Miskolc Mathematical Notes
Vol. 27 (2026), No. 1, pp. 35–57

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2026.5240

SYMMETRIC AND GENERATING FUNCTIONS FOR CERTAIN PRODUCTS OF NUMBERS AND POLYNOMIALS WITH APPLICATION TO BIFURCATION ANALYSIS

CHEMS EDDINE ARROUD, NABIHA SABA, ALI BOUSSAYOUD,
SALAH BOULAARAS, BAGHDADI ALOUI, AND NESRINE HARROUCHE

Received 19 December, 2024

Abstract. In this paper, we establish a novel theorem pertaining to symmetric and generating functions. Utilizing this theorem, we derive new generating functions for products of k-Fibonacci numbers and Fibonacci polynomials with certain (p,q)-numbers, such as (p,q)-Fibonacci and (p,q)-Jacobsthal-Lucas numbers. Furthermore, we examine the bifurcation and chaotic behavior of the generating functions associated with (p,q)-Jacobsthal numbers for specific parameter values of p and q.

2010 *Mathematics Subject Classification:* 05E05; 11B39

Keywords: symmetric functions, generating functions, (p,q)-numbers, Fibonacci polynomials, bifurcation, mathematical model

1. Introduction and preliminaries

The Fibonacci sequence, denoted by $(F_n)_{n\geqslant 0}$, is one of the most well-known and intriguing numerical sequences due to its numerous properties and connections to various fields [15]. It is defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ for every $n \geqslant 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Many generalizations of this sequence have been proposed, some by modifying the initial conditions and others by preserving the recurrence relation.

The k-Fibonacci sequences, denoted by $\{F_{k,n}\}$, are defined recurrently for any positive real number k by the relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$ [7, 10]. Using this recurrence, it is possible to calculate the k-Fibonacci sequence both forward and backward. Substituting n with $-n$ yields the relation $F_{k,-n} = F_{k,-(n-1)} + F_{k,-(n-2)}$. The characteristic equation $x^2 - kx - 1 = 0$ of this sequence has two roots, denoted by α and β. Consequently, the Binet formula leads to

$$F_{k,-n} = (-1)^{n+1} F_{k,n}, \quad \text{for all } n \geq 0, \quad (\text{see [19]}), \tag{1.1}$$

demonstrating a close relationship between the positive and negative indices of the Fibonacci numbers.

On the other hand, the generalized Fibonacci sequence $\{F_{p,q,n}\}_{n\in\mathbb{N}}$, referred to as the (p,q)-Fibonacci sequence, is defined in [25] by

$$F_{p,q,0}=0, \quad F_{p,q,1}=1, \quad \text{and} \quad F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2}, \quad \text{for } n\geq 2.$$

Each term of the (p,q)-Fibonacci sequence is called a (p,q)-Fibonacci number.

It is well known that the (p,q)-Lucas, (p,q)-Pell, (p,q)-Pell-Lucas, (p,q)-Jacobsthal, and (p,q)-Jacobsthal-Lucas numbers, denoted respectively by $\{L_{p,q,n}\}_{n\in\mathbb{N}}$, $\{P_{p,q,n}\}_{n\in\mathbb{N}}$, $\{Q_{p,q,n}\}_{n\in\mathbb{N}}$, $\{J_{p,q,n}\}_{n\in\mathbb{N}}$, and $\{j_{p,q,n}\}_{n\in\mathbb{N}}$, are defined by the following recurrence relations for any positive real numbers p and q (see [5, 14, 22, 24, 26]):

$$\begin{aligned}
&L_{p,q,0}=2, && L_{p,q,1}=p, && \text{and} && L_{p,q,n}=pL_{p,q,n-1}+qL_{p,q,n-2}, && \text{for } n\geq 2,\\
&P_{p,q,0}=0, && P_{p,q,1}=1, && \text{and} && P_{p,q,n}=2pP_{p,q,n-1}+qP_{p,q,n-2}, && \text{for } n\geq 2,\\
&Q_{p,q,0}=2, && Q_{p,q,1}=2p, && \text{and} && Q_{p,q,n}=2pQ_{p,q,n-1}+qQ_{p,q,n-2}, && \text{for } n\geq 2,\\
&J_{p,q,0}=0, && J_{p,q,1}=1, && \text{and} && J_{p,q,n}=pJ_{p,q,n-1}+2qJ_{p,q,n-2}, && \text{for } n\geq 2,\\
&j_{p,q,0}=2, && j_{p,q,1}=p, && \text{and} && j_{p,q,n}=pj_{p,q,n-1}+2qj_{p,q,n-2}, && \text{for } n\geq 2.
\end{aligned}$$

The generating function for the (p,q)-Jacobsthal numbers is given by

$$J_{p,q}(z)=\frac{z}{1-pz-2qz^2}. \tag{1.2}$$

Large classes of polynomials can be defined by Fibonacci-like recurrence relations. One such class, known as Fibonacci polynomials, was studied in 1883 by Catalan and Jacobsthal. These polynomials, denoted by $F_n(x)$, are defined by the recurrence relation

$$\begin{cases} F_n(x)=xF_{n-1}(x)+F_{n-2}(x), & \text{for } n\geq 2,\\ F_0(x)=1, \quad F_1(x)=x. \end{cases}$$

For more details about this sequence, the reader is referred to [8].

In their seminal work [4], Berry, Lewis, and Nye pioneered the study of fractal structures by establishing the self-similarity of the Weierstrass–Mandelbrot function. Building upon this foundation, Benbernou et al. [3] derived regularity conditions for the three-dimensional magnetohydrodynamic equations, while Guariglia and Silvestrov [13] extended wavelet theory through the introduction of fractional wavelets. Subsequent research by Guariglia [11] further demonstrated the significance of fractality in primality theory and image analysis. More recent advancements include the development of Chebyshev wavelet techniques by Guariglia and Guido [12] and the establishment of novel integral inequalities for generalized convex functions by Akdemir et al. [2]. Etemad et al. [9] contributed to this evolving landscape by proving existence results for solutions to multi-order q-difference fractional boundary value problems. Collectively, these studies underscore the growing interplay between fractals, wavelet theory, and analytic inequalities, highlighting their unifying role across pure and applied mathematics.

Now, we present certain essential information and outcomes concerning symmetric functions.

Definition 1 ([1, Definition 2.1])**.** Consider A and B as two alphabets. We define $S_n(A-B)$ as

$$\frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-B)z^n,$$

with the condition $S_n(A-B)=0$ for $n<0$.

Definition 2 ([23, Definition 1.5])**.** Let k be a positive integer and $A=\{a_1,a_2\}$ be a set of given variables. The k^{th} symmetric function $S_k(A)=S_k(a_1+a_2)$ is defined by

$$S_k(A)=S_k(a_1+a_2)=\frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1+a_2)=1,\\ S_1(A) &= S_1(a_1+a_2)=a_1+a_2,\\ S_2(A) &= S_2(a_1+a_2)=a_1^2+a_1a_2+a_2^2,\\ &\vdots \end{aligned}$$

Now, we give some definitions from [17] that will be useful in the sequel.

Definition 3. Assume that $\{a_1,a_2,\ldots,a_n\}$ is an alphabet and k and n are two positive integers. The k^{th} elementary symmetric function, denoted as $e_k(a_1,a_2,\ldots,a_n)$, is the sum of the products of k distinct elements selected from the set $\{a_1,a_2,\ldots,a_n\}$, i.e.,

$$e_k^{(n)}=e_k(a_1,a_2,\ldots,a_n)=\sum_{i_1+i_2+\ldots+i_n=k} a_1^{i_1}a_2^{i_2}\ldots a_n^{i_n} \quad (0\le k\le n),$$

where $i_1,i_2,\ldots,i_n\in\{0,1\}$.

Remark 1. By convention, $e_0(a_1,a_2,\ldots,a_n)=1$, and we set $e_k(a_1,a_2,\ldots,a_n)=0$ for $k<0$ or $k>n$.

Definition 4. Assume that $\{a_1,a_2,\ldots,a_n\}$ is an alphabet and k and n are two positive integers. The k^{th} complete homogeneous symmetric function, denoted as $h_k(a_1,a_2,\ldots,a_n)$, is given by

$$h_k^{(n)}=h_k(a_1,a_2,\ldots,a_n)=\sum_{i_1+i_2+\ldots+i_n=k} a_1^{i_1}a_2^{i_2}\ldots a_n^{i_n} \quad (k\ge 0), \tag{1.3}$$

where $i_1,i_2,\ldots,i_n\ge 0$.

Remark 2. By convention, $h_0(a_1,a_2,\ldots,a_n)=1$, and we set $h_k(a_1,a_2,\ldots,a_n)=0$ for $k<0$.

For $n=2$, the k^{th} complete homogeneous symmetric function (1.3) is given by

$$h_k^{(2)}=h_k(a_1,a_2)=S_k(a_1+a_2)=\frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2},\quad \text{for all } k\in\mathbb{N}_0.$$

Proposition 1 ([19, Proposition 2.1])**.** *The generating function for the elementary symmetric function based on the alphabet $A=\{a_1,a_2,\ldots,a_n\}$ is given by*

$$\sum_{k=0}^{\infty}e_k(a_1,a_2,\ldots,a_n)z^k=\prod_{a\in A}(1+az).$$

Proposition 2 ([19, Proposition 2.2])**.** *The generating function for the complete homogeneous symmetric function based on the alphabet $A=\{a_1,a_2,\ldots,a_n\}$ is given by*

$$\sum_{k=0}^{\infty}h_k(a_1,a_2,\ldots,a_n)z^k=\frac{1}{\prod_{a\in A}(1-az)}.$$

A fundamental relationship exists between elementary symmetric functions and complete homogeneous symmetric functions:

$$\sum_{j=0}^{k}(-1)^j e_j(a_1,a_2,\ldots,a_n)h_{k-j}(a_1,a_2,\ldots,a_n)=0,\quad \forall k>0.$$

Definition 5 ([6, Definition 2])**.** Given an alphabet $A=\{a_1,a_2\}$, the symmetrizing operator $\delta^k_{a_1a_2}$ is defined by

$$\delta^k_{a_1a_2}(f)=\frac{a_1^k f(a_1)-a_2^k f(a_2)}{a_1-a_2},\quad \text{for all } k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}=\{0,1,2,\ldots\}.$$

In this paper, we apply the operator $\delta^{4-l}_{b_1b_2}$ to derive generating functions for certain generalized products of numbers and polynomials. Additionally, we explore the bifurcation and chaotic behavior exhibited by the generating function for (p,q)-Jacobsthal numbers.

This paper is structured as follows: In Section 2, we present our main theorem, which establishes a connection between the symmetric functions defined in the preceding section and the symmetrizing operator. This theorem unifies various previously established generating function results into a single framework. It is used in Section 3 to find the generating functions of the products of (p,q)-numbers with Pell polynomials and k-Fibonacci numbers at positive and negative indices. This section is divided into two parts: Part 1 focuses on calculating some ordinary generating functions of the products of (p,q)-numbers with k-Fibonacci polynomials, while Part 2 focuses on calculating some ordinary generating functions of the products of (p,q)-numbers with Fibonacci polynomials. In Section 4, we investigate the behavior of the family of maps (1.2) corresponding to different values of the parameters p and q. These maps serve as generating functions for sequences of generalized Jacobsthal numbers. It is observed that as the parameters vary, the behavior of these maps evolves from periodicity, through bifurcation, to chaos. Studies on bifurcation and chaotic behavior in nonlinear dynamical systems have become increasingly significant. Numerous studies have focused on chaos in various sequences and polynomials. In [18], the authors studied the bifurcation of Fibonacci generating functions associated with the golden mean. A novel approach involving a chaos-based generating function for Chebyshev polynomials has been explored in [16]. We have analytically determined a new generating function for the Jacobsthal numbers. Furthermore, the chaotic behavior of this generating function is verified through the examination of the bifurcation diagram and the Lyapunov exponent.

2. Main results

In this section, we establish the main theorem of this paper. This result provides a unified framework that encapsulates all previously established results, enabling them to be interpreted as special cases, such as those found in [20].

Theorem 1. *Assume that A and B are two alphabets, denoted by $\{a_1,a_2,\dots,a_k\}$ and $\{b_1,b_2\}$ respectively, then we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(a_1,a_2,\dots,a_k)\,h_{n-l+3}(b_1,b_2)\,z^n \\
&= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\dots,a_k)\,b_1b_2h_{2-l}(b_1,b_2)\,z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^n z^n\right)} \\
&\quad + \frac{e_2(a_1,a_2,\dots,a_k)\,b_1^2b_2^2h_{1-l}(b_1,b_2)z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^n z^n\right)} \\
&\quad - \frac{-e_3(a_1,a_2,\dots,a_k)\,b_1^3b_2^3h_{-l}(b_1,b_2)z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^n z^n\right)} \qquad (2.1)\\
&\quad - \frac{b_1^{4-l}b_2^{4-l}z^{5-l}\sum_{n=0}^{+\infty}(-1)^{n-l+5}e_{n-l+5}(a_1,a_2,\dots,a_k)\,h_n(b_1,b_2)\,z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^n z^n\right)},
\end{aligned}$$

for all n, $k \in \mathbb{N}_0$ and $l \in \{0,1,2,3,4\}$.

Proof. By applying the operator $\delta_{b_1b_2}^{4-l}$ to the series

$$f(b_1z)=\sum_{n=0}^{\infty}h_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n,$$

we have

$$\begin{aligned}\delta_{b_1b_2}^{4-l}f(b_1z)&=\frac{b_1^{4-l}\sum\limits_{n=0}^{\infty}h_n(a_1,a_2,\ldots,a_k)b_1^nz^n-b_2^{4-l}\sum\limits_{n=0}^{\infty}h_n(a_1,a_2,\ldots,a_k)b_2^nz^n}{b_1-b_2}\\&=\sum_{n=0}^{\infty}h_n(a_1,a_2,\ldots,a_k)\left(\frac{b_1^{n-l+4}-b_2^{n-l+4}}{b_1-b_2}\right)z^n\\&=\sum_{n=0}^{\infty}h_n(a_1,a_2,\ldots,a_k)h_{n-l+3}(b_1,b_2)z^n.\end{aligned}$$

On the other hand, by applying the operator $\delta_{b_1b_2}^{4-l}$ to the series

$$f(b_1z)=\frac{1}{\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n},$$

we obtain

$$\delta_{b_1b_2}^{4-l}f(b_1z)=\delta_{b_1b_2}^{4-l}\left(\frac{1}{\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n}\right)$$

$$=\frac{\dfrac{b_1^{4-l}}{\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n}-\dfrac{b_2^{4-l}}{\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_2^nz^n}}{b_1-b_2}$$

$$=\frac{b_1^{4-l}\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_2^nz^n-b_2^{4-l}\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n}{(b_1-b_2)\left(\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_2^nz^n\right)}$$

$$=\frac{\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nb_2^n\frac{b_1^{4-l-n}-b_2^{4-l-n}}{b_1-b_2}z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n\,e_n(a_1,a_2,\ldots,a_k)\,b_2^nz^n\right)}$$

$$= \frac{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$= \frac{\sum_{n=0}^{3-l}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$+ \frac{\sum_{n=5-l}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$= \frac{\sum_{n=0}^{3-l}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{\sum_{n=5-l}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^{4-l} b_2^{4-l}\left(\frac{b_1^{n+l-4}-b_2^{n+l-4}}{b_1-b_2}\right) z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_n)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_n)\, b_2^n z^n\right)},$$

accordingly,

$$\delta_{b_1 b_2}^{4-l} f(b_1 z)$$
$$= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\ldots,a_k)\, b_1 b_2 h_{2-l}(b_1,b_2)\, z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$+ \frac{e_2(a_1,a_2,\ldots,a_k)\, b_1^2 b_2^2 h_{1-l}(b_1,b_2) z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{-e_3(a_1,a_2,\ldots,a_k)\, b_1^3 b_2^3 h_{-l}(b_1,b_2) z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{b_1^{4-l} b_2^{4-l} \sum_{n=5-l}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, h_{n+l-5}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\ldots,a_k)\, b_1 b_2 h_{2-l}(b_1,b_2)\, z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$+ \frac{e_2(a_1,a_2,\ldots,a_k)\, b_1^2 b_2^2 h_{1-l}(b_1,b_2) z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{e_3(a_1,a_2,\ldots,a_k)\, b_1^3 b_2^3 h_{-l}(b_1,b_2) z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{b_1^{4-l} b_2^{4-l} z^{5-l} \sum_{n=0}^{\infty}(-1)^{n-l+5} e_{n-l+5}(a_1,a_2,\ldots,a_k)\, h_n(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}.$$

Therefore,

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,\ldots,a_k)\, h_{n-l+3}(b_1,b_2)\, z^n$$
$$= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\ldots,a_k)\, b_1 b_2 h_{2-l}(b_1,b_2)\, z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$+ \frac{e_2(a_1,a_2,\ldots,a_k)\, b_1^2 b_2^2 h_{1-l}(b_1,b_2) z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{e_3(a_1,a_2,\ldots,a_k)\, b_1^3 b_2^3 h_{-l}(b_1,b_2) z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{b_1^{4-l} b_2^{4-l} z^{5-l} \sum_{n=0}^{\infty}(-1)^{n-l+5} e_{n-l+5}(a_1,a_2,\ldots,a_k)\, h_n(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}.$$

Thus, this completes the proof. □

For $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $l = 3$ and $l = 4$ in Theorem 1, we deduce the following lemmas.

Lemma 1. *Given two alphabets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, then we have*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2) h_n(b_1,b_2) z^n$$

$$= \frac{1 - a_1 a_2 b_1 b_2 z^2}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.2)$$

Based on relationship (2.2), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, a_2) h_{n-1}(b_1, b_2) z^n$$
$$= \frac{z - a_1 a_2 b_1 b_2 z^3}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.3)$$

Lemma 2. *Given two alphabets $B = \{b_1, b_2\}$ and $A = \{a_1, a_2\}$, then we have*

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_{n-1}(b_1, b_2) z^n$$
$$= \frac{(a_1 + a_2) z - a_1 a_2 (b_1 + b_2) z^2}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.4)$$

From (2.4), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, a_2) h_n(b_1, b_2) z^n$$
$$= \frac{(b_1 + b_2) z - b_1 b_2 (a_1 + a_2) z^2}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.5)$$

3. Generating functions for products of (p,q)-numbers with Fibonacci polynomials and k-Fibonacci numbers at positive and negative indices

In this section, we derive new generating functions for products of (p,q)-Fibonacci numbers, (p,q)-Lucas numbers, (p,q)-Pell numbers, (p,q)-Pell–Lucas numbers, (p,q)-Jacobsthal numbers, and (p,q)-Jacobsthal–Lucas numbers with k-Fibonacci numbers and Fibonacci polynomials.

For the case where $A = \{a_1, -a_2\}$ and $B = \{b_1, -b_2\}$, substituting a_2 with $(-a_2)$ and b_2 with $(-b_2)$ into Eqs. (2.2), (2.3), (2.4), and (2.5) yields

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n = \frac{1 - a_1 a_2 b_1 b_2 z^2}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}.$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{z-a_1a_2b_1b_2z^3}{(1-a_1b_1z)(1+a_2b_1z)(1+a_1b_2z)(1-a_2b_2z)}. \quad (3.1)$$

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{(a_1-a_2)z+a_1a_2(b_1-b_2)z^2}{(1-a_1b_1z)(1+a_2b_1z)(1+a_1b_2z)(1-a_2b_2z)}. \quad (3.2)$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_n(b_1,[-b_2])z^n = \frac{(b_1-b_2)z+b_1b_2(a_1-a_2)z^2}{(1-a_1b_1z)(1+a_2b_1z)(1+a_1b_2z)(1-a_2b_2z)}.$$

3.1. *Ordinary generating functions of the products of (p,q)-numbers with k-Fibonacci numbers*

This case consists of three related parts.

First: By making the substitutions

$$\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases} \text{ and } \begin{cases} b_1-b_2=k \\ b_1b_2=1 \end{cases},$$

in Eqs. (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{z-qz^3}{1-pkz-(qk^2+2q+p^2)z^2-pqkz^3+q^2z^4}, \quad (3.3)$$

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{pz+qkz^2}{1-pkz-(qk^2+2q+p^2)z^2-pqkz^3+q^2z^4}, \quad (3.4)$$

respectively, and we have the following results.

Proposition 3. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n}F_{k,n}z^n = \frac{z-qz^3}{1-pkz-(qk^2+2q+p^2)z^2-pqkz^3+q^2z^4}, \quad (3.5)$$

with $F_{p,q,n}F_{k,n} = h_{n-1}(a_1,[-a_2])h_{n-1}(b_1,[-b_2])$.

Theorem 2. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} F_{k,n} z^n = \frac{pz + 2qkz^2 + pqz^3}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4}. \tag{3.6}$$

Proof. By [21], we have $L_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])$. Then, we can see that:

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} F_{k,n} z^n &= \sum_{n=0}^{\infty} \left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right) h_{n-1}(b_1,[-b_2]) z^n \\ &= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2]) h_{n-1}(b_1,[-b_2]) z^n \\ &\quad - p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2]) h_{n-1}(b_1,[-b_2]) z^n,\end{aligned}$$

by using the relationships (3.3) and (3.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} F_{k,n} z^n &= \frac{2\left(pz + qkz^2\right)}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4} \\ &\quad - \frac{p\left(z - qz^3\right)}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4} \\ &= \frac{pz + 2qkz^2 + pqz^3}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4}.\end{aligned}$$

This completes the proof. □

Proposition 4. *By using the change of variable $z = -z$ in Eqs. (3.5) and (3.6) and according to relation (1.1), we give the following new generating functions*

$$\sum_{n=0}^{\infty} F_{p,q,n} F_{k,-n} z^n = \frac{z - qz^3}{1 + pkz - (qk^2 + p^2 + 2q)z^2 + pqkz^3 + q^2z^4}. \tag{3.7}$$

$$\sum_{n=0}^{\infty} L_{p,q,n} F_{k,-n} z^n = \frac{pz - 2qkz^2 + pqz^3}{1 + pkz - (qk^2 + p^2 + 2q)z^2 + pqkz^3 + q^2z^4}. \tag{3.8}$$

By putting $k = 1$ in relationships (3.5), (3.6), (3.7) and (3.8), we obtain the following new generating functions. The calculation results are indicated in Tab.1.

Second: By making the substitutions

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = 2q \end{cases} \text{ and } \begin{cases} b_1 - b_2 = k \\ b_1 b_2 = 1 \end{cases},$$

in Eqs. (3.1) and (3.2), we obtain

TABLE 1. New generating functions for the products of some sequences.

Coefficient of z^n	Generating function
$F_{p,q,n}F_n$	$\frac{z-qz^3}{1-pz-(3q+p^2)z^2-pqz^3+q^2z^4}$
$L_{p,q,n}F_n$	$\frac{pz+2qz^2+pqz^3}{1-pz-(3q+p^2)z^2-pqz^3+q^2z^4}$
$F_{p,q,n}F_{-n}$	$\frac{z-qz^3}{1+pz-(3q+p^2)z^2+pqz^3+q^2z^4}$
$L_{p,q,n}F_{-n}$	$\frac{pz-2qz^2+pqz^3}{1+pz-(3q+p^2)z^2+pqz^3+q^2z^4}$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])\,z^n = \frac{z-2qz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4},$$

$$\sum_{n=0}^{\infty} h_{n}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])\,z^n = \frac{pz+2qkz^2}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4},$$

respectively, and we have the following results.

Proposition 5. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n}F_{k,n}z^n = \frac{z-2qz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}, \tag{3.9}$$

with $J_{p,q,n}F_{k,n} = h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])$.

Theorem 3. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal Lucas numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n}F_{k,n}z^n = \frac{pz+4qkz^2+2pqz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}. \tag{3.10}$$

Proof. We know that

$$j_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2]),\ \text{(see [21])}.$$

So

$$\sum_{n=0}^{\infty} j_{p,q,n}F_{k,n}z^n = \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)h_{n-1}(b_1,[-b_2])z^n$$
$$= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])z^n$$

$$-p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])z^n$$
$$=\frac{2\left(pz+2qkz^2\right)}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}$$
$$-\frac{p\left(z-2qz^3\right)}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}$$
$$=\frac{pz+4qkz^2+2pqz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}.$$

This completes the proof. □

Proposition 6. *By using the change of variable $z=-z$ in Eqs.* (3.9) *and* (3.10) *and according to relation* (1.1)*, we derive the following new generating functions*

$$\sum_{n=0}^{\infty} J_{p,q,n}F_{k,-n}z^n=\frac{z-2qz^3}{1+pkz-(2qk^2+p^2+4q)z^2+2pqkz^3+4q^2z^4}. \tag{3.11}$$

$$\sum_{n=0}^{\infty} j_{p,q,n}F_{k,-n}z^n=\frac{pz-4qkz^2+2pqz^3}{1+pkz-(2qk^2+p^2+4q)z^2+2pqkz^3+4q^2z^4}. \tag{3.12}$$

By setting $k=1$ in relationships (3.9), (3.10), (3.11) and (3.12), we obtain the following new generating functions. The calculation results are indicated in Tab. 2.

TABLE 2. New generating functions for the products of some sequences.

Coefficient of z^n	Generating function
$J_{p,q,n}F_n$	$\frac{z-2qz^3}{1-pz-(6q+p^2)z^2-2pqz^3+4q^2z^4}$
$j_{p,q,n}F_n$	$\frac{pz+4qz^2+2pqz^3}{1-pz-(6q+p^2)z^2-2pqz^3+4q^2z^4}$
$J_{p,q,n}F_{-n}$	$\frac{z-2qz^3}{1+pz-(6q+p^2)z^2+2pqz^3+4q^2z^4}$
$j_{p,q,n}F_{-n}$	$\frac{pz-4qz^2+2pqz^3}{1+pz-(6q+p^2)z^2+2pqz^3+4q^2z^4}$

Third: the substitutions of

$$\begin{cases} a_1-a_2=2p \\ a_1a_2=q \end{cases} \text{ and } \begin{cases} b_1-b_2=k \\ b_1b_2=1 \end{cases},$$

in Eqs. (3.1) and (3.2) we obtain

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])\,z^n$$

$$= \frac{z - qz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}, \quad (3.13)$$

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2])\, h_{n-1}(b_1, [-b_2])\, z^n$$
$$= \frac{2pz + qkz^2}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}, \quad (3.14)$$

respectively, and we have the following results.

Proposition 7. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n} F_{k,n} z^n = \frac{z - qz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}, \quad (3.15)$$

with $P_{p,q,n}F_{k,n} = h_{n-1}(a_1, [-a_2])\, h_{n-1}(b_1, [-b_2])$.

We have the following theorem.

Theorem 4. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,n} z^n = \frac{2pz + 2qkz^2 + 2pqz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}. \quad (3.16)$$

Proof. By referred to [21], we have

$$Q_{p,q,n} = 2h_n(a_1, [-a_2]) - 2ph_{n-1}(a_1, [-a_2]).$$

We see that

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,n} z^n = \sum_{n=0}^{\infty} (2h_n(a_1, [-a_2]) - 2ph_{n-1}(a_1, [-a_2]))\, h_{n-1}(b_1, [-b_2]) z^n$$
$$= 2\sum_{n=0}^{\infty} h_n(a_1, [-a_2])\, h_{n-1}(b_1, [-b_2]) z^n$$
$$- 2p\sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2])\, h_{n-1}(b_1, [-b_2]) z^n.$$

Using relationships (3.13) and (3.14), we obtain

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,n} z^n = \frac{2\left(2pz + qkz^2\right)}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}$$
$$- \frac{2p\left(z - qz^3\right)}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}$$

$$= \frac{2pz+2qkz^2+2pqz^3}{1-2pkz-(qk^2+4p^2+2q)z^2-2pqkz^3+q^2z^4}.$$

This completes the proof. □

Proposition 8. *By using the change of variable $z=-z$ in Eqs.* (3.15) *and* (3.16) *and according to relation* (1.1)*, we give the following new generating functions:*

$$\sum_{n=0}^{\infty} P_{p,q,n}F_{k,-n}z^n = \frac{z-qz^3}{1+2pkz-(qk^2+4p^2+2q)z^2+2pqkz^3+q^2z^4}. \quad (3.17)$$

$$\sum_{n=0}^{\infty} Q_{p,q,n}F_{k,-n}z^n = \frac{2pz-2qkz^2+2pqz^3}{1+2pkz-(qk^2+4p^2+2q)z^2+2pqkz^3+q^2z^4}. \quad (3.18)$$

By taking $k=1$ in relationships (3.15), (3.16), (3.17) and (3.18), we obtain the following new generating functions. The calculation results are indicated in Tab.3

TABLE 3. New generating functions for the products of some sequences.

Coefficient of z^n	Generating function
$P_{p,q,n}F_n$	$\frac{z-qz^3}{1-2pz-(3q+4p^2)z^2-2pqz^3+q^2z^4}$
$Q_{p,q,n}F_n$	$\frac{2pz+2qz^2+2pqz^3}{1-2pz-(3q+4p^2)z^2-2pqz^3+q^2z^4}$
$P_{p,q,n}F_{-n}$	$\frac{z-qz^3}{1+2pz-(3q+4p^2)z^2+2pqz^3+q^2z^4}$
$Q_{p,q,n}F_{-n}$	$\frac{2pz-2qz^2+2pqz^3}{1+2pz-(3q+4p^2)z^2+2pqz^3+q^2z^4}$

3.2. *Ordinary generating functions of the products of (p,q)-numbers with Fibonacci polynomials*

This part consists of three cases.

Case 1: The substitutions of $\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases}$ and $\begin{cases} b_1-b_2=x \\ b_1b_2=1 \end{cases}$ in Eqs. (2.1) and (2.4), gives

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n = \frac{1-qz^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4},$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n = \frac{xz+pz^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4},$$

respectively, and we deduce the following results.

Proposition 9. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with Fibonacci polynomials is given by:*

$$\sum_{n=0}^{\infty} F_{p,q,n}F_n(x)z^n = \frac{xz+pz^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4},$$

with $F_{p,q,n}F_n(x) = h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])$.

Theorem 5. *Let n be a natural number. Then we have the following new generating function for the product of (p,q)-Lucas numbers with Fibonacci polynomials*

$$\sum_{n=0}^{\infty} L_{p,q,n}F_n(x)z^n = \frac{2-pxz-\left(2q+p^2\right)z^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4}.$$

Proof. By [21], we have $L_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n}F_n(x)z^n &= \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)h_n(b_1,[-b_2])z^n \\ &= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_n(b_1,[-b_2])z^n \\ &\quad - p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_n(b_1,[-b_2])z^n \\ &= \frac{2\left(1-qz^2\right)}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4} \\ &\quad - \frac{p\left(xz+pz^2\right)}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4},\end{aligned}$$

after simple calculations, we obtain

$$\sum_{n=0}^{\infty} L_{p,q,n}F_n(x)z^n = \frac{2-pxz-\left(2q+p^2\right)z^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4}.$$

So, the desired result is achieved. □

Case 2: The substitutions of $\begin{cases} a_1-a_2=p \\ a_1a_2=2q \end{cases}$ and $\begin{cases} b_1-b_2=x \\ b_1b_2=1 \end{cases}$ in Eqs. (2.1) and (2.4), gives:

$$\begin{aligned}&\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n \\ &\qquad = \frac{1-2qz^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},\end{aligned}$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\, h_n(b_1,[-b_2])\, z^n$$
$$= \frac{xz+pz^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},$$

respectively, and we deduce the following results.

Proposition 10. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n}F_n(x)\, z^n = \frac{xz+pz^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},$$

with $J_{p,q,n}F_n(x) = h_{n-1}(a_1,[-a_2])\, h_n(b_1,[-b_2])$.

Theorem 6. *Let n be a natural number. Then, we have the following new generating function for the product of (p,q)-Jacobsthal Lucas numbers with Fibonacci polynomials*

$$\sum_{n=0}^{\infty} j_{p,q,n}F_n(x)\, z^n = \frac{2-pxz-\left(4q+p^2\right)z^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4}.$$

Proof. By [21], we have $j_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])$. Then, we can see that

$$\sum_{n=0}^{\infty} j_{p,q,n}F_n(x)\, z^n = \sum_{n=0}^{\infty} \left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right) h_n(b_1,[-b_2])z^n$$
$$= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\, h_n(b_1,[-b_2])z^n$$
$$- p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\, h_n(b_1,[-b_2])z^n$$
$$= \frac{2\left(1-2qz^2\right)}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4}$$
$$- \frac{p\left(xz+pz^2\right)}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},$$

after simple calculations, we obtain

$$\sum_{n=0}^{\infty} j_{p,q,n}F_n(x)\, z^n = \frac{2-pxz-\left(4q+p^2\right)z^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4}.$$

As required. □

Case 3: The substitutions of $\begin{cases} a_1 - a_2 = 2p \\ a_1a_2 = q \end{cases}$ and $\begin{cases} b_1 - b_2 = x \\ b_1b_2 = 1 \end{cases}$ in Eqs. (2.1) and (2.4), gives

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n = \frac{1-qz^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4},$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n = \frac{xz+2pz^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4},$$

respectively, and we deduce the following results.

Proposition 11. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell numbers with Fibonacci polynomials is given by:*

$$\sum_{n=0}^{\infty} P_{p,q,n}F_n(x)\,z^n = \frac{xz+2pz^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4},$$

with $P_{p,q,n}F_n(x) = h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])$.

Theorem 7. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n}F_n(x)\,z^n = \frac{2-2pxz-\left(2q+4p^2\right)z^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4}.$$

Proof. By referred to [21], we have

$$Q_{p,q,n} = 2h_n(a_1,[-a_2]) - 2ph_{n-1}(a_1,[-a_2]).$$

Then, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n}F_n(x)\,z^n &= \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - 2ph_{n-1}(a_1,[-a_2])\right)h_n(b_1,[-b_2])z^n \\
&= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n \\
&\quad -2p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n \\
&= \frac{2\left(1-qz^2\right)}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4} \\
&\quad - \frac{2p\left(xz+2pz^2\right)}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4} \\
&= \frac{2-2pxz-\left(2q+4p^2\right)z^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4}.
\end{aligned}$$

This completes the proof. □

4. Chaotic Behavior and Bifurcation Analysis of Generalized Jacobsthal Numbers

In this section, we examine the dynamical behavior of the proposed generating function and investigate the chaotic nature of the recurrent form of the generalized (p,q)-Jacobsthal numbers (1.2)

$$x_n = \frac{x_{n-1}}{1 - px_{n-1} - 2qx_{n-1}^2}. \tag{4.1}$$

The suggested generating function (4.1) is a discrete-time dynamical system that exhibits sensitive dependence on initial conditions and non-periodic behavior. Through numerical simulations and analysis of the map's iterates, we explore the parameter space where chaos emerges and examine key properties such as bifurcations and period-doubling cascades. These two parameters, p and q, make the study important and unusual. When giving some values to this two parameters, the sequence of iterates generating from the function change the behavior and give transition from periodic to chaotic behavior of the parameter.

4.1. *Bifurcation diagram*

Our study identifies regions of stability and observes the emergence of bifurcation points. The numerical bifurcation diagram provides valuable insights into the transition from stable periodic orbits to chaotic behavior, as shown in Figure.1. For $p = 2.7$ and $-2 \le q \le -1$, the bifurcation diagram shows that the system undergoes a series of bifurcation.

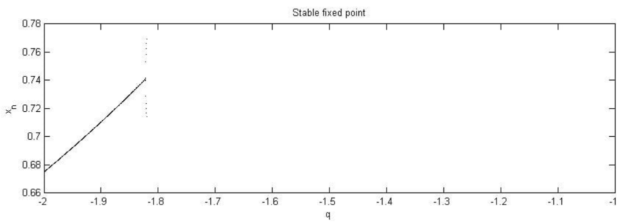

FIGURE 1.

For q between -2 and approximately -1.8, the system exhibits a stable fixed point, which corresponds to a single vertical line in the diagram. At $q = -1.8$, this fixed point undergoes a period-doubling bifurcation, splitting into two stable fixed points see Figure.2. As q increased further, the system exposes additional period-doubling bifurcation, producing periodic orbits of periods $4, 8, 16$ and so on.

Finally, for $-1.3 \le q \le -1$, the system enters a region of chaotic behavior where the attractor is a strange attractor with a fractal structure, as depicted in Figure.3.

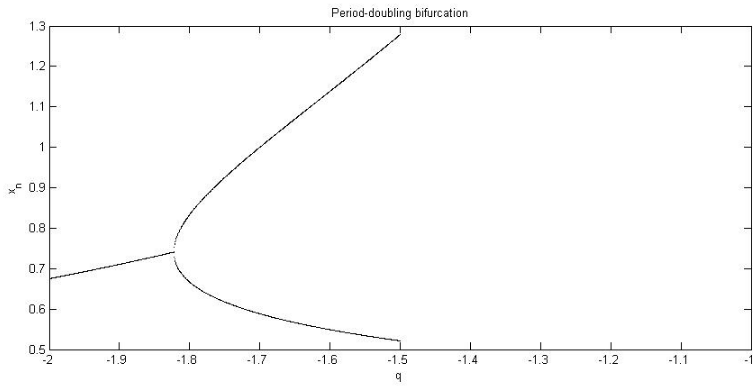

FIGURE 2.

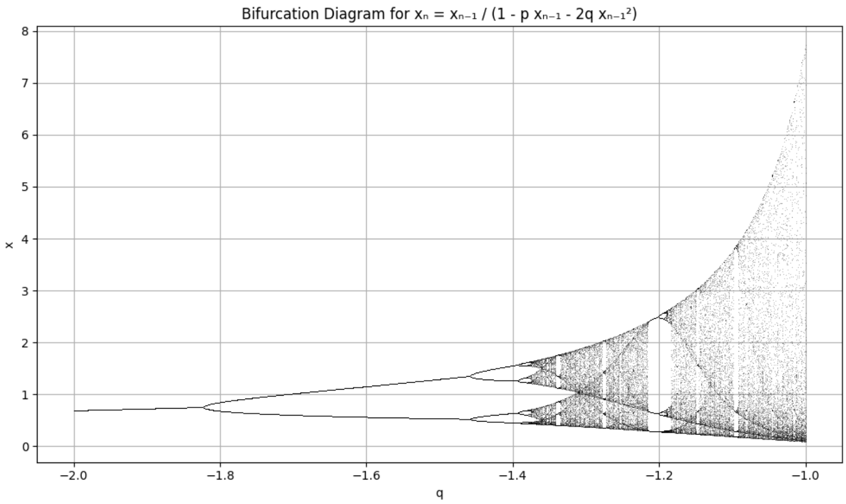

FIGURE 3.

4.2. *Lyapunov Exponent*

To verify the presence of chaos in the proposed generating function, we analyze the Lyapunov exponent. This quantity measures the rate of exponential divergence or convergence of nearby trajectories in a dynamical system. A positive Lyapunov exponent indicates exponential divergence of trajectories, which is a hallmark of chaotic behavior.

The Lyapunov exponent is computed using the formula:

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln\left(\left|\frac{dx_{n+1}}{dx_n}\right|\right), \tag{4.2}$$

which represents the limit, as N approaches infinity, of the average logarithmic derivative of the map over a trajectory of N points.

To evaluate (4.2), we first generate a trajectory of N points using the generating function (4.1). The expression then becomes:

$$\lambda = \lim_{n\longrightarrow\infty} \frac{1}{N} \sum_{n=1}^{N} \ln\left(\left|\frac{1+2qx_n^2}{(1-px_n-2qx_n^2)^2}\right|\right).$$

To demonstrate that the Lyapunov exponent is positive in certain parameter regions, we compute its values numerically. We used MATLAB to compute both the bifurcation diagram and the Lyapunov exponent; the results are presented in Figure .4.

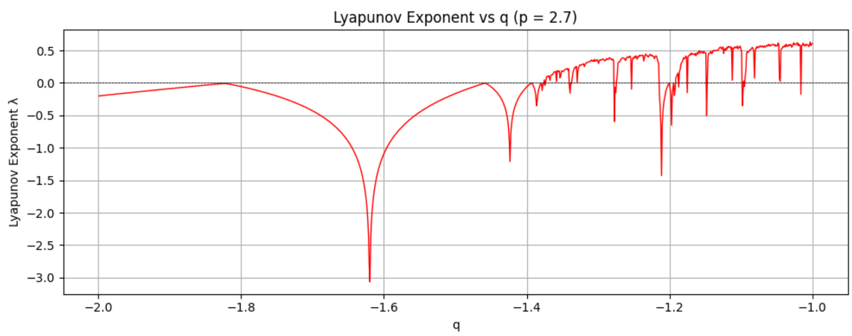

FIGURE 4.

So, the corresponding plot of the Lyapunov exponent provide clear visual evidence of chaotic behavior in the proposed generating map and the chaotic regimes corresponding to the regimes of the bifurcation diagram.

ACKNOWLEDGEMENTS

The authors would like to gratefully acknowledge the anonymous reviewers for their valuable feedback and insightful suggestions, which have significantly improved the quality of this manuscript.

REFERENCES

[1] A. Abderrezzak, "Généralization de la transformation d'euler d'une série formelle." *Adv. Math.*, vol. 103, no. 2, pp. 180–195, 1994, doi: 10.1006/aima.1994.1008.

[2] A. O. Akdemir, S. I. Butt, M. Nadeem, and M. A. Ragusa, "Some new integral inequalities for a general variant of polynomial convex functions." *AIMS Math.*, vol. 7, pp. 20 461–20 489, 2022, doi: 10.3934/math.20221121.

[3] S. Benbernou, S. Gala, and M. Ragusa, "On the regularity criteria for the 3d magnetohydrodynamic equations via two components in terms of BMO space." *Math. Methods Appl. Sci.*, vol. 37, no. 15, pp. 2320–2325, 2014, doi: 10.1002/mma.2981.

[4] M. V. Berry, Z. V. Lewis, and J. F. Nye, "On the Weierstrass-Mandelbrot fractal function." *Proc. R. Soc. Lond., Ser. A*, vol. 370, no. 1743, pp. 459–484, 1980, doi: 10.1098/rspa.1980.0044.

[5] A. Boussayoud and A. Abderrezzak, "Complete homogeneous symmetric functions and Hadamard product." *Ars Combinatoria.*, vol. 144, pp. 81–90, 2019.

[6] A. Boussayoud and M. Kerada, "Symmetric and generating functions." *Int. Electron. J. Pure Appl. Math.*, vol. 7, no. 4, pp. 195–203, 2014, doi: 10.12732/iejpam.v7i4.5.

[7] A. Boussayoud, M. Kerada, S. Araci, and M. Acikgoz, "Symmetric functions of the k-Fibonacci and k-Lucas numbers." *Asian-Eur J. Math.*, vol. 14, no. 03, pp. 1–12, 2021, doi: 10.1142/S1793557121500315.

[8] B. G. S. Doman and J. K. Williams, "Fibonacci and Lucas polynomials." *Math. Proc. Camb. Philos. Soc.*, vol. 90, no. 3, pp. 385–387, 1981, doi: 10.1017/S0305004100058850.

[9] S. Etemad, M. A. Ragusa, S. Rezapour, and A. Zada, "Existence property of solutions for multi-order q-difference FBVPs based on condensing operators and end-point technique." *Fixed Point Theory*, vol. 25, no. 1, pp. 115–142, 2024, doi: 10.24193/fpt-ro.2024.1.08.

[10] S. Falcon and A. Plaza, "The k-fibonacci sequence and the pascal 2-triangle." *Chaos, Solitons & Fractals.*, vol. 33, no. 1, pp. 38–49, 2007, doi: 10.1016/j.chaos.2006.10.022.

[11] E. Guariglia, "Primality, fractality, and image analysis," *Entropy*, vol. 21, no. 3, p. 304, 2019, doi: 10.3390/e21030304.

[12] E. Guariglia and R. Guido, "Chebyshev wavelet analysis," *J. Funct. Spaces*, vol. 2022, no. 1, p. 5542054, 2022, doi: 10.1155/2022/5542054.

[13] E. Guariglia and S. Silvestrov, *Fractional-wavelet analysis of positive definite distributions and wavelets on D'(C).* Springer Proceedings in Mathematics and Statistics, 2016, vol. 179.

[14] H. H. Gulec and N. Taskara, "On the (s,t)-Pell and (s, t)-Pell-Lucas sequences and their matrix representations," *Appl. Math. Lett.*, vol. 25, no. 10, pp. 1554–1559, 2012, doi: 10.1016/j.aml.2012.01.014.

[15] A. F. Horadam, "A generalized Fibonacci sequence." *Am. Math. Mon.*, vol. 68, no. 5, pp. 455–459, 1961, doi: 10.1080/00029890.1961.11989696.

[16] N. Louzzani, A. Boukabou, H. Bahi, and A. Boussayoud, "A novel chaos based generating function of the Chebyshev polynomials and its applications in image encryption," *Chaos, Solitons & Fractals*, vol. 151, p. 111315, 2021, doi: 10.1016/j.chaos.2021.111315.

[17] M. Merca, "A generalization of the symmetry between complete and elementary symmetric functions," *Indian J. Pure Appl. Math.*, vol. 45, pp. 75–90, 2014, doi: 10.1007/s13226-014-0052-0.

[18] M. Özer, A. Čenys, Y. Polatoglu, G. Hacibekiroglu, E. Akat, A. Valaristos, and A. N. Anagnostopoulos, "Bifurcations of Fibonacci generating functions," *Chaos, Solitons and Fractals*, vol. 33, no. 4, pp. 1240–1247, 2007, doi: 10.1016/j.chaos.2006.01.095.

[19] N. Saba and A. Boussayoud, "Ordinary generating functions of binary products of (p,q)-modified pell numbers and k-numbers at positive and negative indices." *J. Sci. Arts.*, vol. 20, no. 3, pp. 627–646, 2020, doi: 10.46939/J.Sci.Arts-20.3-a11.

[20] N. Saba and A. Boussayoud, "New theorem on symmetric functions and their applications on some (p,q)-numbers." *J. Appl. Math. Inform.*, vol. 40, no. 1-2, pp. 243–257, 2022, doi: 10.14317/jami.2022.243.

[21] N. Saba, A. Boussayoud, and A. Abderrezzak, "Symmetric and generating functions of generalized (p,q)-numbers." *Kuwait J. Sci.*, vol. 48, no. 4, 2021, doi: 10.48129/kjs.v48i4.10074.

[22] N. Saba, A. Boussayoud, S. Araci, M. Kerada, and M. Acikgoz, "Construction of a new class of symmetric function of binary products of (p,q)-numbers with 2-orthogonal Chebyshev polynomials," *Bol. Soc. Mat. Mex.*, vol. 27, pp. 1–26, 2021, doi: 10.2298/FIL2103001B.

[23] N. Saba, A. Boussayoud, and M. Kerada, "Generating functions of even and odd Gaussian numbers and polynomials," *J. Sci. Arts*, vol. 1, no. 54, pp. 125–144, 2021, doi: 10.46939/J.Sci.Arts-21.1-a12.

[24] A. Suvarnamani, "Some properties of (p,q)-Lucas number," *Kyungpook Math*, vol. 56, no. 2, pp. 367–370, 2016, doi: 10.5666/KMJ.2016.56.2.367.

[25] A. Suvarnamani and M. Tatong, "Some properties of (p,q)-Fibonacci numbers." *Science and Technology RMUTT Journal.*, vol. 5, no. 2, pp. 17–21, 2015, doi: 10.21660/2017.37.2691.

[26] S. Uygun, "The (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas sequences." *Appl. Math. Sci.*, vol. 70, no. 9, pp. 3467–3476, 2015, doi: 10.12988/ams.2015.52166.

Authors' addresses

Chems Eddine Arroud
Faculty of Exact Sciences and Computer Science, LMEPA Laboratory, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
E-mail address: arroud.math@gmail.com

Nabiha Saba
Higher School of Science and Technology of Computing and Digital of Bejaia, Algeria, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
E-mail address: sabarnhf1994@gmail.com

Ali Boussayoud
Faculty of Exact Sciences and Computer Science, LMAM Laboratory, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
E-mail address: aboussayoud@yahoo.fr

Salah Boulaaras
Department of Mathematics, College of Sciences, Qassim University, Buraydah 51452, Saudi Arabia
E-mail address: s.boularas@qu.edu.s

Baghdadi Aloui
Higher Institute of Industrial Systems of Gabes, Laboratory of Mathematics and Applications LR17ES11, Gabes University, Tunisia
E-mail address: Baghdadi.Aloui@fsg.rnu.tn

Nesrine Harrouche
(**Corresponding author**) Faculty of Exact Sciences and Computer Science, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
E-mail address: nesrine.harrouche@univ-jijel.dz