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SOME GENERALIZATIONS OF MERCER INEQUALITY AND ITS OPERATOR EXTENSIONS

MOHSEN KIAN AND ZAINAB PEYMANI

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Abstract. We give a more general form of the Mercer inequality by replacing some constants by positive operators. As some consequences, our results produce a Jensen operator inequality for superquadratic functions. Moreover, we present some Mercer inequalities of Hermite–Hadamard's type.

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1. INTRODUCTION

Mercer [11] proved a variant of the Jensen inequality for convex functions as follows: If $f\colon [m,M] \to \mathbb{R}$ is a convex function, then

$$f\left(M+m-\sum_{j=1}^{n}\lambda_j x_j\right) \leq f(M)+f(m)-\sum_{j=1}^{n}\lambda_j f(x_j) \tag{1.1}$$

for all $x_j \in [m,M]$ and all $\lambda_j \in [0,1]$ with $\sum_{j=1}^{n}\lambda_j = 1$. An operator extension of (1.1) has been presented in [10]:

$$f\left(M+m-\sum_{j=1}^{n}\Phi_j(A_j)\right) \leq f(M)+f(m)-\sum_{j=1}^{n}\Phi_j(f(A_j)) \tag{1.2}$$

in which $(\Phi_1,\ldots,\Phi_n)$ is a tuple of positive linear maps on $\mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^{n}\Phi_i(I) = I$ and A_j's are self-adjoint operators with spectra in $[m,M]$. Here we denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I is the identity operator. When $f\colon [m,M] \to \mathbb{R}$ is a continuous function and A is a self-adjoint operator with spectrum in $[m,M]$, the operator $f(A)$ is defined by the well-known Gelfand's mapping. This is called the continuous functional calculus, see [4].

Utilising the famous Hermite–Hadamard inequality, the first author and Moslehian [7] presented a variant of the operator Mercer inequality (1.2). Some reverse Mercer operator inequalities have been given in [2]. The authors of [9] introduced logarithmic superquadratic functions. In [12], some sub-additivity inequalities for this class of functions have been presented.

In [14], Moslehian, Mićić, and the first author extended the operator Mercer inequality (1.2) by replacing the scalars m and M by operators and showed that with some conditions on the spectra of operators, the inequality

$$f(\Phi(C))+f(\Phi(B))\le\Phi(f(A))+\Phi(f(D)) \tag{1.3}$$

holds.

Superquadratic functions are introduced in [1] as modifications of convex functions. Since then, this class of functions has been utilised to improve many results concerning convex functions. A function $f\colon [0,\infty)\to\mathbb{R}$ is called superquadratic if for all $x\ge 0$ there exists a constant $C_x\in\mathbb{R}$ such that

$$f(y)\ge f(x)+C_x(y-x)+f(|y-x|)$$

for all $y\ge 0$. These functions enjoy a Jensen type inequality as

$$\begin{aligned}&f(\lambda x+(1-\lambda)y)\\&\qquad\le\lambda f(x)+(1-\lambda)f(y)-\lambda f((1-\lambda)|x-y|)-(1-\lambda)f(\lambda|x-y|)\end{aligned} \tag{1.4}$$

for all $x,y\ge 0$ and $\lambda\in[0,1]$, see [1,9].

The authors of [3] proved a variant of the operator Mercer inequality (1.2) for superquadratic functions: If $f\colon [0,\infty)\to\mathbb{R}$ is a continuous superquadratic function and m,M are positive scalars, then

$$f(M+m-\Phi(A))-\beta(\Phi(A))\le f(m)+f(M)-\Phi(f(A))-\Phi(\beta(A)) \tag{1.5}$$

holds for every positive operator $A\in\mathcal{B}(\mathcal{H})$ with spectrum in $[m,M]$ and every positive linear map Φ on $\mathcal{B}(\mathcal{H})$, when we set the notation

$$\beta(t)=\frac{t-m}{M-m}f(M-t)+\frac{M-t}{M-m}f(t-m),\qquad (t\in[m,M]). \tag{1.6}$$

Recall that an operator $A\in\mathcal{B}(\mathcal{H})$ is called positive if $\langle A\eta,\eta\rangle\ge 0$ for every $\eta\in\mathcal{H}$.

The importance of (1.2), (1.3) and (1.5) is that they are available without the restrictive condition of being operator convex for the function f. Recall that a function $f\colon [m,M]\subseteq\mathbb{R}\to\mathbb{R}$ is *operator convex*, when

$$f(\lambda A+(1-\lambda)B)\le\lambda f(A)+(1-\lambda)f(B)$$

holds for all self-adjoint operators A and B with spectra in $[m,M]$ and every $\lambda\in[0,1]$. It is known that if $f\colon [m,M]\to\mathbb{R}$ is operator convex, then the Jensen operator inequality

$$f(\Phi(A))\le\Phi(f(A)) \tag{1.7}$$

holds for every self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ with spectrum in $[m,M]$ and every unital positive linear map Φ on $\mathcal{B}(\mathcal{H})$. However, if f is convex, but not operator convex, then (1.7) does not hold in general. However, (1.2) is valid for every convex function ((1.5) is valid for every superquadratic function).

In this paper, we study the Mercer inequality and its operator extension for superquadratic functions. In particular, we extend (1.5) by replacing scalars m,M by operators. As applications, a Jensen operator inequality has been presented for superquadratic functions. Moreover, we present a Mercer inequality of Hemite–Hadamard's type. The Hemite–Hadamard inequality asserts that

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t \leq \frac{f(x)+f(y)}{2}$$

holds for every convex function f on $[x,y]$. The reader can refer to [5, 6, 13] for operator versions of this inequality.

2. RESULTS

We begin by presenting a Mercer inequality of Hemite–Hadamard's type for superquadratic functions. We need the following lemma. For simplicity, we use the notation $x\nabla_\lambda y$ for the λ-weighted arithmetic mean $\lambda x + (1-\lambda)y$ of x and y.

Lemma 1. *Let* $0 \leq m < M$ *and let* $f\colon [0,\infty) \to \mathbb{R}$ *be a superquadratic function. Then*

$$\begin{aligned} &f(m+M-x\nabla_\lambda y) + 2\beta(x)\nabla_\lambda\beta(y) \\ &\qquad \leqslant f(m)+f(M)-f(x)\nabla_\lambda f(y) - f((1-\lambda)|x-y|)\nabla_\lambda f(\lambda|x-y|)) \end{aligned}$$

for all $x,y \in [m,M]$ *and every* $\lambda \in [0,1]$.

Proof. For any $x \in [m,M]$, we put $y = m+M-x$ so that $y+x = m+M$. There exists $\lambda \in [0,1]$ such that $y = \lambda m + (1-\lambda)M$. Since f is superquadratic, we cal apply (1.4) to write

$$\begin{aligned} &f(M+m-x) = f(\lambda m + (1-\lambda)M) \\ &\leqslant \lambda f(m) + (1-\lambda)f(M) - \lambda f((1-\lambda)|m-M|) - (1-\lambda)f(\lambda|m-M|) \\ &= f(m)+f(M) \qquad (2.1) \\ &\quad -(\lambda f(M) + (1-\lambda)f(m) + \lambda f((1-\lambda)|M-m|) + (1-\lambda)f(\lambda|M-m|)). \end{aligned}$$

On the other hand, $x = m+M-y = \lambda M + (1-\lambda)m$ and so we have

$$\begin{aligned} f(x) &= f(\lambda M + (1-\lambda)m) \\ &\leq \lambda f(M) + (1-\lambda)f(m) - \lambda f((1-\lambda)|M-m|) - (1-\lambda)f(\lambda|M-m|). \qquad (2.2) \end{aligned}$$

It follows from (2.1) and (2.2) that

$$f(M+m-x)$$

$$\leq f(m)+f(M)-f(x)-2\left(\lambda f((1-\lambda)|M-m|)+(1-\lambda)f(\lambda|M-m|)\right). \quad (2.3)$$

Applying (2.3) with $\lambda = \dfrac{x-m}{M-m}$ we obtain

$$f(M+m-x) \leqslant f(m)+f(M)-f(x)-2\beta(x), \quad (2.4)$$

in which β is defined as (1.6). Now, for every $x,y \in [m,M]$ and every $\lambda \in [0,1]$, using (1.4) we have

$$\begin{aligned} f(M+m-(\lambda x+(1-\lambda)y)) &= f(\lambda(m+M-x)+(1-\lambda)(m+M-y)) \\ &\leq \lambda f(m+M-x)+(1-\lambda)f(m+M-y) \quad (2.5) \\ &\quad -\lambda f((1-\lambda)|x-y|)-(1-\lambda)f(\lambda|x-y|) \end{aligned}$$

as f is superquadratic. Now, by applying (2.4) we get

$$\begin{aligned} &\lambda f(m+M-x)+(1-\lambda)f(m+M-y) \\ &\quad \leqslant \lambda(f(m)+f(M)-f(x)-2\beta(x))+(1-\lambda)(f(m)+f(M)-f(y)-2\beta(y)) \\ &\quad = f(m)+f(M)-(\lambda f(x)+(1-\lambda)f(y))-2(\lambda\beta(x)+(1-\lambda)\beta(y)). \quad (2.6) \end{aligned}$$

It follows from (2.5), (2.6) that

$$\begin{aligned} &f(m+M-(\lambda x+(1-\lambda)y) \\ &\quad \leq f(m)+f(M)-(\lambda f(x)+(1-\lambda)f(y))-2(\lambda\beta(x)+(1-\lambda)\beta(y)) \\ &\quad\quad -(\lambda f((1-\lambda)|x-y|)+(1-\lambda)f(\lambda|x-y|)), \end{aligned}$$

as required. □

In the next result, we present a Mercer inequality of Hermite–Hadamard's type for superquadratic functions. The reader may compare it to [7, Theorem 2.1]. Also refer to the paper [12].

Theorem 1. *Let* $0 \leq m < M$ *and let* f *be a superquadratic function on* $[m,M]$. *Then*

$$\begin{aligned} &f\left(m+M-\frac{x+y}{2}\right)+2\int_0^{1/2} f(u|x-y|)\,\mathrm{d}u \\ &\quad \leq \frac{1}{y-x}\int_x^y f(m+M-u)\mathrm{d}u \quad (2.7) \\ &\quad \leq f(m)+f(M)-\frac{f(x)+f(y)}{2}-(\beta(x)+\beta(y))-2\int_0^1 (1-u)f(u|x-y|)\mathrm{d}u, \end{aligned}$$

and

$$\begin{aligned} f\left(m+M-\frac{x+y}{2}\right)+2\int_0^{1/2} f(u|x-y|)\mathrm{d}u \\ &\leq f(m)+f(M)-\frac{1}{y-x}\int_x^y (f(u)+2\beta(u))\mathrm{d}u \qquad (2.8)\\ &\leq f(m)+f(M)-f\left(\frac{x+y}{2}\right)-\frac{2}{y-x}\int_x^y \beta(u)\mathrm{d}u-2\int_0^{1/2} f(u|x-y|)\mathrm{d}u \end{aligned}$$

for all $x,y\in[m,M]$.

Proof. Assume that $x,y\in[m,M]$ and put $a=M+m-x$ and $b=m+M-y$. Without loss of generality we assume that $x<y$. It follows from (1.4) that

$$\begin{aligned} f\left(m+M-\frac{x+y}{2}\right) &= f\left(\frac{(ta+(1-t)b)+((1-t)a+tb)}{2}\right)\\ &\leq \frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}-f\left(\left|\frac{2t-1}{2}\right||a-b|\right), \end{aligned}$$

since f is superquadratic. Integrating both sides of the above inequality with respect to t over $[0,1]$ yields

$$\begin{aligned} f\left(m+M-\frac{x+y}{2}\right) &\leq \int_0^1 f(m+M-((1-t)x+ty))\mathrm{d}t-\int_0^1 f\left(\left|\frac{2t-1}{2}\right||a-b|\right)\mathrm{d}t\\ &= \frac{1}{y-x}\int_x^y f(m+M-u)\mathrm{d}u-2\int_0^{1/2} f(u|x-y|)\,\mathrm{d}u, \qquad (2.9) \end{aligned}$$

where in the las equality, we employ change of variables in both integrals. On the other hand, it follows from Lemma 1 that

$$\begin{aligned} & f(m+M-(tx+(1-t)y))\\ &\quad \leq f(m)+f(M)-(tf(x)+(1-t)f(y)) \qquad (2.10)\\ &\qquad -2(t\beta(x)+(1-t)\beta(y))-(tf((1-t)|x-y|)+(1-t)f(t|x-y|)). \end{aligned}$$

Noting that

$$\int_0^1 tf((1-t)|x-y|)\mathrm{d}t=\int_0^1 (1-t)f(t|x-y|)\mathrm{d}t$$

and integrating both sides of (2.10) with respect to t over $[0,1]$ we get

$$\begin{aligned} & \int_0^1 f(m+M-(tx+(1-t)y))\mathrm{d}t \qquad (2.11)\\ &\quad \leq f(m)+f(M)-\frac{f(x)+f(y)}{2}-(\beta(x)+\beta(y))-2\int_0^1 (1-t)f(t|x-y|)\mathrm{d}t. \end{aligned}$$

Combining (2.9) and (2.11), we reach (2.7).

Next, it follows from Lemma 1 that

$$f\left(m+M-\frac{a+b}{2}\right) \\ \leqslant f(m)+f(M)-\frac{f(a)+f(b)}{2}-(\beta(a)+\beta(b))-f\left(\left|\frac{a-b}{2}\right|\right) \tag{2.12}$$

holds for all $a,b\in[m,M]$. Let $t\in[0,1]$ and $x,y\in[m,M]$. Replacing a and b, respectively, by $tx+(1-t)y$ and $(1-t)x+ty$ in (2.12), we obtain

$$\begin{aligned} &f\left(m+M-\frac{tx+(1-t)y+(1-t)x+ty}{2}\right)\\ &\leqslant f(m)+f(M)-\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}\\ &\quad-(\beta(tx+(1-t)y)+\beta((1-t)x+ty))-f\left(\left|\frac{tx+(1-t)y-(1-t)x+ty}{2}\right|\right), \end{aligned}$$

or equivalently, we get

$$\begin{aligned} f\left(m+M-\frac{x+y}{2}\right) \leqslant & f(m)+f(M)-\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2} \\ &-(\beta(tx+(1-t)y)+\beta((1-t)x+ty))-f\left(\frac{|1-2t||x-y|}{2}\right). \end{aligned} \tag{2.13}$$

Note that

$$\int_0^1 f(tx+(1-t)y)\mathrm{d}t=\int_0^1 f((1-t)x+ty)\mathrm{d}t=\frac{1}{x-y}\int_x^y f(u)\mathrm{d}u,$$

and

$$\int_0^1 \beta(tx+(1-t)y)\mathrm{d}t=\int_0^1 \beta((1-t)x+ty)\mathrm{d}t=\frac{1}{x-y}\int_x^y \beta(u)\mathrm{d}u.$$

Consequently, the first inequality of (2.8) follows by integrating both sides of (2.13) over $t\in[0,1]$. To obtain the second inequality, we write

$$\begin{aligned} f\left(\frac{x+y}{2}\right)&=f\left(\frac{(tx+(1-t)y)+((1-t)x+ty)}{2}\right)\\ &\leq \frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}-f\left(\left|\frac{1-2t}{2}\right||x-y|\right). \end{aligned}$$

Integrating both sides with respect to t over $[0,1]$ we get

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_x^y f(u)\,\mathrm{d}u-2\int_0^{1/2} f(u|x-y|)\mathrm{d}u.$$

This completes the proof.

$\square$

In a particular case, the Mercer type inequality presented in Theorem 1 concludes a Hermite–Hadamard inequality for superquadratic functions. The next corollary follows from Theorem 1, when we consider $m = x$ and $M = y$.

Corollary 1. *If* $f\colon [0,\infty) \to \mathbb{R}$ *is a superquadratic function, then*

$$\begin{aligned} f\left(\frac{x+y}{2}\right) + 2\int_0^{1/2} f\left(u|x-y|\right) \mathrm{d}u &\leq \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t \\ \leq \frac{f(x)+f(y)}{2} - 2f(0) - 2\int_0^1 (1-u)f(u|x-y|)\mathrm{d}u \end{aligned}$$

for all $0 \leq x < y$.

The power functions $f(t) = t^p$ and $g(t) = -t^q$ are superquadratic, when $p \geq 2$ and $q \in [1,2]$. Hence, the next result follows.

Corollary 2. *Let* $0 \leq m < M$ *and let* $x, y \in [m,M]$. *If* $p \geq 2$, *then*

$$\begin{aligned} &\left(m+M-\frac{x+y}{2}\right)^p + \frac{1}{2^p(p+1)}|x-y|^p \\ &\leq \frac{(M+m-x)^{p+1}-(M+m-y)^{p+1}}{(p+1)(y-x)} \\ &\leq m^p + M^p - \frac{x^p+y^p}{2} - (\beta_p(x)+\beta_p(y)) - 2\frac{|x-y|^p}{(p+1)(p+2)}, \end{aligned} \tag{2.14}$$

in which $\beta_p(x) = \frac{(M-x)(x-m)}{M-m}\left((M-x)^{p-1}+(x-m)^{p-1}\right)$. *If* $p \in [1,2]$, *then* (2.14) *is reversed.*

Let $f(t) = t^2$. Then f is superquadratic as well as subquadratic. This fact together Corollary 2 produce an equation as

$$\begin{aligned} &\left(m+M-\frac{x+y}{2}\right)^2 + \frac{1}{12}|x-y|^2 \\ &= \frac{(M+m-x)^3-(M+m-x)^3}{3(y-x)} \\ &= m^2+M^2-\frac{x^2+y^2}{2} - ((M-x)(x-m)+(M-y)(y-m)) - \frac{|x-y|^2}{6}. \end{aligned}$$

The next proposition gives a generalization of [12, Theorem 2.8].

Proposition 1. *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be a superquadratic function and let* $0 \leq y_1 \leq x_1 \leq x_2 \leq y_2$. *If* $x_1 + x_2 = y_1 + y_2$, *then*

$$f(x_1)+f(x_2) \leq f(y_1)+f(y_2) - 2\frac{y_2-x_1}{y_2-y_1}f(x_1-y_1) - 2\frac{x_1-y_1}{y_2-y_1}f(x_2-y_1). \tag{2.15}$$

Proof. Applying (1.4) with $\lambda = \frac{y_2-x_1}{y_2-y_1}$ we obtain

$$\begin{aligned} f(x_1) &= f\left(\frac{y_2-x_1}{y_2-y_1}y_1 + \frac{x_1-y_1}{y_2-y_1}y_2\right) \\ &\le \frac{y_2-x_1}{y_2-y_1}f(y_1) + \frac{x_1-y_1}{y_2-y_1}f(y_2) - \frac{y_2-x_1}{y_2-y_1}f(x_1-y_1) - \frac{x_1-y_1}{y_2-y_1}f(y_2-x_1). \end{aligned}$$

Similarly with $\lambda = \frac{y_2-x_2}{y_2-y_1}$ we get

$$f(x_2) \le \frac{y_2-x_2}{y_2-y_1}f(y_1) + \frac{x_2-y_1}{y_2-y_1}f(y_2) - \frac{y_2-x_2}{y_2-y_1}f(x_2-y_1) - \frac{x_2-y_1}{y_2-y_1}f(y_2-x_2).$$

The desired inequality now follows from summing two last inequalities. ☐

Corollary 3 ([12, Theorem 2.8])**.** *Let $f\colon [0,\infty) \to \mathbb{R}$ be a superquadratic function. Then*

$$f(a)+f(b) \le f(a+b) - 2\frac{bf(a)+af(b)}{a+b} \tag{2.16}$$

for all $a,b > 0$. In particular, if f is positive, then f is super-additive. If f is non-positive, then $-f$ is sub-additive.

Proof. Without loss of generality we may assume that $a \le b$. Considering $y_1 = 0$, $x_1 = a$, $x_2 = b$ and $y_2 = a+b$, Proposition 1 concludes the desired result. . ☐

Now we present our main result. It is an operator extension of (2.15). It also gives a generalization of the operator Mercer inequality for superquadratic functions, see [3].

Theorem 2. *Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function. Let A,B,C,D be positive operators on a Hilbert space $\mathcal{H}$ such that $A+D = B+C$ and $0 \le A \le mI \le B \le C \le MI \le D$ for some positive scalars m,M. If Φ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} &f(\Phi(B)) + f(\Phi(C)) + \beta(\Phi(B)) + \beta(\Phi(C)) \\ &\quad \le \Phi(f(A)) + \Phi(f(D)) - \Phi(f(m-A)) - \Phi(f(D-M)) \\ &\quad + \frac{\Phi(A-D)+M-m}{M-m} f(M-m) \end{aligned} \tag{2.17}$$

in which $\beta(t)$ is defined by (1.6).

Proof. As f is continuous, the function β is continuous too. Moreover, $\beta(t) = \beta(M+m-t)$ for every $t \in [0,\infty)$. Hence, we can apply the functional calculus to define $\beta(X)$ for every positive operator X.

If $0 \leq s \notin (m,M)$, then $s \in (0,m] \cup [M,\infty)$. First we assume that $s \in [M,\infty)$ and we put $\mu = \frac{M-m}{s-m} \in [0,1]$. Applying (1.4) we obtain

$$\begin{aligned} f(M) &= f(\mu s + (1-\mu)m) \\ &\leq \frac{M-m}{s-m} f(s) + \frac{s-M}{s-m} f(m) - \frac{M-m}{s-m} f(s-M) - \frac{s-M}{s-m} f(M-m) \end{aligned}$$

or equivalently,

$$f(s) - f(s-M) + \frac{M-s}{M-m} f(M-m) \geq \frac{M-s}{M-m} f(m) + \frac{s-m}{M-m} f(M). \tag{2.18}$$

If $s \in [0,m)$, then a similar argument yields

$$f(s) - f(m-s) + \frac{s-m}{M-m} f(M-m) \geq \frac{M-s}{M-m} f(m) + \frac{s-m}{M-m} f(M). \tag{2.19}$$

As $A \leq mI$ and $D \geq MI$, we can apply functional calculus to (2.18) and (2.19), respectively, with $s = D$ and $s = A$ to derive

$$f(D) - f(D-M) + \frac{M-D}{M-m} f(M-m) \geq \frac{M-D}{M-m} f(m) + \frac{D-m}{M-m} f(M)$$

and

$$f(A) - f(m-A) + \frac{A-m}{M-m} f(M-m) \geq \frac{M-A}{M-m} f(m) + \frac{A-m}{M-m} f(M).$$

Applying the positive linear map Φ to both sides of the last two inequalities we reach

$$\begin{aligned} \Phi(f(D)) - \Phi(f(D-M)) + \frac{M-\Phi(D)}{M-m} f(M-m) \\ \geq \frac{M-\Phi(D)}{M-m} f(m) + \frac{\Phi(D)-m}{M-m} f(M) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \Phi(f(A)) - \Phi(f(m-A)) + \frac{\Phi(A)-m}{M-m} f(M-m) \\ \geq \frac{M-\Phi(A)}{M-m} f(m) + \frac{\Phi(A)-m}{M-m} f(M). \end{aligned} \tag{2.21}$$

Next let $t \in [m,M]$ and put $\lambda = \frac{M-t}{M-m}$. It follows from (1.4) that

$$f(t) = f(\lambda m + (1-\lambda)M) \leq \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) - \beta(t), \tag{2.22}$$

where $\beta(t)$ is defined by (1.6). Since Φ is unital and positive, the spectra of operators $\Phi(B)$ and $\Phi(C)$ are contained in $[m,M]$. Accordingly, we can apply the continuous functional calculus to (2.22) with $t = \Phi(B)$ and $t = \Phi(C)$ to get

$$f(\Phi(B)) + \beta(\Phi(B)) \leq \frac{M-\Phi(B)}{M-m} f(m) + \frac{\Phi(B)-m}{M-m} f(M) \tag{2.23}$$

and

$$f(\Phi(C))+\beta(\Phi(C))\leq\frac{M-\Phi(C)}{M-m}f(m)+\frac{\Phi(C)-m}{M-m}f(M).\tag{2.24}$$

Summing (2.23) and (2.24) we get

$$\begin{aligned}&f(\Phi(B))+f(\Phi(C))+\beta(\Phi(B))+\beta(\Phi(C))\\&\leq\frac{2M-\Phi(B+C)}{M-m}f(m)+\frac{\Phi(B+C)-2m}{M-m}f(M)\\&=\frac{2M-\Phi(A+D)}{M-m}f(m)+\frac{\Phi(A+D)-2m}{M-m}f(M)\qquad(\text{by } A+D=B+C)\\&\leq\Phi(f(A))+\Phi(f(D))-\Phi(f(m-A))-\Phi(f(D-M))\\&\quad+\frac{\Phi(A-D)+M-m}{M-m}f(M-m)\end{aligned}$$

where the last inequality follows from summing (2.20) and (2.21). This completes the proof. ☐

The next corollary gives another variant of (2.17). We omit the proof as it is similar to the proof of Theorem 2.

Corollary 4. *With the hypotheses as in Theorem 2:*

$$\begin{aligned}&\Phi(f(B))+f(\Phi(C))+\Phi(\beta(B))+\beta(\Phi(C))\\&\leq\Phi(f(A))+f(\Phi(D))-\Phi(f(m-A))-f(\Phi(D)-M)\\&\quad+\frac{\Phi(A-D)+M-m}{M-m}f(M-m).\end{aligned}$$

As a consequence, the Jensen-Mercer operator inequality for superquadratic functions holds:

Corollary 5 ([3, Theorem 1])**.** *Let $f\colon[0,\infty)\to\mathbb{R}$ be a continuous superquadratic function and let $0<m\leq M$. If C is a positive operator, whose spectrum is contained in $[m,M]$, then*

$$f(M+m-\Phi(C))+\beta(\Phi(C))+f(0)\leq f(m)+f(M)-\Phi(\beta(C))-f(0).$$

Proof. Let C be a positive operator with spectrum in $[m,M]$. Apply Corollary 4 with $A=mI$, $B=(M+m)I-C$ and $D=MI$. ☐

As another consequence, we have the following Jensen operator inequality.

Corollary 6. *Let $f\colon[0,\infty)\to\mathbb{R}$ be a continuous superquadratic function and let $0<m\leq M$. Then*

$$f\left(\frac{A+D}{2}\right)+\beta\left(\frac{A+D}{2}\right)\leq\frac{f(A)+f(D)}{2}-\frac{f(m-A)+f(D-M)}{2}$$

for all positive operators A and D satisfying $A\leq mI\leq\frac{A+D}{2}\leq MI\leq D$.

Remark 1. If the superquadratic function f is positive, then f is convex and Corollary 6 provide an improvement of [14, Corollary 2.7]. For example, if $f(t) = t^p$ with $p \geq 2$, then

$$\left(\frac{A+D}{2}\right)^p + \beta\left(\frac{A+D}{2}\right) \leq \frac{A^p + D^p}{2} - \frac{(m-A)^p + (D-M)^p}{2} \tag{2.25}$$

holds for all positive operators A and D satisfying $A \leq mI \leq \frac{A+D}{2} \leq MI \leq D$. The existence of scalars m, M are necessary in Corollary 6. For example, it is known that the function $f(t) = t^3$ is not operator convex and so one can find positive operators A and D such that the operator

$$\frac{A^3 + D^3}{2} - \left(\frac{A+D}{2}\right)^3$$

is not positive. Accordingly, (2.25) does not hold in general, while the function $f(t) = t^3$ is superquadratic.

Moreover, if f is a non-positive superquadratic function, then Corollary 6 gives a reverse of [14, Corollary 2.7].

Remark 2. An operator version of (2.16) also follows from Theorem 1 as follows:

$$f(B) + f(C) + \beta(B) + \beta(C) \leq f(B+C) - f(B+C-M) \tag{2.26}$$

for all positive operators B, C satisfying $0 < B, C \leq M \leq B+C$ with $M > 0$. To see this apply (2.17) with $A = m = 0$ and $D = B+C$ and note that $f(0) \leq 0$ for every superquadratic function f. It is known that (see e.g. [8]) if $f\colon [0,\infty) \to [0,\infty)$ is an increasing convex function with $f(0) = 0$, then

$$\|f(B) + f(C)\| \leq \|f(B+C)\| \tag{2.27}$$

for all positive operators B and C and every unitarily invariant norm $\|\cdot\|$. We note that every positive superquadratic function f is convex and satisfies $f(0) = 0$. Hence, inequality (2.26) gives an stronger result than (2.27). However, the existence of positive scalar M with $B, C \leq M \leq B+C$ is necessary for (2.26). We give an example of such operators. Let $f(t) = t^3$ and put

$$B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

so that $B, C \leq 3I \leq B+C$. We calculate

$$f(B) = \begin{bmatrix} 14 & -13 \\ -13 & 14 \end{bmatrix} \quad \text{and} \quad f(C) = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix} \quad \text{and} \quad \beta(B) = 5/3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$\beta(C) = \frac{1}{3}\begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f(B+C) = \begin{bmatrix} 77 & -62 \\ -62 & 139 \end{bmatrix} \text{ and } f(B+C-M) = \begin{bmatrix} 5 & -8 \\ -8 & 13 \end{bmatrix}.$$

Accordingly, we have

$$\begin{aligned} f(B)+f(C)+\beta(B)+\beta(C) &\simeq \begin{bmatrix} 27 & -11.33 \\ -11.33 & 42.67 \end{bmatrix} \leq \begin{bmatrix} 72 & -54 \\ -54 & 126 \end{bmatrix} \\ &= f(B+C)-f(B+C-M). \end{aligned}$$

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Authors' addresses

Mohsen Kian
(**Corresponding author**) University of Bojnord, Department of Mathematics, P. O. Box 1339, Bojnord 94531, Iran
E-mail address: `kian@ub.ac.ir`

Zainab Peymani
University of Bojnord, Department of Mathematics, P. O. Box 1339, Bojnord 94531, Iran
E-mail address: `zpeymani168@gmail.com`