

for $x_{n-1} < x < x_n$. In conclusion (13), (14) and (15) imply

$$\|g - V_n g\| \ \le\ c\ \left\{\ n^{-1}\ \|g'\|\ +\ n^{-2}\ \|\varphi^2 g''\|\ \right\},$$

which completes the proof. □

Proof of Theorem 2. We have

$$\begin{aligned}
|(V_n f)(x)| \ \le\ & \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_k}^{x_k + c_1\delta_k} |f(u)|\, du \\
+\ & \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_{k+1} - c_1\delta_k}^{x_{k+1}} |f(u)|\, du \ \le\ \|f\|
\end{aligned}$$

for $x_k < x < x_{k+1}$ $(k = -n, \ldots, n-1)$. Thus

$$\|V_n f\| \ \le\ \|f\|. \tag{2.15}$$

On the other hand, let us denote by $p_n \in \Pi_n$ the best nth degree polynomial approximation to f. Then we know for $f \in C[-1,1]$ (see [2, p. 79, Theorem 7.2.1]) that

$$E_n(f) \ =\ \|f - p_n\| \ \le\ c\ \omega_\varphi^2(f, n^{-1}). \tag{2.16}$$

Moreover, we have the following Bernstein type inequality [2, p. 84, Theorem 7.3.1] for the best approximation polynomial

$$\|\varphi^2 p_n''\| \ \le\ n^2\ \omega_\varphi^2(f, n^{-1}). \tag{2.17}$$

Then, using (16), Lemma 1, (17), (18) and the proof of [4, pp. 83 - 84, Theorem 3.2] we obtain the conclusion of our theorem. □

Proof of Corollary 1. In view of [4, p. 86, Remark 3.4], the proof is a direct consequence of the fact that we can drop the first term on the right hand side of the estimate given in Theorem 2 because of

$$\int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\omega_\varphi^2(f,t)}{t^3}\, dt \;\geq\; \omega_\varphi^2(f,n^{-1}) \cdot \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{dt}{t^3} \;\geq\; c\, n^2\, \omega_\varphi^2(f,n^{-1}), \quad n > 2.$$

□

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