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Almost sure convergence of weighted sums

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ALMOST SURE CONVERGENCE OF WEIGHTED SUMS

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Abstract. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{i,n}, 1 \leq i \leq n\}$ be a triangular array of constants. In this short paper, we establish a general almost sure convergence theorem for the weighted sum $S_n = \sum_{i=1}^n a_{i,n} X_i$. Our results improves the works of Sung [14]. Furthermore, almost sure convergence theorems of S_n for negatively associated random variables, martingale difference sequence and ρ -mixing sequence are obtained.

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1. INTRODUCTION

Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{i,n}, 1 \leq i \leq n\}$ be a triangular array of constants. Define a weighted sum by

$$S_n = \sum_{i=1}^n a_{i,n} X_i.$$

Note that many useful linear statistics are in this form, e.g., least squares estimators, nonparametric regression function estimators, jackknife estimates and so on. Studies on strong laws of S_n are of great interest not only in probability theory but also in statistics.

1.1. Independent random variables

There is a sizable literature on the almost sure (a.s.) convergence of weighted sums S_n when $\{X, X_n, n \geq 1\}$ are assumed to be independent random variables (see for example Choi and Sung [3], Chow [4], Chow and Lai [5], Stout [13], Sung [14, 15], Teicher [16], and Thrum [17]).

Firstly, let us recall the following result of Teicher.

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Theorem 1 (Teicher [16], P.341). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$, for some $1 \leq p \leq 2$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O\left(1/(n^{1/p} \log n)\right), \quad (1.1)$$

then $S_n \rightarrow 0$ a.s.

Choi and Sung [3], Sung [15] and Sung [14] ($p = 1$, $1 < p < 2$ and $p = 2$, respectively) improved the above work of Teicher.

Theorem 2 ([3], Theorem 5). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O(1/n), \quad (1.2)$$

then $S_n \rightarrow 0$ a.s.

Theorem 3 ([15], Theorem 1). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $1 \leq p < 2$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O\left(\frac{1}{n^{1/p} (\log n)^{1-1/p}}\right), \quad (1.3)$$

then $S_n \rightarrow 0$ a.s.

Theorem 4 ([14], Corollary 4). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^2 < \infty$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = o\left(\frac{1}{\sqrt{n \log n}}\right), \quad (1.4)$$

then $S_n \rightarrow 0$ a.s.

By comparing Theorem 3 and Theorem 4, it is natural to ask

Problem 1. *whether the condition (1.4) can be replaced by*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O\left(\frac{1}{\sqrt{n \log n}}\right).$$

1.2. Negatively associated random variables

Definition 1. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA), if for every pair of disjoint subset A and B of $\{1, 2, \dots, n\}$ and any real nondecreasing coordinatewise functions f_1 on \mathbb{R}^A and f_2 on \mathbb{R}^B

$$\text{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever f_1 and f_2 are such that covariance exists. An infinity family of random variables is negatively associated if every finite subfamily is negatively associated.

The notion of negative association was first introduced by Alam and Saxena [1]. Joag-Dev and Proschan [9] showed that many well known multivariate distributions possess the NA property. Some examples include: (a) multinomial, (b) convolution of unlike multinomial, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, (f) negatively correlated normal distribution, (g) permutation distribution, (h) random sampling without replacement, and (i) joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and system reliability, the notion of negative association has received considerable attention recently.

In particular, we refer Jing and Liang [8] for the strong limit theorems for weighted sums of negatively associated random variables as follows.

Theorem 5 ([8], Theorem 2.1). *Suppose that $\mathbb{E}|X|^p < \infty$ for some $p > 0$ and $\mathbb{E}X_1 = 0$ if $p > 1$. Assume that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O(n^{-1/p}).$$

- (1) *if $p > 2$ and $\sum_{i=1}^n a_{i,n}^2 = o((\log n)^{-1})$, then $S_n \rightarrow 0$ a.s.*
- (2) *if $p > 2$ and $\sum_{i=1}^n a_{i,n}^2 = O((\log n)^{-1})$, then $S_n = O(1)$ a.s.*
- (3) *if $0 < p \leq 2$ and $\sum_{i=1}^n |a_{i,n}|^p = O(n^{-\delta})$ for some $\delta > 0$, then $S_n \rightarrow 0$ a.s.*

Based on the above works, we want to ask

Problem 2. *whether the similar strong limits for the weight sums of negative associated random variables hold only in the condition to bound $\max_{1 \leq i \leq n} |a_{i,n}|$.*

So, in this paper, the main purpose is to establish a very general almost sure convergence for weighted sums, which only depends on the bound of the weights $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ and on the almost sure convergence of partial sums. From the general result, Problem 1 and Problem 2 are answered, and some further strong limits theorem are obtained which improve or supplement some relevant results.

2. A GENERAL THEOREM

The main result in the paper is the following very general theorem, which includes many of the results about almost sure convergence, but the proof is very elementary.

Theorem 6. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1}.$$

where $0 < b_n \uparrow +\infty$. If $\{Z_n\}$ satisfies

$$Z_n := b_n^{-1}(X_1 + \cdots + X_n) \rightarrow 0 \quad a.s., \quad (2.1)$$

then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Without loss of generality, we can suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies $a_{1,n} \geq a_{2,n} \geq \dots \geq a_{n,n}$. In fact, if the order does not be satisfied, then we can rearrange the sequence $\{a_{i,n} X_i, 1 \leq i \leq n, n \geq 1\}$.

For $1 \leq i \leq n$, let $\tilde{a}_{i,n} = a_{i,n} b_n$ (which implies $|\tilde{a}_{i,n}| \leq 1$) and

$$c_{i,n} = \begin{cases} \frac{b_i}{b_n} (\tilde{a}_{i,n} - \tilde{a}_{i+1,n}), & \text{if } 1 \leq i \leq n-1 \\ \tilde{a}_{n,n}, & \text{if } i = n. \end{cases}$$

Then we have

$$\begin{cases} c_{i,n} \geq 0, & \text{if } 1 \leq i \leq n-1; \\ \sum_{i=1}^n |c_{i,n}| \leq 3; \\ \lim_{n \rightarrow \infty} |c_{i,n}| = 0, & \text{for every fixed } i, \end{cases} \quad (2.2)$$

and

$$S_n = \sum_{i=1}^n a_{i,n} X_i = \frac{1}{b_n} \sum_{i=1}^n \tilde{a}_{i,n} X_i = \sum_{i=1}^n c_{i,n} Z_i. \quad (2.3)$$

From (2.2), we can claim that for every sequences of real numbers r_n with $r_n \rightarrow 0$, the following holds

$$\sum_{i=1}^n c_{i,n} r_i \rightarrow 0. \quad (2.4)$$

Let us give the proof of (2.4). For any $\varepsilon > 0$, there exists a positive constant N_ε , such that $|r_n| < \varepsilon$, when $n > N_\varepsilon$. So, we have

$$\left| \sum_{i=1}^n c_{i,n} r_i \right| \leq \left| \sum_{i=1}^N c_{i,n} r_i \right| + \left| \sum_{i=N+1}^n c_{i,n} r_i \right| < \left| \sum_{i=1}^N c_{i,n} r_i \right| + 3\varepsilon. \quad (2.5)$$

Furthermore, from the condition that $\lim_{n \rightarrow \infty} |c_{i,n}| = 0$ for every i , we have

$$\left| \sum_{i=1}^N c_{i,n} r_i \right| \rightarrow 0 \quad (2.6)$$

which together with (2.5) implies (2.4). Hence by the condition (2.1) and the relation (2.3), we have $S_n \rightarrow 0$ a.s. \square

3. SOME APPLICATIONS

3.1. Independent random variables

In this subsection, we obtain the almost sure convergence theorems for the weighted sums, which improve Theorem 3 and Theorem 4.

Corollary 1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^2 < \infty$. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = o\left((n \log \log n)^{-1/2}\right). \quad (3.1)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. By the Hartman-Wintner law of the iterated logarithm for the i.i.d sequence and the condition (3.1), we have

$$b_n^{-1}(X_1 + X_2 + \cdots + X_n) \rightarrow 0, \quad a.s.$$

which yields Corollary 1 from Theorem 6. \square

Remark 1. By taking $b_n = \sqrt{n \log n}$, it is easy to see that Corollary 1 improves Sung's result (see Theorem 4).

Corollary 2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $0 < p < 2$. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = O\left(n^{-1/p}\right). \quad (3.2)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. By the Marcinkiewicz-Zygmund strong law of large numbers, the proof of the corollary can be completed. \square

Remark 2. Obviously, Corollary 2 improves the result in Theorem 3.

Remark 3. Recently, Bai and Cheng [2] and Cuzick [6] proved the Marcinkiewicz-Zygmund strong laws of large numbers for the weighted sums

$$n^{-1/p} \sum_{i=1}^n a_{i,n} X_i \rightarrow 0, \quad a.s.$$

when $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^\beta < \infty$ and $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying

$$A_\alpha = \limsup_{n \rightarrow \infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^\alpha = n^{-1} \sum_{i=1}^n |a_{i,n}|^\alpha,$$

where $1 \leq p < 2$, $0 < \alpha, \beta < \infty$ and $1/p = 1/\alpha + 1/\beta$. Bai and Cheng [2] proved it for $1 < p < 2$ by the Bernstein inequality and Cuzick [6] for $p = 1$.

Note that the order of the moment β is higher than the rate of convergence p , so Corollary 2 improves the above results.

3.2. Negatively associated sequence

Corollary 3. *Let $\{X, X_n, n \geq 1\}$ be a sequence of strictly stationary negatively associated random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^2 < \infty$. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = o\left((n \log \log n)^{-1/2}\right). \quad (3.3)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. In [12], the authors obtained the following result: for any $0 < \varepsilon < 1/30$,

$$\limsup_{n \rightarrow \infty} \frac{|X_1 + \cdots + X_n|}{(2\mathbb{E}X^2 n \log \log n)^{1/2}} \leq 1 + 8\varepsilon, \quad a.s.$$

Hence from the condition (3.3), we have

$$b_n^{-1}(X_1 + X_2 + \cdots + X_n) \rightarrow 0, \quad a.s.$$

Based on Theorem 6, we complete the proof of this corollary. \square

Remark 4. By comparing Theorem 5 with Corollary 3, one can find that Corollary 3 complement the works of Jing and Liang for $p = 2$.

3.3. Martingale difference sequence

Corollary 4. *Let $\{X_n, n \geq 0\}$ be a stationary stochastic sequence such that $\mathbb{E}(X_n | \mathcal{F}_{n-1}) = 0$ a.s., all $n \geq 1$, and $\mathbb{E}X_0^2 = 0$. Here \mathcal{F}_n is the σ -field generated by $Z_i, i \leq n$, where $\{\cdots, Z_{-1}, Z_0, Z_1, \cdots\}$ is a stationary stochastic sequence such that $X_k = \phi(Z_k, Z_{k-1}, \cdots)$, $k \geq 0$, for some measurable ϕ . We suppose also that*

$$n^{-1} \sum_{i=1}^n \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}) \rightarrow 1 \quad a.s.$$

Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = o\left((n \log \log n)^{-1/2}\right). \quad (3.4)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Under the assumptions of Corollary 4, we know (see [7, Theorem 1])

$$\limsup_{n \rightarrow \infty} \frac{|X_1 + \cdots + X_n|}{(2n \log \log n)^{1/2}} = 1 \quad a.s.$$

So the desired result can be easily obtained. \square

Remark 5. If we take $Z_i = X_i$ and assume that $\{X_i\}$ is a stationary ergodic martingale difference sequence, then the condition $n^{-1} \sum_{i=1}^n \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}) \rightarrow 1$ a.s. can follow as a consequence of the ergodic theorem.

Corollary 5. Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ denote a sequence of identically distributed martingale differences. Suppose that

- (i) $\mathbb{E}|X_1|^p < \infty$ for some $0 < p < 2$ and $p \neq 1$;
- (ii) $\mathbb{P}(|X_n| \geq x | X_1, \dots, X_{n-1}) \leq \mathbb{P}(|X_1| \geq x)$ a.s. for $p = 1$.

Let $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = O\left(n^{-1/p}\right). \quad (3.5)$$

then if (i) or (ii) holds we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Using the well-know Marcinkiewicz's strong law of large numbers of martingale differences (see [10, P. 53]), the proof is done. \square

3.4. ρ -mixing sequence

A sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called ρ -mixing if the maximal correlation coefficient

$$\rho(n) = \sup_{k \geq 1, X \in L^2(\mathcal{F}_1^k), Y \in L^2(\mathcal{F}_{k+n}^\infty)} |Cov(X, Y)| / \|X\|_2 \|Y\|_2 \rightarrow 0$$

as $n \rightarrow \infty$, where \mathcal{F}_n^m is the σ -field generated by the random variables X_n, X_{n+1}, \dots, X_m . Here $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

Corollary 6. Let $\{X, X_n, n \geq 1\}$ denote a ρ -mixing sequence of identically distributed random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $1 \leq p < 2$. Suppose that

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty$$

and $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = O(n^{-1/p}). \quad (3.6)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Under the conditions of the corollary, the proof can be completed by using the Marcinkiewicz-Zygmund law of large numbers for ρ -mixing sequences (see [11, Corollary 3.1]). \square

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