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HOMOLOGICAL CHARACTERIZATIONS OF G-KRULL DOMAINS AND G-DEDEKIND DOMAINS

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Abstract. Gorenstein Krull domains (G-Krull domains) are defined as domains R that satisfy the following three conditions: (1) For each prime ideal $\mathfrak{p}$ of R of height one, $R_{\mathfrak{p}}$ is a Gorenstein ring. (2) $R = \bigcap R_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all prime ideals of R of height one. (3) Any nonzero element of R lies in only a finite number of prime ideals of height one. In this paper, we aim to characterize G-Krull domains from the perspective of Gorenstein homological algebra, similar to Gorenstein Dedekind domains (G-Dedekind domains). To achieve this objective, we introduce the notion of w-locally Gorenstein projective modules (G-projective modules). An R-module M is called w-locally Gorenstein projective if $M_{\mathfrak{m}}$ is G-projective for any maximal w-ideal $\mathfrak{m}$ of R. We show that a domain R is G-Krull if and only if R is a strong Mori domain and every w-ideal of R is w-locally G-projective. Additionally, we establish that a domain R is G-Dedekind if and only if R is a Noetherian domain and every maximal ideal of R is G-projective.

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1. INTRODUCTION

All rings considered in this paper are commutative with identity. In [7], Qiao and Wang introduced Gorenstein Krull domains (for short, G-Krull domains). A domain R is called a *G-Krull domain* if R satisfies the following three conditions: (1) For each prime ideal $\mathfrak{p}$ of R of height one, $R_{\mathfrak{p}}$ is a Gorenstein ring. (2) $R = \bigcap R_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all prime ideals of R of height one. (3) Any nonzero element of R lies in only a finite number of prime ideals of height one ([7, Page 48, Definition]). R is said to be *Gorenstein* if $\mathrm{id}_{R_{\mathfrak{m}}} R_{\mathfrak{m}} < \infty$ (the injective dimension of $R_{\mathfrak{m}}$ as an $R_{\mathfrak{m}}$-module)

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for any maximal ideal $\mathfrak{m}$ of R (refer to [9, Definition 4.6.12]). They characterized G-Krull domains in the context of various conditions, including the following: a domain R is G-Krull if and only if R is an SM domain, and any nonzero w-ideal of R is a v-ideal. Recently, Xing ([12]) characterized G-Krull domains in terms of w-factor rings: an integral domain is G-Krull if and only if, for any nonzero nonunit, the Gorenstein global dimension of its w-factor ring is zero. SM domains are the analog of Noetherian domains. A domain R is called *strong Mori* (for short, SM) if R satisfies the ascending chain condition on w-ideals of R ([10, Definition 4]). Note that a domain R is a G-Dedekind domain if and only if R is a Noetherian domain and any nonzero ideal of R is a v-ideal ([1, Proposition 1.5]). Thus, G-Krull domains can be viewed as the w-version of G-Dedekind domains. G-Dedekind domains were introduced by Mahdou and Tamekkante in [6]. Recall that a domain R is called *G-Dedekind* if every submodule of a projective R-module is G-projective ([6, Definition 2.1(1)]). An R-module M is said to be *Gorenstein projective* (for short, G-projective) if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \dots$ of projective R-modules with $M = \mathrm{Ker}(P^0 \to P^0)$ such that $\mathrm{Hom}_R(-,Q)$ leaves the sequence exact whenever Q is a projective R-module.

In this paper, we aim to characterize G-Krull domains from the Gorenstein homological algebra point of view, similar to G-Dedekind domains. To attain this objective, for convenience, we introduce the notion of w-locally G-projective modules. An R-module M is called *w-locally G-projective* if $M_{\mathfrak{m}}$ is G-projective for any maximal w-ideal $\mathfrak{m}$ of R. Then we obtain that a domain R is G-Krull if and only if R is an SM domain and every w-ideal of R is w-locally G-projective; if and only if R is an SM domain and every prime (or maximal) w-ideal of R is w-locally G-projective.

Now recall some definitions and notations from [3] and [9]. Let R be an integral domain with quotient field K and let $F(R)$ denote the set of all fractional ideals of R. A *star operation* on R is a mapping $*\colon F(R) \to F(R)$ satisfying: for any $A,B \in F(R)$ and $0 \neq a \in K$, we have

(1) $(aR)_* = aR$ and $(aA)_* = aA_*$;
(2) If $A \subseteq B$, then $A_* \subseteq B_*$;
(3) $A \subseteq A_*$ and $(A_*)_* = A_*$.

For any fractional ideal A of R, A is called a *fractional $*$-ideal* if $A_* = A$, and A is called a *$*$-ideal* if A is an ideal of R and $A_* = A$. For the introduction of star operations, one may consult [3, Section 32 and 34] or [9, Section 7.2]. Examples of classical star operations include the v-operation, where $A_v = (A^{-1})^{-1}$ and $A^{-1} = \{x \in K \mid xA \subseteq R\}$, the t-operation, where

$$A_t = \bigcup\{B_v \mid B \text{ is taken over all finitely generated fractional subideals of } A\},$$

and the w-operation, where

$$A_w = \{x \in A \otimes K \mid xJ \subseteq A \text{ for some finitely generated ideal } J \text{ of } R \text{ with } J^{-1} = R\}.$$

A nonzero ideal $\mathfrak{p}$ of R is said to be *a prime w-ideal* if $\mathfrak{p}$ is both a prime ideal and a w-ideal, denoted by $\mathfrak{p} \in w\text{-Spec}(R)$; and *a maximal w-ideal* if $\mathfrak{p}$ is maximal in the set of all proper w-ideals of R, denoted by $\mathfrak{p} \in w\text{-Max}(R)$. Each maximal w-ideal is prime. The *w-Krull dimension or w-dimension* of R, denoted by $w\text{-dim}(R)$, is the supremum of the heights of all maximal w-ideals of R.

Any unexplained terminology is standard as in [9].

2. Main results

To characterize G-Krull domains in terms of w-locally G-projective modules, we begin with the following lemmas.

Lemma 1 ([8], Proposition 1.8)**.** *Let S be a multiplicative subset of R, M be an R-module, and N be an R_S-module. Then the natural R_S-homomorphism*

$$\theta\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_{R_S}(M_S,N)$$

is an isomorphism.

Lemma 2. *Let S be a multiplicative subset of R, M be an R-module, and N be an R_S-module. Then the natural R_S-homomorphism*

$$\theta\colon \operatorname{Ext}^1_R(M,N) \to \operatorname{Ext}_{R_S}(M_S,N)$$

is an isomorphism.

Proof. This is followed by setting $T := R_S$, $A := M$, $M := T$, and $X := N$ in [9, Theorem 3.3.11]. □

Lemma 3 ([9], Theorem 7.4.13)**.** *A domain R is an SM domain if and only if $R_\mathfrak{m}$ is a Noetherian domain for any maximal w-ideal $\mathfrak{m}$ of R, and each nonzero element of R lies in only finitely many maximal w-ideals of R.*

Theorem 1. *The following statements are equivalent for a domain R.*

(1) *R is an SM domain and $\operatorname{id}_R R_\mathfrak{m} \le 1$ for any maximal w-ideal $\mathfrak{m}$ of R.*
(2) *R is an SM domain and $\operatorname{id}_{R_\mathfrak{m}} R_\mathfrak{m} \le 1$ for any maximal w-ideal $\mathfrak{m}$ of R.*
(3) *R is an SM domain and every w-ideal of R is w-locally G-projective.*
(4) *R is an SM domain and every prime w-ideal of R is w-locally G-projective.*
(5) *$R_\mathfrak{m}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R, and each nonzero element of R lies in only finitely many maximal w-ideals of R.*
(6) *R is a G-Krull domain.*

Proof. (1) ⇒ (2) Let $\mathfrak{m}$ be a maximal w-ideal of R. Then $0 \to R_\mathfrak{m} \to K \to K/R_\mathfrak{m} \to 0$ is an exact sequence of both $R_\mathfrak{m}$-modules and R-modules. Since $\operatorname{id}_R R_\mathfrak{m} \le 1$, we can get that $K/R_\mathfrak{m}$ is an injective R-module. Let $I_\mathfrak{m}$ be an ideal of $R_\mathfrak{m}$, where I is an ideal of R. Then $\operatorname{Ext}^1_{R_\mathfrak{m}}(R_\mathfrak{m}/I_\mathfrak{m}, K/R_\mathfrak{m}) \cong \operatorname{Ext}^1_R(R/I, K/R_\mathfrak{m}) = 0$ by Lemma 2, which implies that $K/R_\mathfrak{m}$ is an injective $R_\mathfrak{m}$-module. Thus $\operatorname{id}_{R_\mathfrak{m}} R_\mathfrak{m} \le 1$.

(2) $\Rightarrow$ (1) Let $\mathfrak{m}$ be a maximal w-ideal of R. If $\mathrm{id}_{R_\mathfrak{m}} R_\mathfrak{m} \leq 1$, then $K/R_\mathfrak{m}$ is an injective $R_\mathfrak{m}$-module. Thus, $K/R_\mathfrak{m}$ is an injective R-module by [9, Exercise 3.16]. So, $\mathrm{id}_R R_\mathfrak{m} \leq 1$.

(2) $\Leftrightarrow$ (5) It follows by [9, Theorem 11.7.7] and Lemma 3.

(3) $\Leftrightarrow$ (5) Note that $I_\mathfrak{m} = (I_w)_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. Then it follows by [9, Theorem 11.7.7] and Lemma 3.

(3) $\Rightarrow$ (4) It is trivial.

(4) $\Rightarrow$ (3) Let

$$\Gamma = \{I \mid I \text{ is a } w\text{-ideal of } R \text{ and } I \text{ is not a } w\text{-locally G-projective } R\text{-module}\}.$$

If Γ is not empty, then Γ has a maximal element by [9, Theorem 6.8.5]. Let $\mathfrak{p}$ be a maximal element of Γ. If $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p}$ is a w-locally G-projective R-module by (4), a contradiction with $\mathfrak{p} \in \Gamma$. Therefore, Γ is empty, i.e., every w-ideal of R is w-locally G-projective.

Next, we show that $\mathfrak{p}$ is a prime ideal of R. If not, then there exist some $a, b \in R \setminus \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Then $\mathfrak{p} \subsetneq (\mathfrak{p} :_R Ra)$ and $\mathfrak{p} \subsetneq (\mathfrak{p} + Ra)_w$. Note that $(\mathfrak{p} :_R Ra)$ is a w-ideal of R. Then both $(\mathfrak{p} :_R Ra)$ and $(\mathfrak{p} + Ra)_w$ are w-locally G-projective R-modules. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathfrak{p} :_R Ra) & = & (\mathfrak{p} :_R Ra) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & L & \longrightarrow & R & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & (\mathfrak{p} + Ra)_w & \longrightarrow & (\mathfrak{p} + Ra)_w/\mathfrak{p} & \longrightarrow & 0,
\end{array}$$

where $\varphi(r) = ra + \mathfrak{p}$ for any $r \in R$. Let $\mathfrak{m}$ be a maximal w-ideal. Then $(\mathfrak{p} :_R Ra)_\mathfrak{m} = (\mathfrak{p}_\mathfrak{m} :_{R_\mathfrak{m}} aR_\mathfrak{m})$ and $((\mathfrak{p} + Ra)_w/\mathfrak{p})_\mathfrak{m} = (\mathfrak{p}_\mathfrak{m} + aR_\mathfrak{m})/\mathfrak{p}_\mathfrak{m}$. Note that

$$0 \longrightarrow (\mathfrak{p}_\mathfrak{m} :_{R_\mathfrak{m}} aR_\mathfrak{m}) \longrightarrow R_\mathfrak{m} \xrightarrow{\varphi_\mathfrak{m}} \frac{\mathfrak{p}_\mathfrak{m} + aR_\mathfrak{m}}{\mathfrak{p}_\mathfrak{m}} \longrightarrow 0$$

is an exact sequence. Then $0 \to (\mathfrak{p} :_R Ra)_\mathfrak{m} \to L_\mathfrak{m} \to (\mathfrak{p} + Ra)_\mathfrak{m} \to 0$ is an exact sequence. Since $(\mathfrak{p} :_R Ra)_\mathfrak{m}$ and $(\mathfrak{p} + Ra)_\mathfrak{m}$ are G-projective $R_\mathfrak{m}$-modules, we can conclude that $L_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module. Note that $0 \to \mathfrak{p}_\mathfrak{m} \to L_\mathfrak{m} \to R_\mathfrak{m} \to 0$ is a split exact sequence of $R_\mathfrak{m}$-modules. Therefore, $\mathfrak{p}_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module. Thus, $\mathfrak{p}$ is a w-locally G-projective R-module, a contradiction with $\mathfrak{p} \in \Gamma$. Hence, $\mathfrak{p}$ is a prime ideal of R.

(5) $\Rightarrow$ (6) Since $R_{\mathfrak{m}}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R, we can conclude that $\dim(R_{\mathfrak{m}}) \leq 1$ by [9, Corollary 11.7.8]. Thus, w-$\dim(R) \leq 1$. Therefore, w-Max(R) is precisely the set of all prime ideals of R of height one. Hence, $R = \bigcap_{\mathfrak{m}\in w\text{-Max}(R)} R_{\mathfrak{m}}$. Therefore, R is a G-Krull domain by (5).

(6) $\Rightarrow$ (5) If R is a G-Krull domain, then w-$\dim(R) = 1$ by [7, Corollary 4.5]. Thus, $X^1(R) = w$-Max(R), where $X^1(R)$ denotes the set of height-one prime ideals of R. Then (5) holds by the definition of G-Krull domains. □

It is well known that a G-Dedekind domain is of finite character, i.e., each nonzero element of a G-Dedekind domain R lies in only finitely many maximal ideals of R. Next, we show that a domain R satisfying that $R_{\mathfrak{m}}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R is not necessarily of w-finite character, i.e., some nonzero element of such a domain R may be contained in infinitely many maximal w-ideals of R.

In [4], Kang called a domain R *t-almost Dedekind* if $R_{\mathfrak{m}}$ is a discrete valuation ring for each maximal t-ideal $\mathfrak{m}$ of R [4, Page 166]. Note that maximal t-ideals and maximal w-ideals coincide [9, Theorem 7.3.4]. Then a domain R is a t-almost Dedekind domain if and only if $R_{\mathfrak{m}}$ is a discrete valuation ring for each maximal w-ideal $\mathfrak{m}$. Note that a domain R is a Krull domain if and only if R is a t-almost Dedekind domain and R is an SM domain by [9, Theorem 7.9.3].

Example 1. Let R be a non-Krull t-almost Dedekind domain. Such a detailed example can be referenced in [4, Page 167, Remark]. Then, for any maximal w-ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is a discrete valuation ring, thus a G-Dedekind domain. However, R does not satisfy that each nonzero element of R lies in only finitely many maximal w-ideals of R. If not, we would get that R is an SM domain by Lemma 3. Thus, R would be a Krull domain, a contradiction.

For many cases, if every prime ideal of R satisfies the property, then every ideal of R satisfies such a property. For example, if every prime ideal of R is finitely generated, then every ideal of R is finitely generated, i.e., R is Noetherian. This is the well-known Cohen theorem. However, if every maximal ideal of R satisfies some property, it is not necessary that every ideal of R does. For example, let $R = \mathbb{Z} + \mathbb{Q}[X]$, where $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Q}$ denotes the rational number field. Then the maximal ideals of R are those of the form pR, where p is a prime element of $\mathbb{Z}$, and the principal ideals $f(X)R$, where $f(X)$ is irreducible in $\mathbb{Q}[X]$ and $f(0) = 1$ ([2, Theorem 4.21]). In this case, every maximal ideal of R is finitely generated, but R is not Noetherian.

It is noteworthy that if a domain R is Noetherian and every maximal ideal of R is G-projective, then every ideal of R is G-projective, i.e., R is G-Dedekind (Corollary 1). Based on the above result, we also can conclude that a domain R is G-Krull if and only if R is an SM domain and every maximal w-ideal of R is w-locally G-projective (Theorem 3).

Lemma 4 ([5], Theorem 211)**.** *Let $(R,\mathfrak{m})$ be a local Noetherian domain, $\mathfrak{p}$ a prime ideal of R with $\mathfrak{p} \subsetneq \mathfrak{m}$, and M a finitely generated R-module. If $\mathrm{Ext}_R^{i+1}(R/Q,M)=0$ for any prime ideal Q properly containing $\mathfrak{p}$, then $\mathrm{Ext}_R^{i}(R/\mathfrak{p},M)=0$.*

Theorem 2. *Let $(R,\mathfrak{m})$ be a local Noetherian domain. If $\mathfrak{m}$ is a G-projective R-module, then R is a G-Dedekind domain.*

Proof. Since $\mathfrak{m}$ is a G-projective R-module, we have $\mathrm{Ext}_R^{i\geq 1}(\mathfrak{m},R)=0$ by [9, Corollary 11.1.3]. Then $\mathrm{Ext}_R^{i\geq 2}(R/\mathfrak{m},R)=0$. By the Generalized Principal Ideal Theorem of Noetherian rings ([9, Theorem 4.3.12]), the height of $\mathfrak{m}$ is finite. Assume that $\mathrm{ht}(\mathfrak{m})=t$. For any prime ideal $\mathfrak{p}$ of R with $\mathrm{ht}(\mathfrak{p})=t-1$, $\mathrm{Ext}_R^{i\geq 2}(R/\mathfrak{p},R)=0$ by Lemma 4. Then for any prime ideal $\mathfrak{p}$ of R with $\mathrm{ht}(\mathfrak{p})=t-2$, $\mathrm{Ext}_R^{i\geq 2}(R/\mathfrak{p},R)=0$, again by Lemma 4. Continuing this process, we get that $\mathrm{Ext}_R^{i\geq 2}(R/\mathfrak{p},R)=0$ for any prime ideal $\mathfrak{p}$ of R. Let M be a finitely generated R-module. Then there exists an ascending chain of submodules of M

$$0=M_0\subseteq M_1\subseteq M_2\subseteq\cdots\subseteq M_{n-1}\subseteq M_n=M$$

such that $M_{i+1}/M_i\cong R/\mathfrak{p}_{i+1}$ for some prime ideal $\mathfrak{p}_{i+1}$ of R, $i=1,2,\ldots,n-1$. For the exact sequence $0\to R/\mathfrak{p}_1\to M_2\to R/\mathfrak{p}_2\to 0$, we get the exact sequence

$$\mathrm{Ext}_R^2(R/\mathfrak{p}_2,R)\to\mathrm{Ext}_R^2(M_2,R)\to\mathrm{Ext}_R^2(R/\mathfrak{p}_1,R).$$

Then $\mathrm{Ext}_R^2(M_2,R)=0$. By the same method, we get that $\mathrm{Ext}_R^2(M,R)=0$. Thus, $\mathrm{id}_R(R)\leq 1$. Therefore, $\mathrm{Ext}_R^1(I,R)=0$. Thus, R is a G-Dedekind domain by [9, Theorem 11.7.7]. □

Corollary 1. *If R is a Noetherian domain and every maximal ideal of R is G-projective, then R is G-Dedekind.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Then $(R_\mathfrak{m},\mathfrak{m}R_\mathfrak{m})$ is a local Noetherian domain. Since $\mathfrak{m}$ is G-projective over R and $\mathfrak{m}$ is super finitely presented (i.e., there exists an exact sequence $\cdots\to P_n\to\cdots\to P_1\to P_0\to\mathfrak{m}\to 0$, where each P_i is a finitely generated projective R-module), we get that $\mathfrak{m}R_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module by [9, Theorem 11.6.17]. Then $R_\mathfrak{m}$ is G-Dedekind by Theorem 2. Hence, the Krull dimension of $R_\mathfrak{m}$ is less than or equal to 1. The same holds for the Krull dimension of R, which implies that every nonzero prime ideal of R is maximal. Thus, R is G-Dedekind by [9, Theorem 11.7.7]. □

Theorem 3. *The following statements are equivalent for a domain R.*

(1) *R is a G-Krull domain.*
(2) *R is an SM domain and every maximal w-ideal of R is w-locally G-projective.*

Proof. (1) ⇒ (2) It follows by Theorem 1.

(2) ⇒ (1) Let $\mathfrak{m}$ be a maximal w-ideal of R. If R is an SM domain, then $R_\mathfrak{m}$ is a Noetherian domain and each nonzero element of R lies in only finitely many maximal

w-ideals of R by Lemma 3. By (2), $\mathfrak{m}R_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module. Then $R_\mathfrak{m}$ is a G-Dedekind domain by Theorem 2. Thus, R is a G-Krull domain by Theorem 1. □

In Theorem 1, it is shown that if R is an SM domain and $\mathrm{id}_R R_\mathfrak{m} \leq 1$ for any maximal w-ideal $\mathfrak{m}$ of R, then R is a G-Krull domain. Next, we show that if R is an SM domain and $\mathrm{id}_R R \leq 1$, then R is just G-Dedekind (Theorem 4).

First, we recall the notion of w-modules. The w-operation on domains was introduced by Wang and McCasland in [10], and then considered for commutative rings by Yin et al. in [13]. Let J be a finitely generated ideal of R. If the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J,R)$ is an isomorphism, then J is called a GV-*ideal*, denoted by $J \in \mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus, $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. And M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). Clearly, R is a GV-torsion-free R-module ([13, Corollary 1.5]). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J,M) = 0$ for any $J \in \mathrm{GV}(R)$ ([13, Definition 2.2]). The w-envelope of a GV-torsion-free module M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. And M is a w-module if and only if $M_w = M$. A w-module M is *of finite type* if $M = N_w$ for some finitely generated submodule N of M ([13, p. 216]). Recall that a GV-torsion-free R-module M is said to be a *strong w-module* if $\mathrm{Ext}^i_R(N,M) = 0$ for each integer $i \geq 1$ and for any GV-torsion R-module N ([11, p. 1918, Definition]).

Proposition 1. *If $\mathrm{id}_R R \leq 1$, then R is a strong w-module.*

Proof. Since R is a w-module, we have that $\mathrm{Ext}^1_R(N,R) = 0$ for any GV-torsion module N by [9, Theorem 6.2.7]. Given that $\mathrm{id}_R R \leq 1$, we also have $\mathrm{Ext}^i_R(N,R) = 0$ for any GV-torsion module N and for any integer $i \geq 2$. Thus, R is a strong w-module. □

Proposition 2. *Let R be a domain. If $\mathrm{id}_R R \leq 1$, then R is a DW domain; that is, every ideal of R is a w-ideal.*

Proof. Let I be a nonzero finitely generated ideal of R. Then there exists an exact sequence

$$0 \to K \to F \to I \to 0,$$

where F is a finitely generated free R-module. Since $\mathrm{id}_R R \leq 1$, we have

$$\mathrm{Ext}^1_R(I,R) \cong \mathrm{Ext}^2_R(R/I,R) = 0.$$

Thus, we obtain the exact sequence

$$0 \to I^* \to F^* \to K^* \to 0,$$

where $M^* = \operatorname{Hom}_R(M,R)$ for any R-module M. Hence, K^* is a finitely generated torsion-free R-module. Therefore, there exists an exact sequence

$$0 \to K^* \to F_1 \to C \to 0,$$

with F_1 a free R-module. Consequently,

$$\operatorname{Ext}_R^1(K^*,R) \cong \operatorname{Ext}_R^2(C,R) = 0,$$

which implies that the sequence

$$0 \to K^{**} \to F^{**} \to I^{**} \to 0$$

is exact. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & F^{**} & \longrightarrow & I^{**} & \longrightarrow & 0 \end{array}$$

By [9, Proposition 2.1.29(1)], both ρ_1 and ρ_3 are monomorphisms, and ρ_2 is an isomorphism by [9, Theorem 2.3.7(3)]. Then, by the Five Lemma ([9, Theorem 1.9.9]), ρ_1 is also an epimorphism. Hence, ρ_1 is an isomorphism, and it follows that I is a w-ideal of R.

Now let A be a nonzero ideal of R. Then

$$A_w = \bigcup_i B_w,$$

where B ranges over the set of finitely generated subideals of A. Since each B is a w-ideal, we have $B_w = B$, and thus $A_w = A$. Therefore, every ideal of R is a w-ideal, and R is a DW domain. □

Theorem 4. *The following statements are equivalent for a domain R with quotient field K.*

(1) *R is an SM domain and* $\operatorname{id}_R R \leq 1$.
(2) *R is a Noetherian domain and* $\operatorname{id}_R R \leq 1$.
(3) *R is an SM domain and* $\operatorname{Ext}_R^1(M,R) = 0$ *for every submodule M of a free module.*
(4) *R is an SM domain, K/R is a w-module, and* $\operatorname{Ext}_R^1(M,R) = 0$ *for every finite type submodule M of a free module.*
(5) *R is an SM domain, K/R is a w-module, and* $\operatorname{Ext}_R^1(I,R) = 0$ *for every w-ideal I of R.*
(6) *R is an SM domain and* $\operatorname{Ext}_R^1(I,R) = 0$ *for every ideal I of R.*
(7) *R is an SM domain and every nonzero ideal I of R is a v-ideal.*
(8) *R is a G-Dedekind domain.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 2.

(2) $\Leftrightarrow$ (8) $\Rightarrow$ (3), and (2) $\Rightarrow$ (7), follow from [9, Theorem 11.7.7].

(3) $\Rightarrow$ (4) It suffices to show that K/R is a w-module. Let J be a GV-ideal of R. Then the exact sequence

$$0 \to J \to R \to R/J \to 0$$

yields $\mathrm{Ext}^1_R(J,R) \cong \mathrm{Ext}^2_R(R/J,R)$. By (3), we obtain $\mathrm{Ext}^1_R(J,R) = 0$, so $\mathrm{Ext}^2_R(R/J,R) = 0$. Now, from the exact sequence

$$0 \to R \to K \to K/R \to 0,$$

we have the long exact sequence

$$\mathrm{Ext}^1_R(R/J,K) \to \mathrm{Ext}^1_R(R/J,K/R) \to \mathrm{Ext}^2_R(R/J,R).$$

Since the last term is zero, it follows that $\mathrm{Ext}^1_R(R/J,K/R) = 0$. Hence, K/R is a w-module.

(4) $\Rightarrow$ (5) This is clear.

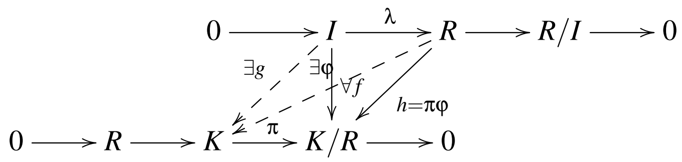
(5) $\Rightarrow$ (1) To show $\mathrm{id}_R R \leq 1$, it suffices to prove that K/R is injective over R, since we have the exact sequence

$$0 \to R \to K \to K/R \to 0$$

and K is injective over R. By (5), for any w-ideal I of R, the sequence

$$0 \to \mathrm{Hom}_R(I,R) \to \mathrm{Hom}_R(I,K) \to \mathrm{Hom}_R(I,K/R) \to 0$$

is exact. Consider the following commutative diagram:



For any $f \in \mathrm{Hom}_R(I,K/R)$, there exists $g \in \mathrm{Hom}_R(I,K)$ such that $f = \pi g$. Since K is injective, there exists $\varphi \in \mathrm{Hom}_R(R,K)$ with $g = \varphi\lambda$. Then $h = \pi\varphi \in \mathrm{Hom}_R(R,K/R)$ satisfies $h\lambda = f$. Hence, $\mathrm{Ext}^1_R(R/I,K/R) = 0$. Since K/R is a w-module, it is injective over R by [9, Theorem 6.8.26], so $\mathrm{id}_R R \leq 1$.

(1) $\Leftrightarrow$ (6) follows from the isomorphism $\mathrm{Ext}^1_R(I,R) \cong \mathrm{Ext}^2_R(R/I,R)$ for every ideal I of R.

(7) $\Rightarrow$ (8) If R is an SM domain and every nonzero ideal I of R is a v-ideal, then each such ideal is also a w-ideal. Thus, R is a DW domain, and hence Noetherian. Therefore, R is a G-Dedekind domain by [9, Theorem 11.7.7]. □

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