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THE (NON-)EXTENDABILITY OF EMBRY'S THEOREM

HRANISLAV STANKOVIĆ

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Abstract. Let A, P and T be bounded linear operators on a Hilbert space such that P is an orthogonal projection commuting with A, and zero is not in the numerical range of T. We prove that if $PT = TA$, then $A = P$. As a consequence, Embry's Theorem on the similarity of normal operators follows easily from our result. Furthermore, we demonstrate that this theorem cannot be extended to quasinormal operators, thereby providing a negative answer to a conjecture posed by Mortad in [9].

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1. INTRODUCTION

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, and let $\mathfrak{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathfrak{B}(\mathcal{H})$, we denote its null space and range by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. For the spectrum of T we use the notation $\sigma(T)$, while the numerical range will be denoted by $\mathcal{W}(T)$. Recall that

$$\mathcal{W}(T) = \{\langle Tx, x \rangle : \|x\| = 1\}.$$

The adjoint of T is denoted by T^*. An operator T is called positive, written $T \geq 0$, if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and it is called self-adjoint (or Hermitian) if $T = T^*$. An operator $P \in \mathfrak{B}(\mathcal{H})$ is called an orthogonal projection if it satisfies $P = P^* = P^2$. It is clear that any orthogonal projection is a positive operator.

An operator T is said to be normal if $T^*T = TT^*$. The theory of normal operators has been studied in great depth, largely due to the powerful structure afforded by the Spectral Theorem, which serves as a foundational tool in their analysis. For a comprehensive introduction to Spectral Theorem and its applications, see [11].

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A fundamental result in the theory of normal operators is the Fuglede–Putnam Theorem [5, 13], which asserts that if $A,B \in \mathfrak{B}(\mathcal{H})$ are normal operators and $T \in \mathfrak{B}(\mathcal{H})$ satisfies $AT = TB$, then it necessarily follows that $A^*T = TB^*$. This theorem highlights the rigidity of normal operators under intertwining and has profound implications in spectral theory. For the different proofs of Fuglede-Putnam Theorem, and in different settings, see [5, 7, 13–15, 17].

Fuglede-Putnam Theorem has many fundamental consequences. For example, it is used to show that the product of two commuting normal operators must be normal (cf. [8]).

Another very interesting result relying on the Fuglede-Putnam Theorem is the following theorem by Embry [4].

Theorem 1 (Embry's Theorem)**.** *[4] Let $A,B \in \mathfrak{B}(\mathcal{H})$ be commuting normal operators. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then $A = B$.*

The importance of this theorem lies in its elegant combination of spectral theory, the geometry of the numerical range, and operator similarity. In the same paper, Embry also demonstrated that neither the commutativity condition on A and B, nor the numerical range condition $0 \notin \mathcal{W}(T)$, can be omitted. Moreover, the same counterexample reveals that the condition $0 \notin \mathcal{W}(T)$ cannot be weakened to $0 \notin \sigma(T)$. For an interesting comparison between Embry's Theorem and Fuglede-Putnam Theorem, see [10].

This paper revisits Embry's Theorem and explores its potential generalizations beyond the class of normal operators.

2. Main results

Our approach begins with a seemingly elementary, yet powerful, characterization involving orthogonal projections and the numerical range.

Theorem 2. *Let $A,P,T \in \mathfrak{B}(\mathcal{H})$ be such that P is an orthogonal projection commuting with A and $0 \notin \mathcal{W}(T)$. If $PT = TA$, then $A = P$.*

Proof. Without loss of generality, we may assume that $P \notin \{0, I\}$. With respect to the decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$, we can write

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

Since P and A commute, we immediately have that $A_2 = 0$ and $A_3 = 0$, i.e.,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}.$$

Condition $PT = TA$ is now equivalent with the system of equations

$$T_1(I - A_1) = 0, \quad T_2(I - A_4) = 0, \quad T_3 A_1 = 0, \quad T_4 A_4 = 0. \tag{2.1}$$

Next, note that T_1 and T_4 are one-to-one. Indeed, let $x \in \mathcal{N}(T_1) \subseteq \mathcal{R}(P)$, and let $x' = \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{H}$. Then,

$$\langle Tx', x' \rangle = \left\langle \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle = \langle T_1 x, x \rangle = 0,$$

and so it must be $x = 0$. Similarly, we can show that $\mathcal{N}(T_4) = \{0\}$. The first and the last equation in (2.1) now imply that $A_1 = I$ and $A_4 = 0$, which clearly demonstrates that $A = P$. □

Remark 1. It is easy to see that the previous theorem also holds if P is any scalar multiple of an orthogonal projection.

As a consequence, we recover Embry's Theorem.

Proof of Theorem 1. Let E_A and E_B be the spectral measures corresponding to A and B, respectively, and let Δ be an arbitrary Borel set in the complex plane. Since $AT = TB$, the Fuglede-Putnam Theorem gives

$$E_A(\Delta)T = TE_B(\Delta).$$

Moreover, since A and B commute, the Fuglede Theorem also ensures that

$$E_A(\Delta)E_B(\Delta) = E_B(\Delta)E_A(\Delta).$$

Theorem 2 now yields that $E_A(\Delta) = E_B(\Delta)$. Since Δ was arbitrary, we conclude that $A = B$. □

A natural attempt to generalize Embry's Theorem and extend it to some super-classes of normal operators is to follow the path of generalizing Fuglede-Putnam Theorem. The crucial insight that allowed certain generalizations is to replace one of the normal operators with its adjoint. Recall that $T \in \mathfrak{B}(\mathcal{H})$ is called p-hyponormal for some $0 < p \leq 1$ if $(TT^*)^p \leq (T^*T)^p$. If $p = 1$, T is simply called a hyponormal operator. Clearly, any normal operator is p-hyponormal. For example, we have the following generalization of the Fuglede-Putnam Theorem:

Theorem 3. *[3, Theorem 7] Let $A, B \in \mathfrak{B}(\mathcal{H})$ be such that A and B^* are p-hyponormal operators. If*

$$AX = XB$$

for some $X \in \mathfrak{B}(\mathcal{H})$, then

$$A^*X = XB^*.$$

For many other generalizations of the Fuglede-Putnam Theorem, see [12] and the references therein.

We may try to extend Embry's Theorem in the same way. However, such a generalization would not fundamentally change the original theorem, as the following discussion shows. We say that $(A,B) \in \mathfrak{B}(\mathcal{H})^2$ has the FP-property if for any $X \in \mathfrak{B}(\mathcal{H})$,

$$AX = XB \quad \Longrightarrow \quad A^*X = XB^*.$$

Also, recall the following result.

Corollary 1. *[16, Corollary 1] Suppose that $(A,B) \in \mathfrak{B}(\mathcal{H})^2$ has the FP-property. If there exists a quasi-affinity $X \in \mathfrak{B}(\mathcal{H})$ (X is one-to-one and has dense range) such that $AX = XB$, then A and B are unitarily equivalent normal operators.*

Theorem 4. *Suppose that a commuting pair $(A,B) \in \mathfrak{B}(\mathcal{H})^2$ has the FP-property. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then A and B are normal, and $A = B$.*

Proof. Since the condition $0 \notin \mathcal{W}(T)$ implies that T is quasi-affinity, the conclusion follows immediately from Corollary 1 and Theorem 1. □

In particular, Theorem 3 yields the following:

Theorem 5. *Let $A,B \in \mathfrak{B}(\mathcal{H})$ be such that A and B^* are p-hyponormal operators, and A and B commute. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then A and B are normal, and $A = B$.*

Another attempt to generalize Embry's Theorem is the following conjecture which appeared in [9] (cf. [12]).

Conjecture 1. *Let $A,B \in \mathfrak{B}(\mathcal{H})$ be two commuting hyponormal (subnormal, quasinormal) operators. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then $A = B$.*

Recall that $T \in \mathfrak{B}(\mathcal{H})$ is called subnormal if it has a normal extension, and quasinormal if it commutes with T^*T. It is also well-known that

$$\text{normal} \Longrightarrow \text{quasinormal} \Longrightarrow \text{subnormal} \Longrightarrow \text{hyponormal}.$$

The following example shows that the Conjecture 1 does not hold even for quasinormal operators.

Example 1. Let $P \in \mathfrak{B}(\mathcal{H})$ be a non-zero positive operator, and let $0 < q < 1$. Define operators $A,B,T \in \mathfrak{B}(\bigoplus_{i=1}^{\infty} \mathcal{H})$ as follows:

$$A = \begin{bmatrix} 0 & & & & \\ qP & 0 & & & \\ 0 & qP & 0 & & \\ & 0 & qP & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & & & & \\ P & 0 & & & \\ 0 & P & 0 & & \\ & 0 & P & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

$$T = \begin{bmatrix} I & 0 & & & \\ 0 & qI & 0 & & \\ & 0 & q^2 I & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

By Brown's characterization of quasinormal operators (see [1] or [2, Chapter 2, Theorem 3.2]), or by direct verification, we have that A and B are quasinormal, and it is evident that $AB = BA$. Using [6, Chapter 1, Proposition 1.8] and the obvious fact that

$$\mathcal{W}(q^n I) = \{q^n\} \quad (n \geq 0),$$

we have that $\mathcal{W}(T)$ is the convex hull of the set $\{q^n : n \geq 0\}$, which is clearly $(0,1]$. Thus, $0 \notin \mathcal{W}(T)$. Finally,

$$AT = \begin{bmatrix} 0 & & & & \\ qP & 0 & & & \\ 0 & q^2 P & 0 & & \\ & 0 & q^3 P & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = TB,$$

while $A \neq B$.

3. Conclusion

In conclusion, our analysis underscores the intrinsic connection between Embry's Theorem and the class of normal operators. Under the current restriction on the numerical range, any meaningful generalization of the theorem appears implausible. Nevertheless, as illustrated in Example 1, although $0 \notin \mathcal{W}(T)$, it is in fact the case that $0 \in \overline{\mathcal{W}(T)}$. This observation suggests that a variant of Embry's Theorem under the stronger condition $0 \notin \overline{\mathcal{W}(T)}$ might be worth investigating.

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Author's address

Hranislav Stanković

Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, Niš, Serbia

E-mail address: hranislav.stankovic@elfak.ni.ac.rs