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EXISTENCE AND STABILITY OF MILD SOLUTIONS FOR HYBRID FRACTIONAL SEMI-LINEAR EVOLUTION EQUATIONS

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Abstract. This paper examines the existence and stability of mild solutions for initial value problems associated with hybrid fractional semi-linear evolution equations. The existence of mild solutions is established using Dhage's fixed point theorem. Additionally, we investigate four distinct types of Mittag-Leffler-Ulam-Hyers stability to analyze the behavior of solutions under perturbations. To demonstrate the applicability of our theoretical findings, we provide a concrete example. These results contribute to the advancement of fractional evolution equations and their stability theory, with potential applications in various fields of applied mathematics and engineering.

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1. INTRODUCTION

Fractional differential equations (FDEs) have garnered significant attention across various scientific fields, including economics, engineering, chemistry, aerodynamics, and the control of dynamical systems. The primary appeal of fractional calculus lies in its ability to capture the complex behaviors of systems that traditional integer-order derivatives fail to represent. By incorporating memory effects and non-local interactions, fractional models provide a more accurate depiction of phenomena, offering deeper insights into the dynamics of systems such as spring pendulums, particle motion in circular cavities, and the spread of epidemics.

The interest in fractional calculus is driven by its capacity to model real-world processes more effectively. Researchers have increasingly recognized its advantages, including the ability to uncover hidden dynamics that are not observable using integer-order derivatives. In this context, Almeida [2] extended the classical Caputo fractional derivative by introducing dependencies on an auxiliary function, which improved model adaptability and accuracy. Recent advancements have also introduced new techniques for solving FDEs, such as the generalized Laplace transform developed by Jarad and Abdeljawad [10], which simplifies the resolution of fractional differential equations in the framework of generalized Caputo derivatives. In parallel, significant progress has been made in understanding the mathematical structure of fractional differential equations. For instance, the study of commutators of fractional integral operators on vanishing-Morrey spaces has revealed new insights into the behavior of these operators [13]. Additionally, the application of fractional calculus to the Lerch zeta function has provided a deeper understanding of the connections between fractional calculus and special functions [9]. The study of hybrid fractional differential equations with fractional proportional derivatives, which combine fractional derivatives with proportional terms, has further enriched the field [1]. Furthermore, investigations into fractional calculus, zeta functions, and Shannon entropy have opened new avenues for research in applied mathematics and information theory [8]. Numerical methods also play a crucial role in solving fractional differential equations. High-order numerical techniques, such as those developed for solving two-dimensional Riesz space fractional advection-dispersion equations, have proven effective in addressing complex dynamical systems [3]. Additionally, the study of fractional derivatives in complex planes has contributed to the advancement of both theoretical and computational aspects of fractional calculus [8]. This growing body of research not only enhances our theoretical understanding of fractional calculus but also expands its applicability to a wide range of real-world problems. As the field continues to evolve, fractional differential equations are poised to provide more precise models for complex systems across multiple disciplines, offering promising solutions to long-standing scientific and engineering challenges.

Inspired by the research presented in [6], we delve into the analysis of the existence and different classifications of Ulam-Hyers stability outcomes for mild solutions for the semi-linear fractional hybrid differential equations that include the $\rho-$Caputo fractional derivative of order $0 \leq \beta \leq 1$.

$$\begin{cases} {}^{C}D^{\beta,\rho}_{0+}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) \qquad t \in [0,T] \quad 0 < \beta < 1, \\ u(0) = 0. \end{cases} \tag{1.1}$$

Where $T > 0$, $\mathcal{A}$ is the infinitesimal generator of $\mathcal{C}_0-$semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathcal{X}$, $g \in \mathcal{C}([0,T] \times \mathcal{X}, \mathcal{X} \setminus \{0\})$ and $h \in \mathcal{C}_c([0,T] \times \mathcal{X}, \mathcal{X})$.

2. Preliminaries

We provide initial context that will be referenced throughout this paper.

Let $\mathcal{X}$ be a Banach space and $\mathcal{C}([0,T],\mathcal{X})$ be the Banach space of continuous functions from $[0,T]$ to $\mathcal{X}$ with the norm $||u|| = \sup_{t\in[0,T]} ||u(t)||$.

Definition 1 ($\rho-$Riemann-Liouville fractional integral [4, Definition 2.1])**.** Let $\beta > 0$, f be an integrable function defined on $[a,b]$ and $\rho\colon [a,b] \to \mathbb{R}$ that is an increasing differentiable function such that $\rho'(t) \neq 0$, for all $t \in [a,b]$.

The ρ-Riemann-Liouville fractional integral operator of order β of a function f is defined by

$$I_{a+}^{\beta,\rho} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \rho'(s)(\rho(t)-\rho(s))^{\beta-1} f(s)\mathrm{d}s.$$

Definition 2 (ρ-Riemann-Liouville fractional derivative [4, Definition 2.2])**.** Let $n \in \mathbb{N}$, $k, \rho \in \mathcal{C}^n([a,b])$ be two functions such that ρ is increasing with $\rho'(t) \neq 0$, for all $t \in [a,b]$.

ρ-Riemann-Liouville fractional derivative of order β of a function f is defined by

$$\mathcal{D}_{a+}^{\beta,\rho} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{1}{\rho'(t)}\frac{d}{dt}\right)^n \int_a^t \rho'(s)(\rho(t)-\rho(s))^{n-\beta-1} f(s)\mathrm{d}s,$$

where $n = [\beta]+1$ and $[\beta]$ denotes the integer part of β.

Definition 3 (ρ-Caputo fractional derivative [4, Definition 2.3])**.** Let $n \in \mathbb{N}$, $f, \rho \in \mathcal{C}^n([a,b])$ be two functions such that ρ is increasing with $\rho'(t) \neq 0$, for all $t \in [a,b]$.

ρ-Caputo fractional derivative of order β of a function f is defined by

$${}^C D_{a+}^{\beta,\rho} f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \rho'(s)(\rho(t)-\rho(s))^{n-\beta-1} f_\rho^{[n]}(s)\mathrm{d}s,$$

where $n = [\beta]+1$, for $\beta \notin \mathbb{N}$. And $f_\beta^{[n]}(t) = \left(\frac{1}{\rho'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)$ on $[a,b]$.

Remark 1.

(1) It is clear that when $\beta = n \in \mathbb{N}$, we have

$${}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = f_\rho^{[n]}(t).$$

(2) If $f \in \mathcal{C}^n([a,b])$ and $\beta > 0$. The relation between the two types of fractional derivatives is given by

$${}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = \mathcal{D}_{a+}^{\beta,\rho}\left(f(t) - \sum_{k=0}^{n-1} \frac{f_\rho^{[k]}(a)}{k!}(\rho(t)-\rho(a))^k\right).$$

Theorem 1. *Given $f \in \mathcal{C}^n([a,b])$ and $\beta > 0$. Then we have*

$$I_{a+}^{\beta,\rho}\, {}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\rho^{[k]}(a)}{k!}(\rho(t)-\rho(a))^k.$$

In particular, if $\beta \in (0,1)$ we have:

$$I_{a+}^{\beta,\rho}\, {}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = f(t) - f(a).$$

Now, we present a generalized integral transform introduced by Jarad and Abdeljawwad [10] which can be used to solve linear FDEs involving $\rho-$Riemann-Liouville and $\rho-$Caputo fractional derivatives.

Definition 4. Let $v,\rho\colon [a,\infty)\to\mathbb{R}$ be e a real-valued function and ρ be a non-negative increasing function such that $\rho'(0)>0$. Then the Laplace transform of v with respect to ρ is defined by

$$\mathcal{L}_\rho\{v(t)\}=\mathfrak{V}(\lambda)=\int_0^\infty \exp\{-\lambda(\rho(t)-\rho(0)\}\rho'(t)v(t)\mathrm{d}t,$$

for all $\lambda\in\mathbb{C}$ such that this integral converges.

Here $\mathcal{L}_\rho$ denotes the Laplace transform with respect to ρ, which we call a *generalized Laplace transform*.

Theorem 2. *Let $\beta>0$ and let v be a function of $\rho-$exponential order $c>0$, piecewise continuous over each finite interval $[a,T]$. Then*

$$\mathcal{L}_\rho\{(I_{a+}^{\beta,\rho}v)(t)\}=\frac{\mathcal{L}_\rho\{v(t)\}}{\lambda^\beta}.$$

Next, we present some information regarding the semigroups of linear operators. These findings are documented in references [7, 12].

For a strongly continuous semigroup, often denoted as C_0-semigroup, represented by $(T(t))_{t\geq 0}$, the infinitesimal generator of $(T(t))_{t\geq 0}$ is defined as follows:

$$\mathcal{A}x=\lim_{t\to 0^+}\frac{T(t)x-x}{x}\qquad x\in\mathcal{X}.$$

The domain of $\mathcal{A}$, denoted as $\mathcal{D}(\mathcal{A})$, is such that

$$\mathcal{D}(\mathcal{A})=\left\{x\in\mathcal{X}:\lim_{t\to 0^+}\frac{T(t)x-x}{x}\text{ exists}\right\}.$$

Theorem 3 ([14, Lemma 2.24])**.** *Let be $(T(t))_{t\geq 0}$ be a C_0-semigroup then there exist constants $w\in\mathbb{R}$ and $M\geq 1$, such that*

$$||T(t)||\leq M\exp\{wt\}\qquad 0\leq t<+\infty.$$

Theorem 4 ([14, Lemma 2.25] (Hille-Yosida))**.** *A linear operator $\mathcal{A}$ is the infinitesimal generator C_0-semigroup of contraction (it means $||T(t)||\leq 1\qquad t\geq 0$) if and only if:*

(1) *$\mathcal{A}$ is closed and $\overline{\mathcal{D}(\mathcal{A})}=\mathcal{X}$.*
(2) *The resolvent set of $\mathcal{A}$, $\rho(A)$ contains $\mathbb{R}_*^+$ and for every $\lambda>0$*

$$\mathcal{R}(\lambda,\mathcal{A})\leq\frac{1}{\lambda},$$

where $\mathcal{R}(\lambda,\mathcal{A}):=(\lambda^\beta I-\mathcal{A})^{-1}=\int_0^\infty\exp\{-\lambda^\beta s\}T(s)x\mathrm{d}s$.

Throughout this paper, let $\mathcal{A}$ denote the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $(T(t))_{t\geq 0}$ on the Banach space $\mathcal{X}$. Consequently, there exists $M \geq 1$ such that $M = \sup_{t\in[0,\infty)} ||T(t)||$.

Definition 5 ([14, Definition 2.13]). The Wright type function is defined by

$$\phi_\beta(s) = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!\Gamma(-\beta k+1-\beta)} = \sum_{k=0}^{\infty} \frac{(-s)^k\Gamma(\beta(k+1))\sin(\pi(k+1)\beta)}{k!},$$

for $0 < \beta < 1$ and $s \in \mathbb{C}$.

Proposition 1 ([14, Proposition 2.14]). *The Wright function ϕ_β is an entire function and has the following properties:*

(1) $\phi_\beta(\delta) \geq 0$ *for* $\delta \geq 0$ *and* $\int_0^\infty \phi_\beta(\delta)\mathrm{d}\delta = 1$;
(2) $\int_0^\infty \phi_\beta(\delta)\delta^p \mathrm{d}\delta = \frac{\Gamma(1+p)}{\Gamma(1+\beta p)}$ *for* $p > -1$;
(3) $\int_0^\infty \phi_\beta(\delta)\exp\{-s\delta\}\mathrm{d}\delta = E_\beta(-s)$, $s \in \mathbb{C}$,
where $E_\beta(s) = \sum_{k=0}^\infty \frac{s^k}{\Gamma(k\beta+1)}$ *is Mittag-Leffler function with one parameter for* $s \in \mathbb{C}$ *and* $\beta > 0$.

Theorem 5 (Gronwall's inequality [14, Theorem 2.11]). *Let μ, ν be two integrable functions and κ be a continuous function on $[a,b]$. Let $\rho \in \mathcal{C}^1([a,b])$ be an increasing function such that $\rho'(t) \neq 0$ for all $t \in [a,b]$. Assume that*

(i) *μ and ν are nonnegative;*
(ii) *κ is nonnegative and nondecreasing.*

If

$$\mu(t) \leq \nu(t) + \kappa(t)\int_a^t (\rho(t)-\rho(s))^{\beta-1}\mu(s)\rho'(s)\mathrm{d}s,$$

then

$$\mu(t) \leq \nu(t) + \int_a^t \sum_{k=1}^{\infty} \frac{\{\kappa(t)\Gamma(\beta)\}^k}{\Gamma(k\beta)}(\rho(t)-\rho(s))^{k\beta-1}\nu(s)\rho'(s)\mathrm{d}s \qquad t \in [a,b].$$

Corollary 1. *Under the hypotheses of Theorem 5, let ν be a nondecreasing function on $[a,b]$. Then we have*

$$\mu(t) \leq \nu(t)E_\beta(\kappa(t)\Gamma(\beta)[\rho(t)-\rho(a)]^\beta),$$

for all $t \in [a,b]$.

Theorem 6 ([5, Corollary 2.1]). *Let $\mathcal{S}$ be a closed, bounded and convex subset of the Banach algebra $\mathcal{X}$. We consider the two operators $\mathcal{I}\colon \mathcal{X} \to \mathcal{X}$ and $\mathcal{J}\colon \mathcal{S} \to \mathcal{X}$ such that*

(i) *$\mathcal{I}$ is Lipschitzian with a lipschitz constant α;*
(ii) *$\mathcal{J}$ is completely continuous;*
(iii) $u = \mathcal{I}u\mathcal{J}v \Rightarrow u \in \mathcal{S} \qquad v \in \mathcal{S}$;

(iv) $\alpha\mathcal{K} < 1$, *where* $\mathcal{K} = ||\mathcal{I}(\mathcal{S})||$.

Then the operator function $u = IuJu$ *has a solution on* $\mathcal{S}$.

3. Existence results

According to Definition 3 and Theorem 1 it is suitable to rewrite the Cauchy problem in the equivalent integral equation

$$u(t) = g(t,u(t))\left\{\frac{1}{\Gamma(\beta)}\int_0^t \rho'(s)(\rho(t)-\rho(s))^{\beta-1}\mathcal{A}\left(\frac{u(s)}{g(t,u(s))}\right)\mathrm{d}s + \frac{1}{\Gamma(\beta)}\int_0^t \rho'(s)(\rho(t)-\rho(s))^{\beta-1}h(s,u(s))\mathrm{d}s\right\}. \tag{3.1}$$

Proof.

$$I_{0+}^{\beta,\rho}\,{}^{C}\mathcal{D}_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = I_{0+}^{\beta,\rho}\left\{\mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t))\right\},$$

then

$$\frac{u(t)}{g(t,u(t))} - \frac{u(0)}{g(0,u(0))} = I_{0+}^{\beta,\rho}\mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + I_{0+}^{\beta,\rho}h(t,u(t)).$$

□

Lemma 1. *If* (3.1) *holds, then we have*

$$u(t) = g(t,u(t))\left\{\beta\int_0^t\int_0^\infty \theta\Phi_\beta(\theta)(\rho(t)-\rho(s))^{\beta-1}T(\rho(t)-\rho(s))\theta) \times h(s,u(s))\rho'(s)d\theta\mathrm{d}s\right\}.$$

Proof. Let $\lambda > 0$. Applying the generalized Laplace transforms to (3.1), we have

$$\mathfrak{U}(\lambda) = \frac{1}{\lambda^\beta}(\mathcal{A}\mathfrak{U}(\lambda) + \mathfrak{H}(\lambda)),$$

where

$$\mathfrak{U}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\}\frac{u(\tau)}{g(\tau,u(\tau))}\rho'(\tau)\mathrm{d}\tau,$$

$$\mathfrak{H}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\}h(\tau,u(\tau))\rho'(\tau)\mathrm{d}\tau.$$

It follows that

$$\mathfrak{U}(\lambda) = \int_0^\infty \exp\{-\lambda^\beta s\}T(s)\mathfrak{H}(\lambda)\mathrm{d}s = \int_0^\infty \beta w^{\beta-1}\exp\{-(\lambda w)^\beta\}T(w^\beta)\mathfrak{H}(\lambda)\mathrm{d}w.$$

Taking $w = \rho(t)-\rho(0)$, we get

$$\begin{aligned}\mathfrak{U}(\lambda) &= \int_0^\infty \beta(\rho(t)-\rho(0))^{\beta-1}\exp\{-\lambda(\rho(t)-\rho(0))^\beta\}T((\rho(t)-\rho(0))^\beta)\mathfrak{H}(\lambda)\rho'(t)\mathrm{d}t\\ &= \int_0^\infty\int_0^\infty \beta(\rho(t)-\rho(0))^{\beta-1}\exp\{-\lambda(\rho(t)-\rho(0))^\beta\}T((\rho(t)-\rho(0))^\beta)\end{aligned}$$

$$\times \exp\{-\lambda(\rho(s)-\rho(0))\}h(s,u(s))\rho'(s)\rho'(t)\mathrm{d}s\mathrm{d}t.$$

We consider the following over-sided stable probability density in [11]

$$\rho_\beta(\theta) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\theta^{-\beta i-1}\frac{\Gamma(\beta i+1)}{i!}\sin(i\pi\beta) \qquad \theta \in (0,\infty),$$

whose integration, is given by

$$\int_0^\infty \exp\{-\lambda\theta\}\rho_\beta(\theta)\mathrm{d}\theta = \exp\{-\lambda^\beta\} \qquad \beta \in (0,1). \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned}
\mathfrak{U}(\lambda) &= \int_0^\infty\int_0^\infty\int_0^\infty \beta\rho_\beta(\theta)(\rho(t)-\rho(0))^{\beta-1}\exp\{-\lambda(\rho(t)-\rho(0))\theta\} \\
&\quad \times T((\rho(t)-\rho(0))^\beta)\exp\{-\lambda(\rho(s)-\rho(0))\}h(s,u(s))\rho'(s)\rho'(t)\mathrm{d}\theta\mathrm{d}s\mathrm{d}t \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta}\exp\{-\lambda(\rho(t)+\rho(s)-2\rho(0))\} \\
&\quad \times T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right)h(s,u(s))\rho'(s)\rho'(t)\mathrm{d}\theta\mathrm{d}s\mathrm{d}t \\
&= \int_0^\infty\int_t^\infty\int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta}\exp\{-\lambda(\rho(\tau)-\rho(0))\} \\
&\quad \times T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right) \\
&\qquad \times h(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)),u(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)))) \\
&\qquad \times \rho'(t)\rho'(\tau)\mathrm{d}\theta\mathrm{d}\tau\mathrm{d}t,
\end{aligned}$$

by Fubini's theorem, we have

$$\begin{aligned}
\mathfrak{U}(\lambda) &= \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\} \\
&\times \left\{\int_0^\tau\int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta}T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right)\right. \\
&\times \left. h(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)),u(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0))))\rho'(t)\mathrm{d}\theta\mathrm{d}t\right\}\rho'(\tau)\mathrm{d}\tau \\
&= \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\} \\
&\quad \times \left\{\int_0^\tau\int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(\tau)-\rho(s))^{\beta-1}}{\theta^\beta}\right. \\
&\quad \times \left. T\left(\frac{(\rho(\tau)-\rho(s))^\beta}{\theta^\beta}\right)h(s,u(s))\rho'(s)\mathrm{d}\theta\mathrm{d}s\right\}\rho'(\tau)\mathrm{d}\tau.
\end{aligned}$$

Now, we can invert the Laplace transform to get

$$\frac{u(t)}{g(t,u(t))} = \beta\int_0^t\int_0^\infty \theta\phi_\beta(\theta)(\rho(t)-\rho(s))^{\beta-1} \times T((\rho(t)-\rho(s))^\beta\theta)h(s,u(s))\rho'(s)\mathrm{d}\theta\mathrm{d}s,$$

where $\phi_\beta(\theta)=\frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\rho_\beta(\theta^{-\frac{1}{\beta}})$ is the probability density function defined in $(0,\infty)$. Thus

$$u(t) = g(t,u(t))\left\{\beta\int_0^t\int_0^\infty \theta\phi_\beta(\theta)(\rho(t)-\rho(s))^{\beta-1} \times T((\rho(t)-\rho(s))^\beta\theta)h(s,u(s))\rho'(s)\mathrm{d}\theta\mathrm{d}s\right\}.$$

□

For any $u\in\mathcal{X}$, define the operator $\omega_\rho^\beta(t,s)$ by

$$\omega_\rho^\beta(t,s)u = \beta\int_0^\infty \theta\phi_\beta(\theta)T((\rho(t)-\rho(s))^\beta\theta)\,u\,\mathrm{d}\theta \qquad 0\le s\le t\le T.$$

Lemma 2. *The operator ω_ρ^β has the following properties:*

(1) *For any fixed $t\ge s\ge 0$ $\omega_\rho^\beta(t,s)$ is bounded linear operator with*

$$\|{\omega_\rho}^\beta(t,s)(\omega)\| \le \frac{\beta.M}{\Gamma(1+\beta)}||u|| = \frac{M}{\Gamma(\beta)}\|u\| \qquad u\in\mathcal{X}.$$

(2) *The operator $\omega_\rho^\beta(t,s)$ is strongly continuous for all $t\ge s\ge 0$, that is for every $u\in\mathcal{X}$ and $0\le s\le t_1\le t_2\le T$ we have*

$$\|\omega_\rho^\beta(t_2,s)u-\omega_\rho^\beta(t_1,s)\|\longrightarrow 0, \qquad \text{as } t_1\longrightarrow t_2.$$

Proof. The proof of this lemma is similar to the one given in [15]. □

Definition 6. A function $u\in\mathcal{C}([0,T],\mathcal{X})$ is called a mild solution of (1.1) if satisfies

$$u(t) = g(t,u(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right\}.$$

Before starting and proving the main result, we introduce the following hypothesis

(C_1) $T(t)$ is compact operator for every $t>0$.

(C_2) For any $r>0$, there exists a function $h_r\in L^\infty([0,T],\mathcal{X})$ such that

$$\sup_{||u||\le r}\|h(t,u)\|\le h_r(t) \qquad t\in[0,T],$$

and there is a constant ζ >0 such that

$$\limsup_{r\to\infty}\frac{||h_r(t)||_{L^\infty}}{r} = \zeta.$$

(C_3) The function $g \in \mathcal{C}_c([0,T]\times\mathcal{X},\mathcal{X}\setminus\{0\})$ is bounded and there exist constants $\mu > 0$ and $L > 0$ such that for all $u,v \in \mathcal{X}$ and $t \in [0,T]$, we have

$$|g(t,u)-g(t,v)| \leq \mu|u-v| \quad |g(t,u)| < L.$$

Theorem 7. *Assume that condition $(C_1)-(C_3)$ hold. Then the problem (1.1) has at least mild solution provided that*

$$\frac{\mu M||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta < 1.$$

Proof. Let $\mathcal{V} = \{u \in \mathcal{X}, ||u|| \leq b\}$ $\qquad b = \frac{LM||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(t)-\rho(0))^\beta$.
We have

$$u(t) = g(t,u(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho{}^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right\}.$$

Then we can transform into $u(t) = Iu(t)\mathcal{J}u(t) \qquad t \in [0,T]$.

Now we prove that all conditions of Theorem 6 are satisfied.

Step 1: Let $u,v \in \mathcal{X}$ then

$$\begin{aligned}| Iu(t)-Iv(t) | &= | g(t,u(t))-g(t,v(t)) | \\ &\leq \mu \mid u(t)-v(t) \qquad t \in [0,T].\end{aligned}$$

Step 2: Firstly, we prove that $\mathcal{J}$ is completely continuous.
Let $u_n, u \in \mathcal{V}$ with $\lim_{n\to+\infty} \|u_n-u\| = 0$. Then

$$h(s,u_n(s)) \to h(s,u(s)), \text{ as } n \to \infty.$$

Therefore

$$\|\mathcal{J}u_n(t)-\mathcal{J}u(t)\| \leq \int_0^t (\rho(t)-\rho(s))^{\beta-1}\frac{M}{\Gamma(\beta)}||h(s,u_n(s))-h(s,u(s))||\rho'(s)\mathrm{d}s.$$

Via Lebesgue dominated convergence theorem, we get

$$\|\mathcal{J}u_n(t)-\mathcal{J}u(t)\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

•$\mathcal{J}(\mathcal{V})$ is uniformly bounded.
Let $u \in \mathcal{V}$,

$$\begin{aligned}\|\mathcal{J}u(t)\| &\leq \int_0^t (\rho(t)-\rho(s))^{\beta-1}\frac{M}{\Gamma(\beta)}\|h_r\|_{L^\infty}\rho'(s)\mathrm{d}s \\ &= \frac{M||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(t)-\rho(0))^\beta \qquad 0 \leq t \leq T \\ &\leq \frac{M||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta.\end{aligned}$$

•$\mathcal{J}(\mathcal{V})$ is equicontinuous.

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in \mathcal{V}$.

$$\begin{aligned}
||\mathcal{J}u(t_2) - \mathcal{J}u(t_1)|| &\leq \int_{t_1}^{t_2} (\rho(t_1) - \rho(s))^{\beta-1} \frac{M}{\Gamma(\beta)} ||h_r||_{L^\infty} \rho'(s) \mathrm{d}s \\
&+ \int_0^{t_1} ||\omega_\rho^\beta(t_2,s)|| \{(\rho(t_2) - \rho(s))^{\beta-1} - (\rho(t_1) - \rho(s))^{\beta-1}\} \|h(s,u(s))\| \rho'(s) \mathrm{d}s \\
&+ \int_0^{t_1} (\rho(t_1) - \rho(s))^{\beta-1} ||(\omega_\rho^\beta(t_2,s) - \omega_\rho^\beta(t_1,s))|| \times ||h(s,u(s))|| \rho'(s) \mathrm{d}s \\
&\leq \frac{-M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)} (\rho(t_1) - \rho(t_2))^\beta + \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)} \Big\{ (\rho(t_1) - \rho(0))^\beta - (\rho(t_2) - \rho(t_1))^\beta \\
&\quad - (\rho(t_1) - \rho(0))^\beta \} + \frac{\|h_r\|_{L^\infty}}{\beta} (\rho(t_1) - \rho(0))^\beta ||\omega_\rho^\beta(t_2,s) - \omega_\rho^\beta(t_1,s)||.
\end{aligned}$$

Thus

$$\|\mathcal{J}u(t_2) - \mathcal{J}u(t_1)\| \to 0 \qquad \text{as } t_2 \to t_1.$$

Step 3: Let $u \in \mathcal{X}$ and $v \in \mathcal{V}$ such that $Iu\mathcal{J}v = u$, prove that $u \in \mathcal{V}$.

$$\begin{aligned}
|u(t)| = |Iu\mathcal{J}v| &= |g(t,u(t))| \times |\mathcal{J}v(t)| \\
&\leq L \times \frac{M||h_r||_{L^\infty}}{\Gamma(\beta+1)} (\rho(T) - \rho(0))^\beta.
\end{aligned}$$

Step 4: Suppose that $\mathcal{S} = \sup\{||\mathcal{J}u||, u \in \mathcal{V}\} \leq b$. Then

$$\mu \times \mathcal{S} \leq \mu \times b < 1.$$

□

4. MITTAG-LEFFLER-ULAM-HYERS STABILITY

For $g \in \mathcal{C}([0,T] \times \mathcal{X}, \mathcal{X} \setminus \{0\})$, $h \in \mathcal{C}_c([0,T] \times \mathcal{X}, \mathcal{X})$, $\psi \in \mathcal{C}([0,T], \mathbb{R}^+)$ and $\varepsilon > 0$. We consider the equation

$$^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) \qquad t \in [0,T] \quad 0 < \beta < 1. \tag{4.1}$$

And the inequalities

$$\left| {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) - \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) - h(t,u(t)) \right| \leq \varepsilon \qquad t \in [0,T]; \tag{4.2}$$

$$\left| {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) - \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) - h(t,u(t)) \right| \leq \psi \qquad t \in [0,T]; \tag{4.3}$$

$$\left| {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) - \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) - h(t,u(t)) \right| \leq \varepsilon\psi \qquad t \in [0,T]. \tag{4.4}$$

Definition 7. We said that the equation (4.1) is Mittag-Leffler-Ulam-Hyers stable with respect to E_β, if there exists a real number $\delta > 0$ such that for each $\varepsilon > 0$ and for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.2) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq \delta\varepsilon E_\beta(t)$, $t \in [0,T]$.

Definition 8. We said that the equation (4.1) is generalized Mittag-Leffler-Ulam-Hyers stable with respect to E_β, if there exists $\mu \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ with $\mu(0) = 0$ such that for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.2) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t)-u(t)| \leq C\mu(t)E_\beta(t)$, $C > 0$ and $t \in [0,T]$.

Definition 9. Equation (4.1) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to ψE_β, if there exists a real number $C_\psi > 0$ such that for each $\varepsilon > 0$ and for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.4) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq C_\psi \varepsilon E_\beta(t)$, $t \in [0,T]$.

Definition 10. Equation (4.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to ψE_β, if there exists a real number $C_\psi > 0$ such that for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.3) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq C_\psi \psi(t) E_\beta(t)$, $t \in [0,T]$.

Remark 2. A function $u \in \mathcal{C}^1([0,T],\mathcal{X})$ is a solution of inequality (4.2) if and only if there exists a function $\varphi \in \mathcal{C}^1([0,T],\mathcal{X})$ (which depend on u) such that

(1) $|\varphi(t)| \leq \varepsilon \qquad t \in [0,T]$.
(2) ${}^C D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) + \varphi(t) \qquad t \in [0,T]$.

Lemma 3. *If $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ is a solution of (4.2), ν is a solution of the following integral inequality*

$$\left|\nu(t) - g(t,\nu(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right|$$
$$\leq \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta.$$

Proof. By the previous remark, we have

$${}^C D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) + \varphi(t) \qquad t \in [0,T].$$

And from Theorem 7, we get

$$\left|\nu(t) - g(t,\nu(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\mathcal{W}_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right|$$
$$\leq L \times \varepsilon \times \frac{M}{\Gamma(\beta)}\{\rho(T)-\rho(0)\}^\beta.$$

□

Theorem 8. *Assume that $h \in \mathcal{C}([0,T]\times\mathcal{X},\mathcal{X})$ and there exists $L_h > 0$ such that $|h(t,u_1)-h(t,u_2)| \leq L_h|u_1-u_2|$, for all $t\in[0,T]$ and $u_1,u_2\in\mathcal{X}$.*
Then equation (4.1) is Mittag-Leffler-Ulam-Hyers stable.

Proof. Let $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ be a solution of inequality (4.2) and let us denote by $u\in\mathcal{C}([0,T],\mathcal{X})$ the unique mild solution of the cauchy problem

$$\begin{cases} {}^{C}D^{\beta,\rho}_{0+}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) & t\in[0,T], \\ u(0)=\nu(0)=0. \end{cases}$$

We have

$$\begin{aligned}
|\nu(t)-u(t)| \leq & \left|\nu(t) - g(t,\nu(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \\
& + \left|g(t,\nu(t))\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right. \\
& \left. - g(t,\nu(t))\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right| \\
& + \left|g(t,\nu(t))\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right. \\
& \left. - g(t,u(t))\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right| \\
\leq & \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta + \frac{LL_hM}{\Gamma(\beta)}\int_0^t (\rho(t)-\rho(s))^{\beta-1}|\nu(s)-u(s)|\rho'(s)\mathrm{d}s \\
& + \frac{M\mu||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta|\nu(t)-u(t)| \\
\leq & \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta \times \frac{1}{d} + \frac{LL_hM}{d\Gamma(\beta)}\int_0^t (\rho(t)-\rho(s))^{\beta-1}|\nu(s)-u(s)|\rho'(s)\mathrm{d}s,
\end{aligned}$$

with

$$d = 1 - \frac{M\mu||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta > 0 \text{ (under the hypothesis of Theorem 7).}$$

By Gronwall's inequality, we get

$$|\nu(t)-u(t)| \leq \frac{LM\varepsilon}{\Gamma(\beta+1)} \times \frac{1}{d}(\rho(T)-\rho(0))^\beta E_\beta\left(\frac{LL_hM}{d}(\rho(t)-\rho(0))^\beta\right) \ (d>0).$$

□

Theorem 9. *Assume that the following conditions hold:*

(i) $h\in\mathcal{C}([0,\infty)\times\mathcal{X},\mathcal{X})$;

(ii) *the function* $\psi \in \mathcal{C}([0,\infty],\mathbb{R}^+)$ *is increasing and there exists* $\lambda > 0$ *such that*

$$\frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta} \leq \lambda\psi(t) \qquad t \in [0,\infty);$$

(iii) $\mu(t)$ *is nonnegative, nondecreasing continuous function defined on* $t \in [0,\infty)$

$$\text{and } |h(t,u_1)-h(t,u_2)| \leq \mu(t)|u_1-u_2|, \quad \text{for all } t \geq 0 \text{ and } u_1,u_2 \in \mathcal{X}.$$

Then, equation (4.1) *is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to* ψE_{β}.

Proof. Let $\nu \in \mathcal{C}^1([0,T],\infty)$ be a solution of equation (4.3). Then, we get

$$\left|\nu(t)-g(t,\nu(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \leq \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta} \leq \lambda\psi(t) \qquad t \in [0,\infty).$$

Let us denote by $u \in \mathcal{C}([0,T],\infty)$ the unique mild solution of the cauchy problem

$$\begin{cases} {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) & \in t[0,\infty), \\ u(0) = \nu(0) = 0. \end{cases}$$

We have

$$u(t) = g(t,u(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right\} \qquad t \in [0,\infty).$$

It follows that

$$\begin{aligned}
|\nu(t)-u(t)| &\leq \left|\nu(t)-g(t,\nu(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \\
&+ \left|g(t,\nu(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right. \\
&\quad \left. - g(t,u(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right| \\
&\leq \lambda\psi(t) + |g(t,\nu(t))|\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)|h(s,\nu(s))-h(s,u(s))|\rho'(s)\mathrm{d}s \\
&\quad + |g(t,\nu(t))-g(t,u(t))|\,||h_r||_{L^{\infty}}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(s))^{\beta-1} \\
&\leq \lambda\psi(t) + L\times\mu(t)\frac{M}{\Gamma(\beta)}\int_0^t(\rho(t)-\rho(s))^{\beta-1}\times|\nu(s)-u(s)|\rho'(s)\mathrm{d}s \\
&\quad + \mu|\nu(t)-u(t)|\,||h_r||_{L^{\infty}}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta}
\end{aligned}$$

$$\leq \frac{\lambda}{d}\psi(t)+\frac{LM\mu(t)}{\Gamma(\beta)}\frac{1}{d}\int_0^t(\rho(t)-\rho(s))^{\beta-1}\times|\nu(s)-u(s)|\rho'(s)\mathrm{d}s, \qquad d>0.$$

By Gronwall's inequality, we get

$$|\nu(t)-u(t)|\leq \frac{\lambda}{d}\psi(t)E_\beta\left(\frac{LM\mu(t)}{d}(\rho(t)-\rho(0))^\beta\right),$$

with $d=1-\mu||h_r||_{L^\infty}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta$. □

5. An illustrative example

Let $\mathcal{X}=L^2([0,\pi])$ equipped with the norm and inner product defined respectively, for all $u,\nu\in L^2([0,\pi])$ by

$$||u||=\left(\int_0^\pi|u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \qquad <u,\nu>=\int_0^\pi u(x)\,\overline{\nu(x)}\,\mathrm{d}x.$$

Consider the following initial-boundary value problem of tire fractional parabolic partial differential equation with nonlinear source term

$$\begin{cases}{}^C\mathcal{D}_{0^+}^{\beta,\rho}\left(\frac{u(x,t)}{e^{-t}u(x,t)}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{u(x,t)}{e^{-t}u(x,t)}\right)+\frac{1}{2}e^{-t}u(x,t) & (t,x)\in[0,1]\times[0,\pi]\\ u(0,t)=u(\pi,t)=0 & t\in[0,1]\\ u(x,0)=0 & x\in[0,\pi].\end{cases}$$

Where $\beta=\frac{2}{3}\quad T=1\quad \rho=t$.

We define an operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{D}(\mathcal{A}):=\{\nu\in\mathcal{E};\nu,\nu^{'}\text{are absolutely continuous and }\nu^{''},\ \nu(0)=\nu(\pi)=0\},$$

and

$$\mathcal{A}u=\frac{\partial^2}{\partial x^2}u.$$

It is well known that $\mathcal{A}$ has a discrete spectrum, the eigenvalue are $-j^2$, $j\in\mathbb{N}$, with corresponding normalized eigenvectors $e_j(z)=\sqrt{\frac{2}{\pi}}\sin(jz)$. Then

$$\mathcal{A}x=\sum_{j=1}^{\infty}-j^2<x,e_j>e_j \qquad x\in\mathcal{D}(A).$$

Thus, $\mathcal{A}$ generates a uniformly bounded analytic semigroup $\{T(t)\}_{t\geq0}$ in $\mathcal{X}$ and it is given by

$$T(t)x=\sum_{j=1}^{\infty}e^{-j^2t}<x,e_j>e_j \qquad x\in\mathcal{X}$$

with

$$||T(t)||\leq e^{-t} \qquad \forall t\geq 0.$$

Hence, we take $M=1$ which implies that $\sup\limits_{t\in[0,\infty)}||T(t)||=1$ and $(\mathcal{C}_1)$ is satisfied.

Then for all $t \in [0,1]$, we have

$$\begin{cases} ||h(t,u)|| = \frac{1}{2}e^{-t}||u||, \\ \sup\limits_{||u||\leq r} ||h(t,u)|| \leq \frac{1}{2}e^{-t}\, r := h_r(t), \\ \limsup\limits_{r\to\infty} \frac{||h_t(t)||_{L^\infty}}{r} = \frac{1}{2} := L. \end{cases}$$

And

$$||g(t,u_1) - g(t,u_2)|| \leq |u_1 - u_2| \qquad u_1, u_2 \in \mathcal{X}.$$

Therefore $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ are satisfied, which is given us

$$\frac{M\mu||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta = \frac{1}{2\Gamma\left(\frac{5}{3}\right)} \simeq 0.45137 < 1.$$

According to Theorem 7. The problem (5) has a unique mold solution on $[0,1]$.

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