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A NOTE ON RAZUMIKHIN THEOREMS IN UNIFORM ULTIMATE BOUNDEDNESS

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Abstract. Uniform boundedness and uniform ultimate boundedness of solutions of retarded functional differential equations are studied by Liapunov functions. The obtained result in this work improves Razumikhin theorems on uniform ultimate boundedness. Some examples are given to illustrate the advantage of the obtained result.

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1. INTRODUCTION

Retarded functional differential equations (RFDEs) are a general type of equations and they include ordinary differential equations and differential difference equations [4]. Significance of boundedness (BD) and ultimate boundedness (UB) of partial solutions of RFDEs is widely known. For some problems, one may be interested only in some variables, and it often is very difficult to solve once for all the problems of whole variables about some complex systems. On the other hand, the BD and the UB of whole variables can be obtained by the BD and the UB of partial variables [4, 6, 7].

In fact, the BD and the UB of partial solutions were studied for ordinary differential equations (ODEs) and difference equations. For some results in the area see, for example, [2, 5–7, 9–11, 13–19] and the references cited therein.

On the results of ODEs, in [5] the authors obtained two Lyapunov-like theorems for the BD and the UB of partial solutions of nonlinear dynamical system

$$\dot{x}_1(t) = f_1(x_1(t),\ x_2(t)),$$
$$\dot{x}_2(t) = f_2(x_1(t),\ x_2(t)),$$

where $x_1 \in D,\ D \subseteq \mathbb{R}^{n_1}$ is an open set, $x_2 \in \mathbb{R}^{n_2}$, $f_1 : D \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : D \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ (see Theorem 3.1 and Theorem 3.2). In [6] the author obtained four

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results (sufficient conditions) for the BD and the UB (dissipation) of partial solutions of the nonlinear autonomous system

$$\begin{aligned}\frac{dy}{dt} &= col\left(\sum_{j=1}^{n} f_{1\,j}(x_j), \cdots, \sum_{j=1}^{n} f_{m\,j}(x_j)\right),\\ \frac{dz}{dt} &= col\left(\sum_{j=1}^{n} f_{m+1\,j}(x_j), \cdots, \sum_{j=1}^{n} f_{n\,j}(x_j)\right)\end{aligned}$$

(see Theorem 1 – Theorem 4), where $y = col(x_1, \cdots, x_m)$, $z = col(x_{m+1}, \cdots, x_n)$; $f_{i\,j}(x_j) \in C(-\infty, +\infty)$. In [7] the authors obtained four results (sufficient conditions) for the uniform ultimate boundedness (uniform dissipation) of partial solutions of the nonlinear nonautonomous system

$$\begin{aligned}\frac{dy}{dt} &= col\left(\sum_{j=1}^{n} f_{1\,j}(t, x_j), \cdots, \sum_{j=1}^{n} f_{m\,j}(t, x_j)\right),\\ \frac{dz}{dt} &= col\left(\sum_{j=1}^{n} f_{m+1\,j}(t, x_j), \cdots, \sum_{j=1}^{n} f_{n\,j}(t, x_j)\right)\end{aligned}$$

(see Theorem 1 – Theorem 4), where $f_{i\,j}(t, x_j) \in C([t_0, +\infty) \times \mathbb{R}, \mathbb{R})$. In monograph [8] the author obtained some results (sufficient and necessary conditions, and sufficient conditions) for the BD of partial solutions of the nonlinear ordinary differential equation

$$\dot{x}(t) = f(t, x(t))$$

(see Theorem 8.38 and Theorem 8.39), where $f \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$.

On the results of the BD and the UB of partial solutions of difference equations, in [18] the authors obtained a result (sufficient conditions) for the uniform ultimate boundedness (UUB) of partial solutions of the large-scale discrete system

$$x_i(\tau + 1) = f_i[\tau, x_i(\tau)] + g_i[\tau, x_1(\tau), \cdots, x_m(\tau)] \ \ (i = 1, 2, \cdots, m),$$

where $\tau \in I = \{\tau_0 + k : k = 0, 1, 2, \cdots\}, x_i \in \mathbb{R}^{n_i}$, $f_i : I \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}, g_i : I \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \to \mathbb{R}^{n_i}$, $\sum_{i=1}^{m} n_i = n$ (see Theorem 3). In [15] the authors gave some results (sufficient conditions) for the BD and the UB of partial solutions of discrete-time nonlinear dynamical system

$$\begin{aligned}x_1(k + 1) &= f_1(x_1(k),\ x_2(k)),\\ x_2(k + 1) &= f_2(x_1(k),\ x_2(k)),\end{aligned}$$

where $k \in \{0, 1, 2, \cdots\}, x_1 \in D,\ D \subseteq \mathbb{R}^{n_1}$ is an open set, $x_2 \in \mathbb{R}^{n_2}, f_1 : D \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}, f_2 : D \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ (see Theorem 3.2 and Theorem 3.3). In [9] the author obtained some results for the BD and the UB of partial solutions of the difference equation

$$z(\tau+1) = f(\tau, z(\tau))$$

(see Theorem 1-5), where $z \in \mathbb{R}^n,\ \tau \in I = \{0, 1, 2, \cdots\}, f : I \times \mathbb{R}^n \to \mathbb{R}^n$.

On the BD and the UB of partial solutions of ODEs and difference equations, one obtained quite a few results. Nonetheless, the results for uniform BD and UUB of partial solutions of RFDEs are still very few [3, 4]. In this paper, using Liapunov function (Liapunov functions are simpler than Liapunov functionals), a result of uniform BD and UUB of the partial coordinates or all the coordinates of the solutions of RFDEs is given. Our result improves the well-known Razumikhin theorems on UUB [4]. The result is easier to apply; its application area is widened. Moreover, some examples are given to illustrate the application and advantage of the obtained result at the end.

2. Preliminaries

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty,\ \infty)$, $\mathbb{R}^+ = [0,\ \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$ (in this paper, if $x \in \mathbb{R}^n$ is a column vector, then $|x|$ denotes the Euclidean length of x: $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$), $C = C([-r,\ 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-r,\ 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We designate the norm of an element $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T$ in C by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. If

$$\sigma \in \mathbb{R}, A \geq 0 \quad \text{and} \quad x \in C([\sigma - r, \sigma + A], \mathbb{R}^n),$$

then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta), \theta \in [-r,\ 0]$.

Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous and consider the retarded functional differential equation [1, 4]

$$\dot{x}(t) = f(t,\ x_t). \tag{2.1}$$

We will assume that there is a unique solution $x(t, t_0, \phi)$ $(x(t_0, \phi)(t))$ of the Equation (2.1) through $(t_0,\ \phi) \in \mathbb{R} \times C$. Let

$$\begin{aligned}
&x = (x_1,\ x_2,\ \cdots, x_n)^T \in \mathbb{R}^n, \phi = (\phi_1,\ \phi_2,\ \cdots, \phi_n)^T \in C,\\
&x_{i\sim j} = (x_i,\ x_{i+1},\ \cdots, x_j)^T \in \mathbb{R}^{j+1-i} \quad (1 \leq i \leq j \leq n),\\
&x(t) = x(t, t_0, \phi),\ x_{i\sim j}(t) = x_{i\sim j}(t, t_0, \phi),
\end{aligned}$$

$$\begin{aligned}
&y(t) = x_{1\sim m}(t),\ z(t) = x_{m+1\sim n}(t);\\
&C_{i\sim j} = C([-r,\ 0], \mathbb{R}^{j+1-i}), \phi_{i\sim j} = (\phi_i,\ \phi_{i+1},\ \cdots, \phi_j)^T \in C_{i\sim j}.
\end{aligned}$$

If $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ is a continuous function, then $\dot{V}(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0))$, the derivative of V along the solutions of Equation (2.1) is defined to be

$$\dot{V}(t,\phi_{1\sim m}(0),\phi_{m+1\sim n}(0)) = \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h, x_{1\sim m}(t,\phi)(t+h), x_{m+1\sim n}(t,\phi)(t+h)) - V(t,\phi_{1\sim m}(0),\phi_{m+1\sim n}(0))].$$

In this paper, suppose that if $(x_{1\sim m}, x_{m+1\sim n})^T \neq 0$, then $V(t, x_{1\sim m}, x_{m+1\sim n}) > 0$ for all $t \in \mathbb{R}$.

Definition 1. The partial solutions $x_{1\sim m}(t, t_0, \phi)$ $(1 \leq m \leq n)$ of the Equation (2.1) are uniformly bounded if , for any $B_1 > 0$, there is a $B_2 = B_2(B_1) > 0$ such that, for all $t_0 \in \mathbb{R}$, $\phi \in C$, and $|\phi| \leq B_1$, we have

$$|x_{1\sim m}(t, t_0, \phi)| \leq B_2(B_1)$$

for all $t \geq t_0$.

Definition 2. The partial solutions $x_{1\sim m}(t, t_0, \phi)$ of the Equation (2.1) are uniformly ultimately bounded if there is a $B > 0$ such that, for any $B_3 > 0$, there is a constant $T(B_3) > 0$ such that

$$|x_{1\sim m}(t, t_0, \phi)| \leq B$$

for $t \geq t_0 + T(B_3)$ for all $t_0 \in \mathbb{R}$, $\phi \in C, |\phi| \leq B_3$.

3. Main result

Without loss of generality, we suppose $r > 0$. Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$. Suppose $u, v, w, q : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, nondecreasing functions, $u(s), v(s), w(s)$ positive for $s > 0$, $u(0) = v(0) = w(0) = q(0) = 0$, $u(s) \to \infty$ as $s \to \infty$. Suppose there is a constant $c > 0$ such that $\dot{q}(s) \geq c \ for \ s \geq 0$.

Theorem 1. *Let the following conditions hold.*

(1) *There are positive integers m and k with $1 \leq m \leq k \leq n$. There is a continuous function $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that*

$$u(|x_{1\sim m}|) \leq V(t, x_{1\sim m}, x_{m+1\sim n}) \leq v(|x_{1\sim k}|) \tag{3.1}$$

for $t \in \mathbb{R}, x \in \mathbb{R}^n$.

(2) *There is a continuous, nondecreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$p(s) > q(s) \ for \ s > 0. \tag{3.2}$$

(3) *Let* $x_{m+1\sim n} \in \mathbb{R}^{n-m}$ *be a vector, and let*

$$x(t) = x_{1\sim n}(t) = (y^T(t), z^T(t))^T = (x^T_{1\sim m}(t), x^T_{m+1\sim n}(t))^T$$

be the solution of the Equation (2.1). *If there is a constant* $H > 0$ *and if* $|x_{1\sim m}(t)| \geq H$ *for all* $t \geq t_0$ *and*

$$q(V(s, y(s), x_{m+1\sim n})) < p(V(t, y(t), x_{m+1\sim n}))$$

for all $t \geq t_0$, $s \in [t-r, t]$, *then*

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq -w(|y(t)|) \tag{3.3}$$

for all $t \geq t_0$.

Under these conditions, the coordinates $x_{1\sim m}(t, t_0, \phi)$ *of the solutions of the Equation* (2.1) *are uniformly bounded and uniformly ultimately bounded.*

Remark 1. The well-known Razumikhin theorems on UUB may be deduced from the above Theorem 1 (the function $q(s) \equiv s$, $m = k = n$).

4. Proof of Theorem 1

In order to prove Theorem 1 in this paper, we need three lemmas.

Although $|x_{1\sim n}(t)| \geq H$ is assumed in Lemma 2 and Lemma 3 below, this paper is interested only in the issues of the uniform boundedness and the uniform ultimate boundedness of the partial solutions $x_{1\sim m}(t)$ ($m \leq n$). Therefore, condition $|x_{1\sim m}(t)| \geq H$ is assumed in Theorem 1. In fact, if $|x_{1\sim m}(t)| \geq H$, then $|x_{1\sim n}(t)| \geq H$ ($|x_{1\sim n}(t)| \geq |x_{1\sim m}(t)| \geq H$). Consequently, the Theorem 1 in this paper can be proved by Lemma 3 below (see page 11).

Let $\mathbb{R}_0 = [t_0, \infty)$. If

$$\overline{V}(t, y_t, z_t) = \sup_{\theta\in[-r,\,0]} V(t+\theta, y(t+\theta), z(t)) \tag{4.1}$$

for $(t, x(t+\theta)) \in \mathbb{R}_0 \times C$, $\theta \in [-r, 0]$ ($x(t+\theta) = (y_t^T, z_t^T)^T \in C$), then there is a $\theta_0 = \theta_0(t)$ in $[-r, 0]$ such that

$$\overline{V}(t, y_t, z_t) = V(t+\theta_0(t), y(t+\theta_0(t)), z(t)) \tag{4.2}$$

and either $\theta_0(t) = 0$ or $\theta_0(t) < 0$ and

$$V(t+\theta, y(t+\theta), z(t)) < V(t+\theta_0(t), y(t+\theta_0(t)), z(t)) \tag{4.3}$$

if $\theta_0(t) < \theta \leq 0$.

Lemma 1. *Let $(y^T(t), z^T(t))^T$ be the solution of Equation (2.1). If $\theta_0 = \theta_0(t) < 0$ for some $t \in \mathbb{R}_0$, then*

$$\dot{\overline{V}}(t, y_t, z_t) = 0. \tag{4.4}$$

Proof. If

$$V(t+h+\theta, y(t+h+\theta), z(t+h)) > V(t+\theta_0, y(t+\theta_0), z(t))$$

for $\theta \in (\theta_0, 0]$, and sufficiently small $h > 0$, then we would have

$$\lim_{h\to 0+} V(t+h+\theta, y(t+h+\theta), z(t+h)) \geq \lim_{h\to 0+} V(t+\theta_0, y(t+\theta_0), z(t)),$$

and using the facts that V, $y(t+h+\theta)$, and $z(t+h)$ are continuous, we get

$$V(t+\theta, y(t+\theta), z(t)) \geq V(t+\theta_0, y(t+\theta_0), z(t))$$

for $\theta \in (\theta_0, 0]$, which contradicts the (4.3). So

$$V(t+h+\theta, y(t+h+\theta), z(t+h)) \leq V(t+\theta_0, y(t+\theta_0), z(t)) \tag{4.5}$$

for $\theta \in (\theta_0, 0]$, and sufficiently small $h > 0$. From this, we have

$$\overline{V}(t+h, y_{t+h}, z_{t+h}) = \overline{V}(t, y_t, z_t) \tag{4.6}$$

for sufficiently small $h > 0$ [4]. Therefore, $\dot{\overline{V}}(t, y_t, z_t) = 0$. The proof of the Lemma 1 is therefore complete. □

Lemma 2. *Let the following conditions hold.*

(1) *There is a continuous function $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that*

$$u(|x_{1\sim m}|) \leq V(t, x_{1\sim m}, x_{m+1\sim n}) \leq v(|x_{1\sim k}|) \ (1 \leq m \leq k \leq n) \tag{4.7}$$

for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

(2) *Let $x(t) = x_{1\sim n}(t) = (y^T(t), z^T(t))^T = (x^T_{1\sim m}(t), x^T_{m+1\sim n}(t))^T$ be the solution of the Equation (2.1). If $|x_{1\sim n}(t)| \geq H$ for all $t \geq t_0$ and*

$$V(t+\theta, y(t+\theta), z(t)) \leq V(t, y(t), z(t))$$

for all $t \geq t_0$, $\theta \in [-r, 0]$, then

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq 0 \tag{4.8}$$

for all $t \geq t_0$.

Under these conditions, the coordinates $x_{1\sim m}(t, t_0, \phi)$ of the solutions of the Equation (2.1) are uniformly bounded .

Proof. Let $B_1 \geq H$ be given, $t_0 \in \mathbb{R}$, $\phi \in C$, and $|\phi| \leq B_1$. Choose $B_2 > B_1$ such that $u(B_2) \geq v(B_1)$. Consider any solution $x(t)$ of (2.1). Using (4.2), (4.7) and our choice of $u(B_2)$, we get

$$\begin{aligned}&\overline{V}(t_0, y_{t_0}, z_{t_0}) = V(t_0+\theta_0, y(t_0+\theta_0), z(t_0)) \leq \\ &v(|(y^T(t_0+\theta_0), x^T_{m+1\sim k}(t_0))^T|) \leq v(|\phi|) \leq v(B_1) \leq u(B_2).\end{aligned} \tag{4.9}$$

We claim that

$$\overline{V}(t, y_t, z_t) \le u(B_2) \quad \text{for all } t \ge t_0. \tag{4.10}$$

If this were not so, then there would exist a t^*, $t^* > t_0$, such that

$$\overline{V}(t^*, y_{t^*}, z_{t^*}) > u(B_2). \tag{4.11}$$

Since (4.9), (4.11) and $\overline{V}(t, y_t, z_t)$ is continuous in t, there exists $t_1 \in [t_0, t^*]$ such that

$$\overline{V}(t_1, y_{t_1}, z_{t_1}) = u(B_2) < \overline{V}(t^*, y_{t^*}, z_{t^*}). \tag{4.12}$$

Therefore the mean value theorem yields there exists $\bar{t} \in [t_1, t^*]$ such that

$$\dot{\overline{V}}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) > 0. \tag{4.13}$$

If $\theta_0 < 0$, then using Lemma 1, we have $\dot{\overline{V}}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = 0$, which contradicts the (4.13). If $\theta_0 = 0$, then using (4.2), (4.7), (4.12), and our choice of $u(B_2)$, we have

$$v(B_1) \le u(B_2) \le \overline{V}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = V(\bar{t}, y(\bar{t}), z(\bar{t})) \le v(|x_{1\sim k}(\bar{t})|) \le v(|x(\bar{t})|).$$

Using the fact that v is nondecreasing, we get $B_1 \le |x(\bar{t})|$ $(\,|x(\bar{t})| \ge B_1 \ge H\,)$. If $\theta_0 = 0$, then using (4.1) and (4.2), we have

$$V(\bar{t}+\theta, y(\bar{t}+\theta), z(\bar{t})) \le \sup_{\theta\in[-r,\,0]} V(\bar{t}+\theta, y(\bar{t}+\theta), z(\bar{t})) =$$
$$\overline{V}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = V(\bar{t}+\theta_0, y(\bar{t}+\theta_0), z(\bar{t})) = V(\bar{t}, y(\bar{t}), z(\bar{t})) \tag{4.14}$$

for $\theta \in [-r, 0]$, $\bar{t} \ge t_0$. From (4.8), (4.14) and $|x(\bar{t})| \ge H$, we have $\dot{\overline{V}}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = \dot{V}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) \le 0$, which contradicts the (4.13). So $\overline{V}(t, y_t, z_t) \le u(B_2)$ for all $t \ge t_0$.

From (4.1), (4.7) and (4.10), we obtain

$$u(|y(t)|) \le V(t, y(t), z(t)) =$$
$$V(t+0, y(t+0), z(t)) \le \sup_{\theta\in[-r,\,0]} V(t+\theta, y(t+\theta), z(t)) =$$
$$\overline{V}(t, y_t, z_t) \le u(B_2)$$

for all $t \ge t_0$. Using the fact that u is nondecreasing, we get $|y(t)| \le B_2$ for all $t \ge t_0$. The proof of the Lemma 2 is therefore complete. □

Lemma 3. *Let the following conditions hold.*

(1) *There are positive integers m and k with $1 \le m \le k \le n$. There is a continuous function $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that*

$$u(|x_{1\sim m}|) \le V(t, x_{1\sim m}, x_{m+1\sim n}) \le v(|x_{1\sim k}|) \tag{4.15}$$

for $t \in \mathbb{R}, x \in \mathbb{R}^n$.

(2) *There is a continuous, nondecreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$p(s) > q(s) \text{ for } s > 0. \tag{4.16}$$

(3) *Let* $x(t) = x_{1\sim n}(t) = (y^T(t), z^T(t))^T = (x^T_{1\sim m}(t), x^T_{m+1\sim n}(t))^T$ *be the solution of the Equation* (2.1). *If* $|x_{1\sim n}(t)| \geq H$ *for all* $t \geq t_0$ *and*

$$q(V(t+\theta, y(t+\theta), z(t))) < p(V(t, y(t), z(t)))$$

for all $t \geq t_0$, $\theta \in [-r, 0]$, *then*

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq -w(H) \tag{4.17}$$

for all $t \geq t_0$.

Under these conditions, the coordinates $x_{1\sim m}(t, t_0, \phi)$ *of the solutions of the Equation* (2.1) *are uniformly bounded and uniformly ultimately bounded.*

Proof. First, we show the uniform boundedness.

If $|x(t)| \geq H$ and $V(t+\theta, y(t+\theta), z(t)) \leq V(t, y(t), z(t))$ for $\theta \in [-r, 0]$, $t \geq t_0$, then using (4.16) and the fact that q is nondecreasing, we have

$$q(V(t+\theta, y(t+\theta), z(t))) \leq q(V(t, y(t), z(t))) < p(V(t, y(t), z(t))) \tag{4.18}$$

for $\theta \in [-r, 0]$, $t \geq t_0$; this follows since $V(t, y(t), z(t)) > 0$ $(|x(t)| \geq H > 0)$. Hypothesis (4.17) implies $\dot{V}(t, y(t), z(t)) \leq -w(H) \leq 0$. Lemma 2 implies the coordinates $x_{1\sim m}(t, t_0, \phi)$ of the solutions of the Equation (2.1) are uniformly bounded.

We now show the uniform ultimate boundedness. Choose $B > H$ such that $u(B) > v(H)$. Let $B_3 \geq H$ be given. The same argument as in the proof of Lemma 2 shows there is a $B_4 > B$ such that for any $t_0 \in \mathbb{R}, t \geq t_0$, and $\phi \in C$ with $|\phi| \leq B_3$, the B_4 and $u(B_4)$ satisfy

$$\begin{aligned} &u(B) < u(B_4), \ \ \overline{V}(t, y_t, z_t) \leq u(B_4) \ \text{ and} \\ &|y(t)| \leq B_4 \ \text{ for all } \ t \geq t_0 \ \text{ (see (4.10))}. \end{aligned} \tag{4.19}$$

If $\tilde{x} = (B_4, 0, \cdots, 0)^T \in \mathbb{R}^n$, then using (4.15) and (4.19), we have

$$\begin{aligned} &0 < u(B) < u(B_4) = u(|\tilde{x}_{1\sim m}|) \leq \\ &V(t, \tilde{x}_{1\sim m}, \tilde{x}_{m+1\sim n}) \leq v(|\tilde{x}_{1\sim k}|) = v(B_4). \end{aligned} \tag{4.20}$$

From the properties of the functions $p(s)$ and $q(s)$, there is a number $a > 0$ such that

$$p(s) - q(s) > a \ \text{ for } \ u(B) \leq s \leq v(B_4). \tag{4.21}$$

Let N be the first nonnegative integer such that

$$q(u(B)) + Na \geq q(v(B_4)), \tag{4.22}$$

and let

$$T_l = \frac{l\, q(v(B_4))}{c\, w(H)} \quad (l = 0, 1, 2, \cdots, N). \tag{4.23}$$

First, we show that

$$q(V(t, y(t), z(t))) \leq q(u(B)) + (N - l)a \tag{4.24}$$

for all $t \geq t_0 + lr + T_l$, $l = 0, 1, 2, \cdots, N$. From (4.19)and (4.20), we have

$$V(t, y(t), z(t)) \leq \overline{V}(t, y_t, z_t) \leq u(B_4) \leq v(B_4) \ \text{ for } \ t \geq t_0. \tag{4.25}$$

If $l = 0$, then using (4.22) and (4.25), we have

$$q(V(t, y(t), z(t))) \leq q(\overline{V}(t, y_t, z_t)) \leq q(v(B_4)) \leq$$
$$q(u(B)) + Na = q(u(B)) + (N-0)a$$

for all $t \geq t_0 = t_0 + 0 \cdot r + T_0$. That is, the inequality (4.24) is true for the $l = 0$.

We now show that inequality (4.24) is true for $l = 1$. If

$$q(u(B)) + (N-1)a < q(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0))) \tag{4.26}$$

for $\tilde{t}_0 \geq t_0$, then, since $q(u(B)) \leq q(u(B)) + (N-1)a < q(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)))$, it follows that

$$u(B) < V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)). \tag{4.27}$$

From (4.25) and (4.27), we obtain

$$u(B) < V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)) \leq v(B_4). \tag{4.28}$$

Using (4.15), (4.27), and our choice of $v(H)$, we have

$$v(H) < u(B) < V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)) \leq v(|x_{1\sim k}(\tilde{t}_0)|) \leq v(|x(\tilde{t}_0)|), \tag{4.29}$$

and using the fact that v is nondecreasing, we get $H \leq |x(\tilde{t}_0)|$. Using (4.21), (4.22), (4.25), (4.26) and (4.28), we get

$$p(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0))) > q(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0))) + a >$$
$$q(u(B)) + (N-1)a + a = q(u(B)) + Na \geq q(v(B_4)) \geq$$
$$q(\overline{V}(\tilde{t}_0, y_{\tilde{t}_0}, z_{\tilde{t}_0}) \geq q(V(\tilde{t}_0 + \theta, y(\tilde{t}_0 + \theta), z(\tilde{t}_0))) \tag{4.30}$$

for θ in $[-r, 0]$. Hypothesis (4.17) implies

$$\dot{V}(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)) \leq -w(H) < 0 \tag{4.31}$$

for $\tilde{t}_0 \geq t_0$. Therefore, if there is a $t_0^* \in [t_0, t_0 + T_1)$ such that

$$q(V(t_0^*, y(t_0^*), z(t_0^*))) \leq q(u(B)) + (N-1)a,$$

then using (4.26) and (4.31), we have

$$q(V(t, y(t), z(t))) \leq q(u(B)) + (N-1)a \tag{4.32}$$

for $t \geq t_0 + r + T_1$. If

$$q(u(B)) + (N-1)a < q(V(t, y(t), z(t)))$$

for all $t \in [t_0, t_0 + T_1)$, then using (4.23) and (4.31), we have

$$q(V(t_0 + T_1, y(t_0 + T_1), z(t_0 + T_1))) =$$
$$q(V(t_0, y(t_0), z(t_0))) + \dot{q}(V(\xi, y(\xi), z(\xi)))\,\dot{V}(\xi, y(\xi), z(\xi))\,T_1 =$$

$$q(V(t_0,y(t_0),z(t_0))) + \dot{q}(V(\xi,y(\xi),z(\xi)))\,\dot{V}(\xi,y(\xi),z(\xi))\,\frac{q(v(B_4))}{c\,w(H)} \le$$
$$q(V(t_0,y(t_0),z(t_0))) - \dot{q}(V(\xi,y(\xi),z(\xi)))\,\frac{q(v(B_4))\,w(H)}{c\,w(H)} =$$
$$q(V(t_0,y(t_0),z(t_0))) - \dot{q}(V(\xi,y(\xi),z(\xi)))\,\frac{q(v(B_4))}{c} \le$$
$$q(V(t_0,y(t_0),z(t_0))) - c\,\frac{q(v(B_4))}{c} =$$
$$q(V(t_0,y(t_0),z(t_0))) - q(v(B_4)) \le q(v(B_4)) - q(v(B_4)) = 0 \le$$
$$q(u(B)) + (N-1)a.$$

Therefore, we have

$$q(V(t,y(t),z(t))) \le q(u(B)) + (N-1)a \quad \text{for all } t \ge t_0 + r + T_1. \tag{4.33}$$

Thus, the inequality (4.24) is true for the $l = 1$.

The arguments are essentially the same as before, we are able to show that

$$q(V(t,y(t),z(t))) \le q(u(B)) + (N-K)a$$

for all $t \ge t_0 + Kr + T_K, K = 2,3,\cdots,N$ [4, 12]. Therefore, we have

$$q(V(t,y(t),z(t))) \le q(u(B)) + (N-l)a$$

for all $t \ge t_0 + lr + T_l, l = 0,1,2,\cdots,N$. For $l = N$, we have

$$q(V(t,y(t),z(t))) \le q(u(B)) + (N-N)a = q(u(B))$$

for all $t \ge t_0 + Nr + T_N$. Hypothesis (4.15) implies

$$q(u(|y(t)|)) \le q(V(t,y(t),z(t))) \le q(u(B))$$

for all $t \ge t_0 + Nr + T_N$ and hence

$$|(x_1(t,t_0,\phi),\ x_2(t,t_0,\phi),\ \cdots, x_m(t,t_0,\phi))^T| \le B$$

for all $t \ge t_0 + Nr + T_N$. Let $T = Nr + T_N$. Thus, this completes the proof of uniform ultimate boundedness. The proof of the Lemma 3 is therefore complete. □

Now we are in the position to prove our Theorem 1.

Proof of Theorem 1. If

$$|x_{1\sim m}(t)| \ge H \tag{4.34}$$

for all $t \ge t_0$ and

$$q(V(s,y(s),x_{m+1\sim n})) < p(V(t,y(t),x_{m+1\sim n})) \tag{4.35}$$

for all $t \ge t_0, s \in [t-r,t],\ x_{m+1\sim n} \in \mathbb{R}^{n-m}$, then using (c), we have

$$\dot{V}_{(2.1)}(t,y(t),z(t)) \le -w(|y(t)|) \tag{4.36}$$

for all $t \geq t_0$. From (4.34) and (4.35), we obtain

$$|x_{1\sim n}(t)| \geq |x_{1\sim m}(t)| \geq H \tag{4.37}$$

for all $t \geq t_0$ and

$$q(V(t+\theta, y(t+\theta), z(t))) < p(V(t, y(t), z(t))) \tag{4.38}$$

for all $t \geq t_0$, $\theta \in [-r, 0]$. Using (4.36), (4.37), and the fact that w is nondecreasing, we get

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq -w(|y(t)|) = -w(|x_{1\sim m}(t)|) \leq -w(H) \tag{4.39}$$

for all $t \geq t_0$. Lemma 3 implies the coordinates $x_{1\sim m}(t, t_0, \phi)$ of the solutions of the Equation (2.1) are uniformly bounded and uniformly ultimately bounded (see the (4.37), the (4.38), and the (4.39)). The proof of Theorem 1 is therefore complete. $\square$

5. Examples

We give the following examples in order to illustrate the application and advantage of the Theorem 1 in this paper [4, 20, 21]. It is easy to see that the well-known Razumikhin theorems on uniform ultimate boundedness cannot apply to the following examples.

Example 1. Consider the scalar equation

$$\begin{aligned}\dot{x}(t) = -\sum_{j=1}^{n} b_j(t) x^{2k-1}(t)\ [\alpha \arctan \beta x^{2m}(t-\tau_j(t)) + \gamma x^{2m}(t-\tau_j(t))] \\ -a(t)x^{2k-1}(t)\ [\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] + \tilde{p}(t),\end{aligned} \tag{5.1}$$

where $\tilde{p}(t)$, $a(t)$, $b_j(t)$, and $\tau_j(t)$ are bounded continuous functions on $\mathbb{R}$, $\alpha, \beta, \gamma =$ const., $\alpha\beta > 0, \gamma > 0,\ k = 1, 2, \cdots, K,\ m = 1, 2, \cdots, M$.

We make the following assumptions on Equation (5.1): there is a $q_0 \in (0, 1)$ such that

$$a(t) \geq \delta > 0,\ \sum_{j=1}^{n} |b_j(t)| \leq \delta q_0,\ 0 \leq \tau_j(t) \leq r \quad (j = 1, 2, \cdots, n)$$

for all $t \in \mathbb{R}$, $q_0, \delta, r =$ const.

Under the above hypotheses, we will show that the solutions of the Equation (5.1) are uniformly bounded and uniformly ultimately bounded.

In fact, since $q_0 \in (0,1)$, there is a $q_1 > 1$ such that $q_0 q_1 < 1$. If

$$p(s) = q_1 q(s),\ q(s) = \alpha \arctan \beta s + \gamma s,\ V(x) = x^{2m},$$

$|x(t)| \geq H =$ const. > 0, and

$$p(V(x(t))) > q(V(x(s))),\ s \in [t-r,\ t]\ (\Rightarrow q_1(\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)) > \alpha \arctan \beta x^{2m}(t-\tau_j(t)) + \gamma x^{2m}(t-\tau_j(t)),\ j = 1,2,\cdots,n),$$

then

$$\begin{aligned}
\dot{V}_{(5.1)}(x(t)) &= 2mx^{2(k+m-1)}(t)\{-a(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad - \sum_{j=1}^{n} b_j(t)[\alpha \arctan \beta x^{2m}(t-\tau_j(t)) + \gamma x^{2m}(t-\tau_j(t))]\} \\
&\quad + 2m\tilde{p}(t)x^{2m-1}(t) \\
&\leq 2mx^{2(k+m-1)}(t)\{-a(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad + \sum_{j=1}^{n} |b_j(t)| \cdot q_1[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)]\} + 2m\tilde{p}(t)x^{2m-1}(t) \\
&= 2mx^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \times \\
&\quad [-a(t) + q_1 \sum_{j=1}^{n} |b_j(t)|] + 2m\tilde{p}(t)x^{2m-1}(t) \\
&\leq 2mx^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \times \\
&\quad [-\delta + q_0 q_1 \delta] + 2m\tilde{p}(t)x^{2m-1}(t) \\
&\leq -2m\delta(1-q_0 q_1)x^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad + 2m\tilde{p}(t)x^{2m-1}(t).
\end{aligned}$$

By choosing $H_1 \geq H =$ const > 0 appropriately ($\tilde{p}(t)$ is bounded continuous function on $\mathbb{R}$, constant $\delta(1-q_0 q_1) > 0$, $\alpha\beta > 0, \gamma > 0$), we obtain a positive constant μ such that

$$2m\left\{\delta(1-q_0 q_1) - \frac{\tilde{p}(t)}{x^{2k-1}(t)\left[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)\right]}\right\} \geq \mu$$

for $|x(t)| \geq H_1$ and $p(V(x(t))) > q(V(\phi(\theta)))$, $\theta \in [-r,\ 0]$. From this, we have

$$\begin{aligned}
\dot{V}_{(5.1)}(x(t)) &\leq -2m\delta(1-q_0 q_1)x^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad + 2m\tilde{p}(t)x^{2m-1}(t)
\end{aligned}$$

$$= -2m\left\{\delta(1-q_0q_1) - \frac{\tilde{p}(t)}{x^{2k-1}(t)\left[\alpha\arctan\beta x^{2m}(t)+\gamma x^{2m}(t)\right]}\right\}$$
$$\times x^{2(k+m-1)}(t)[\alpha\arctan\beta x^{2m}(t)+\gamma x^{2m}(t)]$$
$$\leq -\mu x^{2(k+m-1)}(t)[\alpha\arctan\beta x^{2m}(t)+\gamma x^{2m}(t)]$$

for $|x(t)| \geq H_1$ and $p(V(x(t))) > q(V(\phi(\theta)))$, $\theta \in [-r, 0]$. Therefore, the Theorem 1 implies the solutions of the Equation (5.1) are uniformly bounded and uniformly ultimately bounded.

Example 2. Consider the second-order system

$$\dot{x}(t) = 2y(t),$$

$$\dot{y}(t) = -f(x(t)) - \Phi(t, y(t))a^{y^2(t)} \tag{5.2}$$

$$+\tilde{p}(t) + y(t)\int_{-r}^{0} g(x(t+\theta))a^{y^2(t+\theta)}d\theta,$$

where $a =$ constant > 1.

We make the following assumptions on System (5.2):

(a) $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is continuous, Φ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{R}$) into bounded sets and there are constants $a_0 > 0$, $H > 0$, such that

$$\frac{\Phi(t,y)}{y} > a_0 > 0 \text{ for } t \in \mathbb{R},\ |y| \geq H.$$

(b) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x)sgnx \to \infty\ as\ |x| \to \infty$.
(c) $\tilde{p} : \mathbb{R} \to \mathbb{R}$ is bounded and continuous.
(d) $g : \mathbb{R} \to \mathbb{R}$ is continuous and there is a constant $L > 0$, such that

$$|g(x)| \leq L \text{ for all } x \in \mathbb{R}.$$

(e) $Lr < a_0$.

It is always assumed that a uniqueness result holds for the solutions of System (5.2). Under the above hypotheses, we will show that the second coordinate of the solutions of the System (5.2) is uniformly bounded and uniformly ultimately bounded.

In fact, since $Lr < a_0$, there is a $q_1 > 1$ such that $q_1 Lr < a_0$. If

$q(s) = a^s - 1,\ p(s) = q_1 q(s),\ V(x,y) = F(x) + y^2,\ F(x) = \int_0^x f(s)ds,$

$|y(t)| \geq H$, and

$q(V(x, y(s))) < p(V(x, y(t))),\ s \in [t-r,\ t]\ (q(V(x, y(t+\theta))) <$

$p(V(x, y(t))),\ \theta \in [-r,\ 0])$

$$\Big(\Rightarrow a^{F(x)+y^2(t+\theta)} - 1 < q_1(a^{F(x)+y^2(t)} - 1)$$

$$\Rightarrow a^{F(x)+y^2(t+\theta)} < a^{F(x)+y^2(t+\theta)} + (q_1 - 1) < q_1 a^{F(x)+y^2(t)}$$

$$\Rightarrow a^{y^2(t+\theta)} < q_1 a^{y^2(t)} \Big),$$

then

$$\begin{aligned} \dot{V}_{(5.2)}(x(t), y(t)) &= 2y(t)[-\Phi(t, y(t))a^{y^2(t)} + \tilde{p}(t) + \\ &\quad y(t)\int_{-r}^{0} g(x(t+\theta))a^{y^2(t+\theta)}d\theta] \\ &\leq 2y^2(t)[-a_0 a^{y^2(t)} + \int_{-r}^{0} |g(x(t+\theta))|a^{y^2(t+\theta)}d\theta)] \\ &\quad +2|y(t)|\,|\tilde{p}(t)| \\ &\leq -2(a_0 - q_1 Lr)y^2(t)a^{y^2(t)} + 2|y(t)|\,|\tilde{p}(t)|, \end{aligned}$$

By choosing $H_1 \geq H$ appropriately ($\tilde{p}(t)$ is bounded continuous function, constant $(a_0 - q_1 Lr) > 0$), we obtain a positive constant μ such that

$$2\left[(a_0 - q_1 Lr) - \frac{|\tilde{p}(t)|}{|y(t)|\, a^{y^2(t)}}\right] \geq \mu$$

for $|y(t)| \geq H_1$ and $q(V(x, y(s))) < p(V(x, y(t))),\ s \in [t-r,\ t]$. From this, we have

$$\begin{aligned} \dot{V}_{(5.2)}(x(t), y(t)) &\leq -2(a_0 - q_1 Lr)y^2(t)a^{y^2(t)} + 2|y(t)|\,|\tilde{p}(t)| \\ &= -2\left[(a_0 - q_1 Lr) - \frac{|\tilde{p}(t)|}{|y(t)|\, a^{y^2(t)}}\right] y^2(t)a^{y^2(t)} \\ &\leq -\mu y^2(t)a^{y^2(t)} \end{aligned}$$

for $|y(t)| \geq H_1$ and $q(V(x, y(s))) < p(V(x, y(t))),\ s \in [t-r,\ t]$. Therefore, the Theorem 1 implies the second coordinate of the solutions of the System (5.2) is uniformly bounded and uniformly ultimately bounded.

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