Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 579–590

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.5218

EXISTENCE OF SOLUTIONS FOR FRACTIONAL BOUNDARY VALUE PROBLEM VIA GLOBAL MINIMIZATION THEOREM

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Received 30 November, 2024

Abstract. The goal of this paper is to investigate the existence of solutions for the following boundary value problem involving the Riemann-Liouville fractional derivative:

$$\begin{cases} D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}\left(D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(x)\right)\right)\right) = f\left(x,u(x)\right), \qquad x\in[0,T], \\ u(0)=u(T)=0, \\ D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(0)\right) = D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(T)\right) = 0, \end{cases}$$

where $0<\alpha\leq 1$, $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and $D^{\alpha}_{0^+}, D^{\alpha}_{T^-}$ are the left and right fractional Riemann-Liouville derivatives of order α.

2010 *Mathematics Subject Classification:* 46C05; 74G65; 47H09; 47G40

Keywords: minimization theorem, Hilbert space, nonexpansive potential operator, BVP

1. Introduction

Many phenomena in engineering and technological discipline, that include various field of sciences, can be described by fractional differential equation; see for example [1, 2, 7, 8, 12, 13, 17, 18].

Numerous researchers have explored the intricate connections between fractional calculus and various scientific disciplines. For instance, the study conducted in [14] examines the relationship between fractional calculus, zeta functions, and Shannon entropy, offering valuable insights into information theory and thermodynamics. In the realm of mathematics, the work presented in [15] provides a deeper understanding of special functions and their analytical properties. Likewise, the research in [19] extends traditional complex differentiation to fractional orders, unveiling new structural properties and potential applications.

From a functional analysis standpoint, the study of commutators of fractional integral operators on null Morrey spaces [23] sheds light on significant operator-theoretic properties with implications for harmonic analysis and partial differential equations. Additionally, investigations into elliptic boundary value problems under zero mean oscillation assumptions establish connections between fractional differential operators and modern function space theory, leading to refined estimates for PDE solutions [22].

The existence of solutions of boundary value problems associated to a nonlinear fractional differential equation have been studied widely (see [3, 20, 24, 25]). The selection of the appropriate fractional derivative for a boundary value problem depends on various factors, including analytical properties, boundary conditions, and the specific requirements of the model under consideration. Several studies have investigated fractional-order differential equations using fixed point theory, yet only a limited number have explored them through variational methods. The Riemann-Liouville fractional derivative is particularly well-suited for such problems, as it aligns with fractional Sobolev spaces, making it a valuable tool in variational analysis and boundary value problems. It is worth noting that the study of equations involving both left and right fractional differential operators has attracted significant interest, leading to the development of various theorems and techniques. Follow the previous contributions in this direction and inspired by the theorem proposed in [11], Boucenna and Moussaoui proved [9] the existence of positive solutions for a following boundary value problem via a topological-variational theorem

$$\begin{cases} D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(x)\right) = f\left(x,u(x)\right), & x \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\frac{1}{2} < \alpha \leq 1$, $f\colon\ [0,T]\times\mathbb{R}$ a continuous function and $D^{\alpha}_{0^+}, D^{\alpha}_{T^-}$ are respectively the left and right fractional Riemann-Liouville derivatives of order α. A similar problem was studied in [16] by using Minimization Principle and Mountain Pass Theorem. Ricceri's critical point theorem was used in [6] to prove existence of infinitely many solutions for a perturbed nonlinear fractional boundary value problem depending on two parameters and involving both left and right Riemann-Liouville derivatives. Later on, in [10], the authors considered an appropriate Banach space where they have used Minimization Principle and Mountain Pass Theorem to prove the existence of at least one solution for the problem

$$\begin{cases} D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}\left(D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(x)\right)\right)\right) = f\left(x,u(x)\right), & x \in [0,T], \\ u(0) = u(T) = 0, \\ D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(0)\right) = D^{\alpha}_{T^-}\left(D^{\alpha}_{0^+}u(T)\right) = 0, \end{cases}$$

where $0 < \alpha \leq 1$ and $f\colon\ [0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Ayadi et al. [5] proved a new global minimization theorem using the notion of a nonexpansive potential operator. Our contribution consists in working in the Banach space introduced in [10] and applying the global minimization theorem proved in [5] to find the global minimum for a suitable functional. To this end, we will firstly give the needed tools for our proof and in the second section, we present the proof of our main result.

2. PRELIMINARIES

The definitions and properties of the fractional calculus which are used in the sequel are taken from [17, 21].

Definition 1 ([17, Section 2.1],[21, Section 2.3.2])**.** Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function u denoted by $I^{\alpha}_{a^+}u$ and $I^{\alpha}_{b^-}u$, respectively, are defined by

$$I^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\mathrm{d}s, \qquad t > a$$

and

$$I^{\alpha}_{b^-}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)\mathrm{d}s, \quad t < b$$

provided that the right-hand side is pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2 ([17, Section 2.1], [21, Section 2.3.3])**.** Let u be a function defined on $[a,b]$. For $n-1 \leq \alpha < n\, (n \in \mathbb{N}^*)$, the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function u denoted by $D^{\alpha}_{a^+}u$ and $D^{\alpha}_{b^-}u$, respectively, are defined by

$$\begin{aligned} D^{\alpha}_{a^+}u(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}_{a^+}u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-s)^{n-\alpha-1}u(s)\mathrm{d}s, \qquad t > a \end{aligned}$$

and

$$\begin{aligned} D^{\alpha}_{b^-}u(t) &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}_{b^-}u(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b (s-t)^{n-\alpha-1}u(s)\mathrm{d}s, \qquad t < b \end{aligned}$$

provided that the right-hand side is pointwise defined. In particular, for $\alpha = n$, $D^{\alpha}_{a^+}u(t) = D^n u(t)$ and $D^{\alpha}_{b^-}u(t) = (-1)^n D^n u(t)$, $t \in [a,b]$.

Property 1 ([17, property 2.1], [21, Section 2.3.3, Section 2.3.4])**.** *For* $\alpha > 0$ *and for some* $\beta > -1, \quad \alpha - 1 > 0$, *we have*

(1) $D^{\alpha}_{a^+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$, $t > a$ *and*

$$D^{\alpha}_{b^-}(b-t)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-t)^{\beta-\alpha}, \qquad t < b.$$

(2) $D^{\alpha-1}_{a^+}(t-a)^{\alpha-1} = \Gamma(\alpha)$, $t > a$ *and* $D^{\alpha-1}_{b^-}(b-t)^{\alpha-1} = \Gamma(\alpha)$, $t < b$.

(3) $D^{\alpha}_{a^+}(t-a)^{\alpha-j} = 0$, $t > a$ *and* $D^{\alpha}_{b^-}(b-t)^{\alpha-j} = 0$, $t < b$ *for* $j = 1,2,\ldots,[\alpha]+1$.

(4) *If C is a constant and $\beta=0$, then*

$$D^{\alpha}_{a^+}C=\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha},\ t>a \ \text{ and } \ D^{\alpha}_{b^-}C=\frac{C}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.$$

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1 ([17, Lemma 2.4],[21, Section 2.3.3])**.** *If $\alpha>0$ and $u\in L^p(a,b)\,(1\leq p\leq+\infty)$, then the following equalities*

$$D^{\alpha}_{a^+}I^{\alpha}_{a^+}u=u \quad \text{and} \quad D^{\alpha}_{b^-}I^{\alpha}_{b^-}u=u$$

hold almost everywhere on $[a,b]$.

Lemma 2 ([17, Corollary 2.1])**.** *Let $\alpha>0$ and $n=[\alpha]+1$.*

(1) *The equality $D^{\alpha}_{a^+}u(t)=0$ is valid if and only if*

$$u(t)=\sum_{j=1}^{n}c_j(t-a)^{\alpha-j},$$

where $c_j\in\mathbb{R}$ $(j=1,2,\dots,n)$ are arbitrary constants.

(2) *The equality $D^{\alpha}_{b^-}u(t)=0$ is valid if and only if*

$$u(t)=\sum_{j=1}^{n}d_j(b-t)^{\alpha-j},$$

where $d_j\in\mathbb{R}$ $(j=1,2,\dots,n)$ are arbitrary constants.

In this paper, we will use the Banach spaces defined in [10]. We point out that the reader can find more informations about this subject in [10] and references therein.

Definition 3 ([10, Definition 2])**.** Let $0<\alpha\leq 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p}=\left(\|u\|^p_{L^p}+\|D^{\alpha}_{0^+}u\|^p_{L^p}\right)^{\frac{1}{p}}$$

for all $u\in E_0^{\alpha,p}$ with $\|u\|_{L^p}=(\int_0^T|u(t)|^p\mathrm{d}t)^{\frac{1}{p}}$. We put

$$E_0^{\alpha,p}=\Big\{u\in L^p([0,T]);\, D^{\alpha}_{0^+}u\in L^p([0,T]);\, u(0)=u(T)=0\Big\}.$$

Definition 4 ([10, Definition 3.2, Remark 3.3])**.** The fractional derivative space $E_0^{2\alpha,p}$ is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{2\alpha,p}=\left(\|u\|^p_{L^p}+\|D^{\alpha}_{0^+}u\|^p_{L^p}+\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|^p_{L^p}\right)^{\frac{1}{p}}$$

for all $u\in E_0^{2\alpha,p}$. We put

$$E_0^{2\alpha,p}=\{u\in L^p([0,T])\ :\ D^{\alpha}_{0^+}u\in L^p([0,T]),\, D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\in L^p([0,T],\mathbb{R}),$$

$$u(0)=u(T)=0,\ D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(0))=D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(T))=0\}\,.$$

The space $E_0^{2\alpha,p}$ can be endowed with respect to the equivalent norm

$$\|u\|_{2\alpha,p}=\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}=\Big(\int_0^T|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(t))|^p\mathrm{d}t\Big)^{\frac{1}{p}}.$$

The following properties which are taken from [10], are necessary for all proofs in this paper.

Lemma 3 ([10, Lemma 3.4]). *For any $u\in L^p([0,T],\mathbb{R})$, we have*

$$\|I^{\alpha}_{0^+}u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|u\|_{L^p}$$

and

$$\|I^{\alpha}_{T^-}u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|u\|_{L^p}.$$

Proposition 1 ([10, Proposition 3.5]). *For all $u\in E_0^{2\alpha,p}$, we have*

$$I^{\alpha}_{0^+}D^{\alpha}_{0^+}u=u\ \text{ and }\ I^{\alpha}_{T^-}D^{\alpha}_{T^-}u=u.$$

Proposition 2 ([10, Proposition 3.6]). *For any $u\in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{0^+}u\|_{L^p}.$$

Moreover, if $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|u\|_{\infty}\le\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|D^{\alpha}_{0^+}u\|_{L^p},$$

where $\|u\|_{\infty}=\sup\limits_{t\in[0,T]}|u(t)|$.

Proposition 3 ([10, Proposition 3.7]). *For any $u\in E_0^{2\alpha,p}$, we have*

$$\|D^{\alpha}_{0^+}u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}.$$

Moreover, if $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|u\|_{\infty}\le\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\cdot\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}.$$

Proposition 4 ([10, Proposition 3.10]). *If $u\in E_0^{\alpha,p}$ and $v\in C_0^{\infty}([0,T])$, then*

$$\int_0^T D^{\alpha}_{0^+}u(t)v(t)\mathrm{d}t=\int_0^T u(t)D^{\alpha}_{T^-}v(t)\mathrm{d}t.$$

Proposition 5 ([10, Proposition 3.11]). *If $u \in E^{2\alpha,p}$ and $v \in C_0^\infty([0,T])$, then*

$$\int_0^T D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))v(t)\mathrm{d}t = \int_0^T u(t)D_{T^-}^{\alpha}(D_{0^+}^{\alpha}v(t))\mathrm{d}t.$$

We mention that the proof of our main result is based on the following global minimization theorem, see [5].

Let H a real Hilbert space and $(.,.)$ the scalar product.

Theorem 1 ([5, Theorem 4]). *Let $\varphi\colon H \longrightarrow \mathbb{R}$ be a functional such that:*

(1) *φ is twice Gâteaux differentiable on H.*
(2) *For all $u \in H$, $\|(I' - \varphi'')(u)\| \leq 1$.*
(3) *$(I - \varphi')(C) \subset C$ for some convex nonempty closed and bounded subset C of H.*

Then, φ has a global minimum on H. Indeed, there exists a $u_0 \in C$ such that

$$\varphi(u_0) = \inf_H \varphi.$$

In particular, $\varphi'(u_0) = 0$.

3. Main Results

We aim to use Theorem 1 to prove existence of at least a solution of the following problem:

$$\begin{cases} D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\right) = f\left(x,u(x)\right), \qquad x \in [0,T], \\ u(0) = u(T) = 0, \\ D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(0)\right) = D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(T)\right) = 0, \end{cases} \tag{3.1}$$

where $0 < \alpha \leq 1$ and $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $D_{0^+}^{\alpha}, D_{T^-}^{\alpha}$ are the left and right fractional Riemann-Liouville derivatives of order α, respectively. The idea is to reformulate the Problem (3.1) as a variational problem where we look for the critical point for some functional which will be defined in the sequel. Existence of such critical point is ensured by Theorem 1 when the nonlinear term f satisfies certain conditions. One can see that if u is a solution of the Problem ((3.1)), then u is a fixed point of the operator A defined by

$$\mathrm{A}u(x) = I_{0^+}^{\alpha}\left(I_{T^-}^{\alpha}\left(I_{0^+}^{\alpha}\left(I_{T^-}^{\alpha}f\left(x,u(x)\right)\right)\right)\right) \tag{3.2}$$

and satisfies the problem

$$\begin{cases} D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}Au(x)\right)\right)\right) = f\left(x,u(x)\right), \qquad x \in [0,T], \\ \mathrm{A}u(0) = \mathrm{A}u(T) = 0, \\ D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}Au(0)\right) = D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}Au(T)\right) = 0. \end{cases} \tag{3.3}$$

On the other hand, a weak solution of the Problem (3.1) in the space $E_0^{2\alpha}$ satisfies

$$\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right) \mathrm{d}x - \int_0^T f\left(x, u(x)\right) v(x) \,\mathrm{d}x = 0 \tag{3.4}$$

for all $v \in E_0^{2\alpha}$.

Proposition 6 ([10, Proposition 4.2]). *All critical point of the functional φ defined on $E_0^{2\alpha}$ by*

$$\varphi(u) = \frac{1}{2}\int_0^T |D_{T^-}^{\alpha}(D_{0^+}^{\alpha} u(t))|^2 \mathrm{d}t - \int_0^T F(t, u(t))\mathrm{d}t,$$

where $F(t,u) = \int_0^u f(t,s)\mathrm{d}s$ is a solution of the boundary value problem (3.1).

Let mention that by (3.3) (3.4), we have

$$\begin{aligned}
&\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right) \mathrm{d}x = \int_0^T f\left(x, u(x)\right) v(x) \,\mathrm{d}x \\
&= \int_0^T D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} \mathrm{A}u(x)\right)\right)\right) v(x) \,\mathrm{d}x,
\end{aligned}$$

then

$$\begin{aligned}
&\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right) \mathrm{d}x \\
&- \int_0^T D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} \mathrm{A}u(x)\right)\right)\right) v(x) \,\mathrm{d}x = 0.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right) \mathrm{d}x - \int_0^T f\left(x, u(x)\right) v(x) \,\mathrm{d}x = 0 \\
&\Leftrightarrow \quad \langle u, v\rangle_{2\alpha} - \langle \mathrm{A}u, v\rangle_{2\alpha} = 0 \quad \Leftrightarrow \quad \langle u - \mathrm{A}u, v\rangle_{2\alpha} = 0.
\end{aligned}$$

Beside, it was proven in [10] that if $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function such that

(C_1) $\left|f(x,y)\right| \leq a(t)|y|^{\theta_1} + b(t),$ with $a(t), b(t), \in L^1\left([0,T]\right),$ $\theta_1 > 0$ hold,

then the functional

$$\varphi(u) = \frac{1}{2}\|u\|_{2\alpha} - \int_0^T F\left(x, u(x)\right) \mathrm{d}x$$

is well defined and Gâteaux differentiable on the Hilbert space $E_0^{2\alpha}$ with

$$\begin{aligned}
\varphi'(u).v &= \langle u, v\rangle_{2\alpha} - \int_0^T f\left(x, u(x)\right) v(x) \,\mathrm{d}x, \qquad \forall v \in E_0^{2\alpha}, \\
&= \langle u - \mathrm{A}u, v\rangle_{2\alpha}.
\end{aligned}$$

That is

$$\varphi'(u) = (I - \mathrm{A})(u),$$

which means that a fixed point of A is a critical point of φ. In this sense, we will apply Theorem 1 to prove existence of critical point in $E_0^{2\alpha}$ for the functional φ. Let f' be a Carathéodory function, where f' is the derivative of f with respect to its second variable and satisfying:

(C_2) $\left|f'(x,y)\right| \leq p(t)|y|^{\theta_2} + q(t)$, with $p(t), q(t) \in L^1([0,T])$, $\theta_2 > 0$.

The following lemma is an important tool for our proof.

Lemma 4. *If $(C_1), (C_2)$ are satisfied, then φ is twice Gâteau differentiable and*

$$\left(\left(\varphi''(u)\right) v\right) z = \int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} z(x)\right)\right) \mathrm{d}x - \int_0^T f'\left(x, u(x)\right) v(x) z(x)\, \mathrm{d}x$$

for all $v, z \in E_0^{2\alpha}$.

Proof. Since (C_1) hold, then:

$$\varphi'(u).v = \langle u, v\rangle_{2\alpha} - \int_0^T f\left(x, u(x)\right) v(x)\, \mathrm{d}x,$$

for all $v \in E_0^{2\alpha}$. In a same way, if (C_2) hold, then φ' is Gâteaux differentiable. Setting

$$\mathrm{K}(u) = \frac{1}{2}\|u\|_{2\alpha}, \quad \mathrm{J}(u) = \int_0^T F\left(x, u(x)\right) \mathrm{d}x$$

and by using the calculation rules in [4], we have

$$\begin{aligned}\left(\left(\varphi''(u)\right) v\right) z &= \frac{d}{\mathrm{d}t}\left(\frac{d}{\mathrm{d}s}\mathrm{K}(u+tv+sz)\right)\Big|_{t=s=0} - \frac{d}{\mathrm{d}t}\left(\frac{d}{\mathrm{d}s}\mathrm{J}(u+tv+sz)\right)\Big|_{t=s=0} \\ &= \langle z, v\rangle_{2\alpha} - \int_0^T f'(x,u(x)) z(x) v(x)\, \mathrm{d}x, \qquad \forall v, z \in E_0^{2\alpha}.\end{aligned}$$

□

By a subsolution and a supersolution of (3.4), we mean respectively $\mu, \zeta \in E_0^{2\alpha}$ satisfying respectively

$$\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\mu(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right) \mathrm{d}x - \int_0^T f\left(x, \mu(x)\right) v(x)\, \mathrm{d}x \leq 0$$

and

$$\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\zeta(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha} v(x)\right)\right) \mathrm{d}x - \int_0^T f\left(x, \zeta(x)\right) v(x)\, \mathrm{d}x \geq 0$$

for all $v \in E_0^{2\alpha}$, $v \geq 0$. Now, we assume that

(C_3) For each fixed $x \in [0,T]$, $f(x,y)$ is a nondecreasing function of y for

$$\mu(x) \leq y \leq \zeta(x).$$

(C_4) $|f'(x,y)| \leq \dfrac{(\Gamma(\alpha+1))^4}{T^{4\alpha}}, \quad \forall x \in [0,T], \forall y \in \mathbb{R},\ 0 \leq \alpha < 1.$

The following theorem is our main result.

Theorem 2. *If $(C_1)-(C_4)$ hold, then Problem (3.1) has at least a weak solution in $E_0^{2\alpha}$.*

Proof. Let us consider the functional φ defined on $E_0^{2\alpha}$ by

$$\varphi(u) = \frac{1}{2}\|u\|_{2\alpha} - \int_0^T F(x, u(x))\, \mathrm{d}x$$

with $\varphi' = I - \mathrm{A}$, and A the operator defined by (3.2). We show that all conditions of Theorem 1 are satisfied.

(1) By Lemma 4, φ is twice Gâteaux differentiable.

(2) $\|(I' - \varphi'')u\| \leq 1$. Indeed, from Propositions 2, and 3 and Holder's inequality, we have

$$\begin{aligned}
\Big\|(I' - \varphi'')(u)\Big\| &= \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \Big|(I' - \varphi'')(u).v.z\Big|, \\
&\leq \frac{(\Gamma(\alpha+1))^4}{T^{4\alpha}} \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \left(\int_0^T |v(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^T |z(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{(\Gamma(\alpha+1))^4}{T^{4\alpha}} \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \|v\|_{L^2}.\|z\|_{L^2} \\
&\leq \frac{(\Gamma(\alpha+1))^4}{T^{4\alpha}} \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 \\
&\quad \cdot \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \|v\|_{2\alpha}.\|z\|_{2\alpha} \\
&\leq 1.
\end{aligned}$$

(3) Let

$$D = \big[\mu,\ \zeta\big] = \Big\{u \in E_0^{2\alpha} : \mu(x) \leq u(x) \leq \zeta(x),\ \forall x \in [0,T]\Big\}.$$

We can easily check that D is a bounded, closed and convex subset of the space $E_0^{2\alpha}$. Moreover,

$$\big(I - \varphi'(D)\big) \subset D.$$

Indeed, let $u \in D$, then from the assumption (C_3) and the definition of A, μ, ζ, we have

$$\langle \mu, v\rangle_{2\alpha} \leq \big((I - \varphi')(u), v\big) \leq \langle \zeta, v\rangle_{2\alpha}, \qquad \forall v \in E_0^{2\alpha},\ \text{and}\ \ v \geq 0.$$

Finally, by the maximum principle, we get

$$\mu \leq (I - \varphi')(u) \leq \zeta, \forall u \in D,$$

that is

$$(I-\varphi')(D)\subset D.$$

All conditions of Theorem 1 hold, then the functional φ has a critical point which is the weak solution of the Problem (3.1). $\square$

Example 1. Let us consider the following boundary value problem:

$$\begin{cases} D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u(x)\right)\right)\right)=f\left(x,u(x)\right), & x\in\left[0,\frac{\pi}{12}\right], \\ u(0)=u\left(\frac{\pi}{12}\right)=0, \\ D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u(0)\right)=D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u\left(\frac{\pi}{12}\right)\right)=0, \end{cases} \tag{3.5}$$

where

$$f(x,y)=\begin{cases} \frac{1}{2}\frac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}(y+1)\sin(x), & \text{if } x\in\left[0,\frac{\pi}{12}\right] \text{ and } y\le 0, \\ \frac{1}{2}\frac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}(y-1)\sin(x), & \text{if } x\in\left[0,\frac{\pi}{12}\right] \text{ and } y> 0. \end{cases}$$

For each fixed $x\in\left[0,\frac{\pi}{12}\right]$, $f(x,y)$ is a nondecreasing function of all $y\in\mathbb{R}$. Moreover, we can easily check that $\mu(x)=0$ and $\zeta(x)=\frac{\pi}{15}$ are subsolution and supersolution of (3.5), respectively. Indeed, we have

$$\begin{aligned} f(x,0)&=\frac{1}{2}\frac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}\sin(x)\ge 0=D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u(0)\right)\right)\right) \\ &=D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\mu(x)\right)\right)\right). \end{aligned}$$

On the other hand,

$$f\left(x,\zeta(x)\right)=f\left(x,\frac{\pi}{15}\right)=\frac{1}{2}\frac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}\left(\frac{\pi}{15}-1\right)\sin(x)<0.$$

By Property 1 and since $x\in\left[0,\frac{\pi}{12}\right]$, we have

$$D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\zeta(x)\right)\right)\right)=\frac{\frac{\pi}{15}}{\Gamma\left(\frac{1}{5}\right)}(x)^{\frac{-4}{5}}>0,$$

then

$$D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\zeta(x)\right)\right)\right)\ge f\left(x,\zeta(x)\right).$$

All the assumptions in the statement of Theorem 2 are satisfied, so Problem (3.5) has at least a nontrivial solution.

ACKNOWLEDGEMENTS

The authors express their gratitude to the anonymous referees for their helpful suggestions and corrections.

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