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CHARACTERIZATION OF LIPSCHITZ FUNCTIONS VIA COMMUTATORS OF MAXIMAL OPERATORS ON SLICE SPACES

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Abstract. Let $0 \leq \alpha < n$, M_α be the fractional maximal operator, $M^\sharp$ be the sharp maximal operator and b be the locally integrable function. Denote by $[b, M_\alpha]$ and $[b, M^\sharp]$ be the commutators of the fractional maximal operator M_α and the sharp maximal operator $M^\sharp$. In this paper, we show some necessary and sufficient conditions for the boundedness of the commutators $[b, M_\alpha]$ and $[b, M^\sharp]$ on slice spaces when the function b is the Lipschitz function, by which some new characterizations of the non-negative Lipschitz function are obtained.

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1. Introduction and main results

Let T be the classical singular integral operator and b be the locally integrable function, the commutator $[b, T]$ is defined by

$$[b, T]f(x) = bTf(x) - T(bf)(x).$$

The well-known result of Coifman, Rochberg and Weiss[6] showed that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ was introduced by John and Nirenberg [14], which is defined as the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $f_Q := \frac{1}{|Q|} \int_Q f(x) dx$. In 1978, Janson[12] obtained some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the commutator $[b, T]$ and proved that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and

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only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $(0<\beta<1)$, where $1<p<n/\beta$ and $1/p-1/q=\beta/n$ (see also Paluszyński[17]). Recently, the commutators have been studied intensively by many authors, which plays an important role in harmonic analysis and partial differential equations (see, for example, [1,5,10,11,19,21]).

As usual, a cube $Q \subset \mathbb{R}^n$ always means its sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure of Q and χ_Q the characteristic function of Q. For $1 \le p \le \infty$, we denote by p' the conjugate index of p, namely, $p'=p/(p-1)$. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$.

Let $0 \le \alpha < n$, for a locally integrable function f, the maximal operator M_α is given by

$$M_\alpha(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|^{1-\alpha/n}}\int_Q|f(y)|dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

When $\alpha=0$, M_0 is the classical Hardy-Littlewood maximal operator M, and M_α is the classical fractional maximal operator when $0<\alpha<n$.

The sharp maximal operator $M^\sharp$ was introduced by Fefferman and Stein [9], which is defined as

$$M^\sharp f(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|\,dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The maximal commutator of the fractional maximal operator M_α with the locally integrable function b is given by

$$M_{\alpha,b}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|^{1-\alpha/n}}\int_Q|b(x)-b(y)||f(y)|dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The nonlinear commutators of the fractional maximal operator M_α and sharp maximal operator $M^\sharp$ with the locally integrable function b are defined as

$$[b,M_\alpha](f)(x)=b(x)M_\alpha(f)(x)-M_\alpha(bf)(x)$$

and

$$[b,M^\sharp](f)(x)=b(x)M^\sharp(f)(x)-M^\sharp(bf)(x).$$

When $\alpha=0$, we simply write by $[b,M]=[b,M_0]$ and $M_b=M_{0,b}$. We also remark that the commutators $M_{\alpha,b}$ and $[b,M_\alpha]$ essentially differ from each other. For example, maximal commutator $M_{\alpha,b}$ is positive and sublinear, but nonlinear commutators $[b,M_\alpha]$ and $\left[b,M^\sharp\right]$ are neither positive nor sublinear. The study of the mapping properties of commutators of maximal operators has been widely explored, we refer the readers to see [8,18,20,22,23,25] and therein references.

To state our results, we first present some definitions and notations.

Definition 1. Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C|x-y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of the function b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

In 2019, Auscher and Mourgoglou [2] introduced the slice space $(E_2^p)_t(\mathbb{R}^n)$ with $0 < t < \infty$ and $1 < p < \infty$, they studied the weak solutions of boundary value problems with a t-independent elliptic systems in the upper half plane. Recently, Auscher and Prisuelos-Arribas[3] obtained the boundedness of some classical operators on the slice space $(E_r^p)_t(\mathbb{R}^n)$ with $0 < t < \infty$ and $1 < p, r < \infty$.

For $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Definition 2. Let $0 < t < \infty$ and $1 < r, p < \infty$. The slice space $(E_r^p)_t(\mathbb{R}^n)$ is defined as the set of all locally r-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E_r^p)_t(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(\frac{1}{|Q(x,t)|}\int_{Q(x,t)} |f(y)|^r dy\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} < \infty.$$

If we take $r = p$, then the slice space $(E_r^p)_t(\mathbb{R}^n)$ is the Lebesgue space $L^p(\mathbb{R}^n)$. For a cube Q, we denote by $\|f\|_{(E_r^p)_t(Q)} = \|f\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)}$.

For a fixed cube Q and $0 \leq \alpha < n$, the maximal operator with respect to Q of a function f is given by

$$M_{\alpha,Q}(f)(x) = \sup_{Q \supseteq Q_0 \ni x} \frac{1}{|Q_0|^{1-\alpha/n}} \int_{Q_0} |f(y)| dy,$$

where the supremum is taken over all the cubes Q_0 with $Q_0 \subseteq Q$ and $Q_0 \ni x$. Moreover, we denote by $M_Q = M_{0,Q}$ when $\alpha = 0$.

In 2017, Zhang [24] showed some characterizations via the boundedness of the commutator $[b, M]$ on Lebesgue spaces, where the function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.

Theorem 1. *[24] Let $0 < \beta < 1$ and b be a locally integrable function. If $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$, then the following statements are equivalent:*

(1) *$b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.*

(2) *$[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

(3) *there exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_{Q} |b(x) - M_{Q}(b)(x)|^{q} \, \mathrm{d}x \right)^{1/q} \leq C.$$

Next, we recall the result of [22], which showed some characterizations via the boundedness of the commutator $[b,M]$ on slice spaces, where the function b belongs to the Lipschitz space $\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ and $b \geq 0$.

Theorem 2. *[22] Let* $0 < \beta < 1$, $0 < t < \infty$ *and* b *be a locally integrable function. If* $1 < p < r < \infty$, $1 < q < s < \infty$ *and* $\beta/n = 1/p - 1/r = 1/q - 1/s$, *then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *and* $b \geq 0$.
(2) $[b,M]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.
(3) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_{Q}(b)(\cdot)\|_{(E_r^s)_t(Q)} \leq C.$$

Our first result can be stated as follows.

Theorem 3. *Let* $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $0 < t < \infty$ *and* b *be a locally integrable function. If* $1 < p < r < \infty$, $1 < q < s < \infty$ *and* $(\alpha+\beta)/n = 1/p - 1/r = 1/q - 1/s$, *then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *and* $b \geq 0$.
(2) $[b,M_{\alpha}]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.
(3) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_{Q}(b)(\cdot)\|_{(E_r^s)_t(Q)} \leq C. \tag{1.1}$$

(4) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|^{1+\beta/n}} \int_{Q} |b(x) - M_{Q}(b)(x)| \, dx \leq C. \tag{1.2}$$

Here is the second result.

Theorem 4. *Let* $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $0 < t < \infty$ *and* b *be a locally integrable function. If* $1 < p < r < \infty$, $1 < q < s < \infty$ *and* $(\alpha+\beta)/n = 1/p - 1/r = 1/q - 1/s$, *then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$.
(2) $M_{\alpha,b}$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.
(3) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - b_{Q}\|_{(E_r^s)_t(Q)} \leq C. \tag{1.3}$$

(4) *There exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{1+\beta/n}}\int_{Q}|b(x)-b_{Q}|\,dx\leq C. \tag{1.4}$$

Finally, we obtain the following result.

Theorem 5. *Let* $0<\beta<1$, $0<t<\infty$ *and* b *be a locally integrable function. If* $1<p<r<\infty$, $1<q<s<\infty$ *and* $\beta/n=1/p-1/r=1/q-1/s$, *then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_{\beta}(\mathbb{R}^{n})$ *and* $b\geq 0$.

(2) $\left[b,M^{\sharp}\right]$ *is bounded from* $(E_{p}^{q})_{t}(\mathbb{R}^{n})$ *to* $(E_{r}^{s})_{t}(\mathbb{R}^{n})$.

(3) *There exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{\beta/n+1/s}}\left\|b(\cdot)-2M^{\sharp}(b\chi_{Q})(\cdot)\right\|_{(E_{r}^{s})_{t}(Q)}\leq C. \tag{1.5}$$

(4) *There exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{1+\beta/n}}\int_{Q}\left|b(x)-2M^{\sharp}(b\chi_{Q})(x)\right|dx\leq C. \tag{1.6}$$

2. PRELIMINARIES

To prove our results, we give some necessary lemmas in this section. It is well-known that the Lipschitz space $\dot{\Lambda}_{\beta}(\mathbb{R}^{n})$ coincides with some Morrey-Companato spaces (see [13] for example) and can be characterized by mean oscillation as the following lemma, which is due to DeVore and Sharpley [7] and Paluszyński [17].

Lemma 1. *Let* $0<\beta<1$ *and* $1\leq q<\infty$. *The space* $\dot{\Lambda}_{\beta,q}(\mathbb{R}^{n})$ *is defined as the set of all locally integrable functions* f *such that*

$$\|f\|_{\dot{\Lambda}_{\beta,q}}=\sup_{Q}\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_{Q}|f(x)-f_{Q}|^{q}\,dx\right)^{1/q}<\infty.$$

Then, for all $0<\beta<1$ *and* $1\leq q<\infty, \dot{\Lambda}_{\beta}(\mathbb{R}^{n})=\dot{\Lambda}_{\beta,q}(\mathbb{R}^{n})$ *with equivalent norms.*

Lemma 2. *[26] Let* $0\leq\alpha<n$, Q *be a cube in* $\mathbb{R}^{n}$ *and* f *be locally integrable. Then*

$$M_{\alpha}(f\chi_{Q})(x)=M_{\alpha,Q}(f)(x),\ \textit{for all } x\in Q.$$

The following lemma is given by Lu, Wang and Zhou[15], they obtained that the boundedness of the fractional maximal operator M_{α} on slice spaces.

Lemma 3. *Let* $0<t<\infty$, $1<p<r<\infty$ *and* $1<q<s<\infty$ *with* $\alpha/n=1/p-1/r=1/q-1/s$ *for* $0\leq\alpha<n$. *If* $f\in(E_{p}^{q})_{t}(\mathbb{R}^{n})$, *then*

$$\|M_{\alpha}f\|_{(E_{r}^{s})_{t}(\mathbb{R}^{n})}\leq C\|f\|_{(E_{p}^{q})_{t}(\mathbb{R}^{n})},$$

where the positive constant C *is independent of* f *and* t.

Lemma 4. *[16] Let $0<t<\infty$, $1<p,r<\infty$ and Q be a cube in $\mathbb{R}^n$. Then*

$$\|\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)} \sim |Q|^{1/p}.$$

Lemma 5. *[4] For any fixed cube Q, let $E=\{x\in Q: b(x)\le b_Q\}$ and $F=\{x\in Q: b(x)>b_Q\}$. Then the following equality is true:*

$$\int_E |b(x)-b_Q|\,dx = \int_F |b(x)-b_Q|\,dx.$$

3. Proofs of Theorems 3-5

Proof of Theorem 3. (1) $\Rightarrow$ (2): Assume $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\ge 0$. For any locally integral function f, we have

$$\begin{aligned}
|[b,M_\alpha](f)(x)| &= |b(x)M_\alpha(f)(x)-M_\alpha(bf)(x)|\\
&\le \sup_{Q\ni x}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b(x)-b(y)||f(y)|dy\\
&\le C\|b\|_{\dot{\Lambda}_\beta}\sup_{Q\ni x}\frac{1}{|Q|^{1-(\alpha+\beta)/n}}\int_Q |f(y)|dy\\
&\le C\|b\|_{\dot{\Lambda}_\beta}M_{\alpha+\beta}(f)(x).
\end{aligned}$$

By Lemma 3, we obtain that $[b,M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3): We divide the proof into two cases based on the scope of α.

Case 1. Assume $0<\alpha<n$. For any fixed cube Q,

$$\begin{aligned}
&\frac{1}{|Q|^{\beta/n+1/s}}\|b(\cdot)-M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)}\\
&\le \frac{1}{|Q|^{\beta/n+1/s}}\left\|b(\cdot)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(\cdot)\right\|_{(E_r^s)_t(Q)}\\
&\quad+\frac{1}{|Q|^{\beta/n+1/s}}\left\||Q|^{-\alpha/n}M_{\alpha,Q}(b)(\cdot)-M_Q(b)(\cdot)\right\|_{(E_r^s)_t(Q)}\\
&:= I+II.
\end{aligned}$$

For I. By the definition of $M_{\alpha,Q}$, we can see

$$M_{\alpha,Q}(\chi_Q)(x)=|Q|^{\alpha/n},\ \text{for all } x\in Q. \tag{3.1}$$

Using Lemma 2, for any $x\in Q$, we have

$$M_\alpha(\chi_Q)(x)=M_{\alpha,Q}(\chi_Q)(x)=|Q|^{\alpha/n}, M_\alpha(b\chi_Q)(x)=M_{\alpha,Q}(b)(x).$$

Thus, for any $x\in Q$,

$$\begin{aligned}
b(x)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(x) &= |Q|^{-\alpha/n}\left(b(x)|Q|^{\alpha/n}-M_{\alpha,Q}(b)(x)\right)\\
&= |Q|^{-\alpha/n}\left(b(x)M_\alpha(\chi_Q)(x)-M_\alpha(b\chi_Q)(x)\right)
\end{aligned}$$

$$= |Q|^{-\alpha/n}\,[b,M_\alpha]\,(\chi_Q)\,(x).$$

Since $[b,M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then by Lemma 4 and noting that $(\alpha+\beta)/n = 1/q-1/s$, we have

$$\begin{aligned} I &= |Q|^{-\beta/n-1/s}\left\|b(\cdot)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(\cdot)\right\|_{(E_r^s)_t(Q)} \\ &= |Q|^{-(\alpha+\beta)/n-1/s}\left\|[b,M_\alpha]\,(\chi_Q)\,(\cdot)\right\|_{(E_r^s)_t(Q)} \\ &\le C|Q|^{-(\alpha+\beta)/n-1/s}\left\|\chi_Q\right\|_{(E_p^q)_t(\mathbb{R}^n)} \\ &\le C. \end{aligned}$$

Next, we estimate II. Similar to (3.1), by Lemma 2.3 and noting that

$$M_Q\,(\chi_Q)\,(x) = \chi_Q(x), \text{ for all } x\in Q,$$

it is easy to see

$$M\,(\chi_Q)\,(x) = \chi_Q(x) \text{ and } M\,(b\chi_Q)\,(x) = M_Q(b)(x), \text{for any } x\in Q. \tag{3.2}$$

Then, by (3.1) and (3.2), for any $x\in Q$, we obtain

$$\begin{aligned} &\left||Q|^{-\alpha/n}M_{\alpha,Q}(b)(x)-M_Q(b)(x)\right| \\ &\quad\le |Q|^{-\alpha/n}\left|M_\alpha\,(b\chi_Q)\,(x)-|b(x)|M_\alpha\,(\chi_Q)\,(x)\right| \\ &\qquad + |Q|^{-\alpha/n}\left||b(x)|\,M_\alpha\,(\chi_Q)\,(x)-M_\alpha\,(\chi_Q)\,(x)M\,(b\chi_Q)\,(x)\right| \\ &\quad= |Q|^{-\alpha/n}\left|M_\alpha\,(|b|\chi_Q)\,(x)-|b(x)|\,M_\alpha\,(\chi_Q)\,(x)\right| \\ &\qquad + |Q|^{-\alpha/n}M_\alpha\,(\chi_Q)\,(x)\left||b(x)|\,M\,(\chi_Q)\,(x)-M\,(b\chi_Q)\,(x)\right| \\ &\quad= |Q|^{-\alpha/n}\left|[|b|,M_\alpha](\chi_Q)(x)\right| + \left|[|b|,M]\,(\chi_Q)\,(x)\right|. \end{aligned}$$

Since $[b,M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$ and we can see that $b\in\dot{\Lambda}_\beta\,(\mathbb{R}^n)$ implies $|b|\in\dot{\Lambda}_\beta\,(\mathbb{R}^n)$. By the definitions of $[b,M_\alpha]$ and M_α, we have, for any $x\in Q$,

$$\begin{aligned} \left|[|b|,M_\alpha]\,(\chi_Q)\,(x)\right| &\le \sup_{Q'\ni x}\frac{1}{|Q'|^{1-\alpha/n}}\int_{Q'}|b(x)-b(y)||\chi_Q(y)|\,dy \\ &\le \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\sup_{Q'\ni x}\frac{1}{|Q'|^{1-(\alpha+\beta)/n}}\int_{Q'}|\chi_Q(y)|\,dy \\ &\le \|b\|_{\dot{\Lambda}_\beta}M_{\alpha+\beta}\,(\chi_Q)\,(x) \\ &= \|b\|_{\dot{\Lambda}_\beta}|Q|^{(\alpha+\beta)/n}\chi_Q(x). \end{aligned}$$

Similarly, we can see

$$\left|[|b|,M](\chi_Q)(x)\right| \le \|b\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\chi_Q(x), \text{ for any } x\in Q.$$

Thus, for any $x \in Q$,

$$\left| |Q|^{-\alpha/n} M_{\alpha,Q}(b)(x) - M_Q(b)(x) \right| \le C \|b\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \chi_Q(x).$$

Then, by Lemma 4, we have

$$\begin{aligned} II &= |Q|^{-\beta/n-1/s} \left\| |Q|^{-\alpha/n} M_{\alpha,Q}(b)(\cdot) - M_Q(b)(\cdot) \right\|_{(E_r^s)_t(Q)} \\ &\le C|Q|^{-1/s} \|\chi_Q\|_{(E_r^s)_t(Q)} \le C. \end{aligned}$$

This gives the desired estimate

$$|Q|^{-\beta/n-1/s} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \le C,$$

which leads us to (1.1) since Q is arbitrary and the constant C is dependent of Q.

Case 2. Assume $\alpha = 0$. For any fixed cube Q and any $x \in Q$, by (3.2), we can see

$$b(x) - M_Q(b)(x) = b(x) M(\chi_Q)(x) - M(b\chi_Q)(x) = [b, M](\chi_Q)(x).$$

Assume that $[b, M]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$ and $\beta/n = 1/q - 1/s$, then by Lemma 4, we have

$$\begin{aligned} |Q|^{-\beta/n-1/s} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} &= |Q|^{-\beta/n-1/s} \|[b, M](\chi_Q)(\cdot)\|_{(E_r^s)_t(Q)} \\ &\le C|Q|^{-\beta/n-1/s} \|\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)} \le C, \end{aligned}$$

which implies (1.1).

(3) $\Rightarrow$ (4): Assume (1.1) holds, then for any fixed cube Q, by Hölder's inequality and (1.1), we can see

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b)(x)| \, dx \\ &\quad \le \frac{C}{|Q|^{1+\beta/n}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)} \\ &\quad \le \frac{C}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \le C, \end{aligned}$$

where the constant C is independent of Q. Thus we have (1.2).

(4) $\Rightarrow$ (1): To prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, by Lemma 1, it suffices to show that there is a constant $C > 0$ such that for any fixed cube Q,

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| \, dx \le C.$$

For any fixed cube Q, let $E = \{x \in Q : b(x) \le b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. Since for any $x \in E$, we have $b(x) \le b_Q \le M_Q(b)(x)$, then

$$|b(x) - b_Q| \le |b(x) - M_Q(b)(x)|. \tag{3.3}$$

By Lemma 5 and (3.3), we obtain

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,\mathrm{d}x &= \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-b_Q|\,\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-M_Q(b)(x)|\,\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,\mathrm{d}x \le C.\end{aligned}$$

Thus we obtain $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. Next, we will prove $b\ge 0$, it suffices to show $b^-=0$, where $b^-=-\min\{b,0\}$. Let $b^+=|b|-b^-$, then $b=b^+-b^-$. For any fixed cube Q and $x\in Q$, we observe that

$$0\le b^+(x)\le |b(x)|\le M_Q(b)(x),$$

then it is easy to see

$$0\le b^-(x)\le M_Q(b)(x)-b^+(x)+b^-(x)=M_Q(b)(x)-b(x).$$

Combining with the above estimates and (1.2), we obtain

$$\begin{aligned}\frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x &\le \frac{1}{|Q|}\int_Q |M_Q(b)(x)-b(x)|\\ &\le |Q|^{\beta/n}\left(\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,dx\right)\le C|Q|^{\beta/n}.\end{aligned}$$

Thus, $b^-=0$ follows from Lebesgue's differentiation theorem.

This completes the proof of Theorem 3. □

Proof of Theorem 4. (1) ⇒ (2): Assume $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q\subset\mathbb{R}^n$, we have

$$\begin{aligned}M_{\alpha,b}(f)(x) &= \sup_{Q\ni x}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b(x)-b(y)||f(y)|dy\\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}M_{\alpha+\beta}f(x).\end{aligned}$$

By Lemma 3, we obtain that $M_{\alpha,b}$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) ⇒ (3): For any fixed cube $Q\subset\mathbb{R}^n$ and all $x\in Q$, we have

$$\begin{aligned}|b(x)-b_Q| &\le \frac{1}{|Q|}\int_Q |b(x)-b(y)|dy\\ &= \frac{1}{|Q|^{\alpha/n}}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b(x)-b(y)|\chi_Q(y)dy\\ &\le |Q|^{-\alpha/n}M_{\alpha,b}(\chi_Q)(x).\end{aligned}$$

Since $M_{\alpha,b}$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then by Lemma 4 and noting that $(\alpha+\beta)/n = 1/q - 1/s$, we obtain

$$\begin{aligned}\frac{1}{|Q|^{\beta/n+1/s}}\|b(\cdot)-b_Q\|_{E_r^s)_t(Q)} &\leq |Q|^{-(\alpha+\beta)/n-1/s}\|M_{\alpha,b}(\chi_Q)(\cdot)\|_{(E_r^s)_t(Q)}\\ &\leq C|Q|^{-(\alpha+\beta)/n-1/s}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)} \leq C,\end{aligned}$$

which implies (1.3) since the cube $Q\subset\mathbb{R}^n$ is arbitrary.

(3) $\Rightarrow$ (4): Assume (1.3) holds, we will prove (1.4). For any fixed cube Q, by Hölder's inequality and Lemma 4, it is easy to see

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,dx &\leq \frac{C}{|Q|^{1+\beta/n}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)}\|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)}\\ &\leq \frac{C}{|Q|^{\beta/n+1/s}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)} \leq C.\end{aligned}$$

(4) $\Rightarrow$ (1): It follows from Lemma 1 directly, thus we omit the details.

The proof of Theorem 4 is finished. □

Proof of Theorem 5. (1) $\Rightarrow$ (2): Assume $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\geq 0$. For any locally integral function f, the following estimate was given in [25]:

$$\left|[b,M^\sharp]f(x)\right| \leq C\|b\|_{\dot{\Lambda}_\beta}M_\beta(f)(x).$$

Then, by Lemma 3, we obtain that $\left[b,M^\sharp\right]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3): Assume $[b,M^\sharp]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, we will prove (1.5). For any fixed cube Q, we have (see [4] for details)

$$M^\sharp(\chi_Q)(x) = 1/2, \text{ for all } x\in Q.$$

Then, for all $x\in Q$,

$$\begin{aligned}b(x)-2M^\sharp(b\chi_Q)(x) &= 2\left(b(x)M^\sharp(\chi_Q)(x) - M^\sharp(b\chi_Q)(x)\right)\\ &= 2[b,M^\sharp](\chi_Q)(x).\end{aligned}$$

Since $[b,M^\sharp]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then applying Lemma 4 and noting that $\beta/n = 1/q-1/s$, we obtain

$$\begin{aligned}|Q|^{-\beta/n-1/s}\left\|b(\cdot)-2M^\sharp(b\chi_Q)(\cdot)\right\|_{(E_r^s)_t(Q)} &= 2|Q|^{-\beta/n-1/s}\left\|[b,M^\sharp](\chi_Q)\right\|_{(E_r^s)_t(Q)}\\ &\leq C|Q|^{-\beta/n-1/s}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)} \leq C,\end{aligned}$$

where the constant C is independent of Q. Then we achieve (1.5).

(3) $\Rightarrow$ (4): Assume (1.5) holds, we will prove (1.6). For any fixed cube Q, it follows from Hölder's inequality and (1.5) that

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q \left|b(x)-2M^\sharp(b\chi_Q)(x)\right|dx$$

$$\leq C|Q|^{-\beta/n-1/s}\left\|b(\cdot)-2M^{\sharp}\left(b\chi_{Q}\right)(\cdot)\right\|_{(E_{p}^{q})_{t}(Q)}\leq C,$$

which implies (1.6) since the constant C is independent of Q.

$(4)\Rightarrow(1)$: We first prove $b\in\dot{\Lambda}_{\beta}\left(\mathbb{R}^{n}\right)$. For any fixed cube Q, the following estimate was given in [4]:

$$\frac{1}{|Q|}\int_{Q}|b(x)-b_{Q}|\,dx\leq\frac{2}{|Q|}\int_{Q}\left|b(x)-2M^{\sharp}\left(b\chi_{Q}\right)(x)\right|dx.$$

Then by (1.6), we have

$$\frac{1}{|Q|^{1+\beta/n}}\int_{Q}|b(x)-b_{Q}|\,dx\leq\frac{2}{|Q|^{1+\beta/n}}\int_{Q}\left|b(x)-2M^{\sharp}\left(b\chi_{Q}\right)(x)\right|dx\leq C,$$

which leads to $b\in\dot{\Lambda}_{\beta}\left(\mathbb{R}^{n}\right)$ by Lemma 1.

Now, let us prove $b\geq0$. It suffices to show $b^{-}=0$, where $b^{-}=-\min\{b,0\}$ and let $b^{+}=|b|-b^{-}$. For any fixed cube Q, we have (see [4] for details)

$$|b_{Q}|\leq2M^{\sharp}\left(b\chi_{Q}\right)(x),\text{ for any }x\in Q.$$

Then, for all $x\in Q$,

$$2M^{\sharp}\left(b\chi_{Q}\right)(x)-b(x)\geq|b_{Q}|-b(x)=|b_{Q}|-b^{+}(x)+b^{-}(x).$$

By (1.6), we obtain

$$|b_{Q}|-\frac{1}{|Q|}\int_{Q}b^{+}(x)dx+\frac{1}{|Q|}\int_{Q}b^{-}(x)dx\leq C|Q|^{\beta/n},\tag{3.4}$$

where the constant C is independent of Q.

Let the side length of Q tends to 0 (then $|Q|\to0$) with $x\in Q$. By Lebesgue's differentiation theorem, we obtain that the limit of the left-hand side of (3.4) equals to

$$|b(x)|-b^{+}(x)+b^{-}(x)=2b^{-}(x)=2\left|b^{-}(x)\right|.$$

Moreover, the right-hand side of (3.4) tends to 0. Thus, we have $b^{-}=0$.

The proof of Theorem 5 is completed. □

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