Miskolc Mathematical Notes
Vol. 27 (2026), No. 1, pp. 59–75
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2026.5213

NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEMS FOR SEQUENTIAL DIFFERENTIAL EQUATION INVOLVING A FRACTIONAL MIXED DERIVATIVES

BELQASSIM AZZOUZ, MOUSTAFA BEDDANI, AND HAMID BEDDANI

Received 26 November, 2024

Abstract. The study of fractional differential equations occupies an important place in various fields of science. In this paper, we investigate the existence result for a nonlocal integral boundary value problems for a sequential differential equation involving a fractional mixed derivatives. Our method consists to define an extended space on which we can apply the Mönch fixed point theorem via the noncompactness measure. In addition, the compactness of the solution set is studied using the sequential method. Finally, an example is given to illustrate the results obtained.

2010 *Mathematics Subject Classification:* 26A33; 34A08; 47H08; 47H10

Keywords: measure of noncompactness, Banach space, fixed point theorem, χ-Hilfer fractional derivative

1. INTRODUCTION

The aim of this paper is to study the existence and the compactness of the solution set for a sequential fractional differential equation with nonlocal integral boundary value conditions. More precisely, we consider the following problem:

$$ {}^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^{+}}\Big(D^{\chi}y(\xi)-\kappa y(\xi)\Big)=\hbar\Big(\xi,y(\xi),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^{+}}y(\xi)\Big),\ \ \xi\in(\underline{\xi},\overline{\xi}], \tag{1.1}$$

$$ I^{1-\gamma,\chi}_{\underline{\xi}^{+}}D^{\chi}y(\underline{\xi}^{+})=\sum_{i=1}^{n}\zeta_i D^{\chi}y(\xi_i),\quad y(\underline{\xi})=0, \tag{1.2}$$

where:

- κ is a real number,
- ${}^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^{+}}$ denotes the χ-Hilfer fractional derivative of order ρ and parameter σ such that $0<\rho<1$ and $0\le\sigma\le 1$,
- ${}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^{+}}$ is the χ-Caputo fractional derivative of order $\gamma=\rho+\sigma-\sigma\rho$,

- E is a Banach space and $\hbar\colon (\underline{\xi},\overline{\xi}] \times E^2 \to E$ is a function that satisfies certain conditions (see Section 3),
- $\chi \in \mathcal{C}^1([\underline{\xi},\overline{\xi}],\mathbb{R})$ such that $\chi'(\xi) > 0$ for all $\xi \in [\underline{\xi},\overline{\xi}]$,
- $D^\chi = \frac{1}{\chi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}$, $\underline{\xi}$, $\overline{\xi} \in \mathbb{R}_+^*$ with $\underline{\xi} < \overline{\xi}$ and $\xi_i \in (\underline{\xi},\overline{\xi}), i = 1,\ldots,n$ such that

$$\Gamma(\gamma) \neq \sum_{i=1}^{n} \zeta_i (\chi(\xi_i) - \chi(\underline{\xi}))^{\gamma-1},$$

  where Γ is the gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \ \ (x > 0)$.

The domain of fractional differential equations becomes a very important tool for understanding many physical phenomena. Moreover, their contributions to mathematical analysis help us to obtain certain appreciable results in the economic and engineering fields. For details, we refer the reader to [3, 4, 6, 14, 17, 20, 22, 23].

On the other hand, fractional differential equations with nonlocal conditions are of great importance in several branches of applied analysis. For example, in [13], the author claimed that the nonlocal conditions can be more effective than others to describe some physical situations. Furthermore, there is an extensive literature that focused on the study of the existence, uniqueness and stability for nonlocal fractional differential equations involving Riemann and Hilfer derivatives [7, 8, 26]. In the references [5, 10, 11], the authors studied the topological properties of some fractional differential equations, especially the compactness and the stability.

Recently, in [24], the authors studied the following nonlocal boundary value problems of sequential $\psi-$Hilfer-type fractional differential equations:

$$\left({}^H\mathcal{D}^{\alpha,\beta,\psi} + k\,{}^H\mathcal{D}^{\alpha-1,\beta,\psi}\right)x(t) = f\Big(t,x(t)\Big), \quad t \in [a,b],$$

$$x(a) = 0 \ \text{ and } \ x(b) = \sum_{i=1}^{n} \mu_i \int_a^{\varsigma_i} \psi'(s)x(s)\mathrm{d}s + \sum_{j=1}^{m} \theta_j x(\xi_j),$$

where ${}^H\mathcal{D}^{\alpha,\beta,\psi}$ is the ψ-Hilfer fractional derivative of order α, $1 < \alpha < 2$ and parameter β, $0 \leq \beta \leq 1$, $k \in \mathbb{R}$, $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$, μ_i, $\theta_j \in \mathbb{R}$, ς_i, $\xi_j \in (a,b]$ and ψ is a positive increasing function on $(a,b]$, which has a continuous derivative $\psi'(t)$ on (a,b). See also the work discussed by Ragusa [21], on the inclusion of the commutators of fractional integral operators to vanishing Morrey spaces. For other interesting papers which consider fractional differential problems, we mention [1, 12, 15, 16, 19].

The present work is organized as follows: In Section 2, we give some general results and preliminaries. The Section 3 presents two important results concerning the existence of solutions and compactness of (1.1)-(1.2) applying the fixed point theorem. An example to reinforce our work in Section 4.

2. Basic results and background

In this section, we will give some concepts and notations about the functional spaces, fractional calculus, noncompactness measure which are used throughout this paper. we denote by $\mathcal{C}([\underline{\xi},\overline{\xi}])$ (resp. by $L^1([\underline{\xi},\overline{\xi}])$) the space of E-valued continuous functions (resp. the space of E-Bochner's integrable functions) with the following norm

$$\|u\|_\infty = \sup\Big\{\|u(\xi)\|,\ \xi \in [\underline{\xi},\overline{\xi}]\Big\}\ \Big(\text{resp. } \|u\|_{L^1} = \int_{\underline{\xi}}^{\overline{\xi}} \|u(\xi)\| \mathrm{d}\xi\Big).$$

Let $\mathcal{C}_{1-\gamma}([\underline{\xi},\overline{\xi}])$ be the Banach spaces of functions from $(\underline{\xi},\overline{\xi}]$ into E which is defined as:

$$\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]) = \Big\{u \in \mathcal{C}((\underline{\xi},\overline{\xi}]):\ (\chi(.)-\chi(\underline{\xi}))^{1-\gamma}u(.) \in \mathcal{C}([\underline{\xi},\overline{\xi}],E)\Big\}.$$

with his norm $\|u\|_{\gamma,\chi}$, that is given by

$$\|u\|_{\gamma;\chi} = \sup_{\xi\in(\underline{\xi},\overline{\xi}]} (\chi(\xi)-\chi(\underline{\xi}))^{1-\gamma}\|u(\xi)\|.$$

Next, we denote by $\mathcal{C}^1_{1-\gamma;\chi}((\underline{\xi},\overline{\xi}])$ the space of functions (γ,χ) - continuously differentiable defined as follows

$$\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]) = \Big\{u\colon (\underline{\xi},\overline{\xi}] \to E:\ u(.) \in \mathcal{C}([\underline{\xi},\overline{\xi}]) \text{ and } D^\chi u(.) \in \mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\Big\}.$$

We note that the space $\mathcal{C}^1_{1-\gamma;\chi}((\underline{\xi},\overline{\xi}])$ with the norm $\|u\|^1_{\gamma;\chi} = \|u\|_\infty + \|D^\chi u\|_{\gamma;\chi}$ is a Banach space.

In the following, for all $\eta > -1$, we put $\Psi_\eta(r,s) = (\chi(r)-\chi(s))^\eta$, for all $s,r \in [\underline{\xi},\overline{\xi}]$ with $r > s$ and $\Psi^*_\eta = (\chi(\overline{\xi})-\chi(\underline{\xi}))^\eta$.

First, we introduce the notions of χ-fractional derivative according to the Riemann-Liouville and Hilfer concept and their properties.

Definition 1 ([17, 25])**.** Let $\ell \in L^1([\underline{\xi},\overline{\xi}])$ and $\chi \in \mathcal{C}^1([\underline{\xi},\overline{\xi}])$ such that $\chi'(\xi) > 0$, for all $\xi \in [\underline{\xi},\overline{\xi}]$,

(i) the χ-Riemann- Liouville fractional integral of order $\rho > 0$ of the function ℓ is defined by

$$\mathfrak{I}^{\rho;\chi}_{\underline{\xi}^+}\ell(\xi) = \frac{1}{\Gamma(\rho)} \int_{\underline{\xi}}^{\xi} \chi'(s)\Psi_{\rho-1}(\xi,s)\ell(s)\mathrm{d}s,$$

(ii) the χ-Riemann- Liouville fractional derivative of order $\rho > 0$ of the function ℓ is defined by

$$^{RL}\mathcal{D}^{\rho;\chi}_{\underline{\xi}^+}\ell(\xi) = \frac{1}{\Gamma(n-\rho)} \left(\frac{1}{\chi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n \left(\int_{\underline{\xi}}^{\xi} \chi'(s)\Psi_{n-\rho-1}(\xi,s)\ell(s)\mathrm{d}s\right),$$

where $n=[\rho]+1$ such that $[\rho]$ represents the integer part of the real number ρ.

Definition 2 ([17,25]). Let $\chi\in\mathcal{C}^1([\underline{\xi},\overline{\xi}],\mathbb{R})$ be a function satisfying $\chi'(\xi)>0$, for all $\xi\in[\underline{\xi},\overline{\xi}]$. The χ-Hilfer fractional derivative of a function ℓ of order $0<\rho<1$ and type $0\leq\sigma\leq 1$ is given by

$$^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}\ell(\xi)=\mathfrak{I}^{\sigma(1-\rho),\chi}_{\underline{\xi}^+}\left(\frac{1}{\chi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\mathfrak{I}^{(1-\sigma)(1-\rho),\chi}_{\underline{\xi}^+}\ell(\xi)=\mathfrak{I}^{1-\gamma,\chi RL}_{\underline{\xi}^+}D^{\gamma,\chi}_{\underline{\xi}^+}\ell(\xi),$$

where $\gamma=\rho+\sigma(1-\rho)$.

Lemma 1 ([17]). *Let $\rho,\mu\in\mathbb{R}^*_+$ and $\xi>\underline{\xi}$, then*

(i_1) $\mathfrak{I}^{\rho,\chi}_{\underline{\xi}^+}\Psi_{\mu-1}(\xi,\underline{\xi})=\frac{\Gamma(\mu)}{\Gamma(\rho+\mu)}\Psi_{\rho+\mu-1}(\xi,\underline{\xi})$.

(i_2) $^{RL}\mathcal{D}^{\rho,\chi}_{\underline{\xi}^+}\Psi_{\mu-1}(\xi,\underline{\xi})=\frac{\Gamma(\mu)}{\Gamma(\mu-\rho)}\Psi_{\mu-\rho-1}(\xi,\underline{\xi}),0<\rho<1,\ \mu>1$, *in the case when* $\rho=\mu$, *we get* $^{RL}\mathcal{D}^{\rho,\chi}_{\underline{\xi}^+}\Psi_{\mu-1}(\xi,\underline{\xi})=0$.

We consider the following auxiliary spaces

$$\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])=\Big\{u\colon(\underline{\xi},\overline{\xi}]\to E/u\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}]),\ ^{RL}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}u\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])\Big\},$$

$$\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])=\Big\{u\colon(\underline{\xi},\overline{\xi}]\to E/u\in\mathcal{C}([\underline{\xi},\overline{\xi}]),\ D^{\chi}u\in\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])\Big\}\ \text{and}$$

$$\mathcal{C}^{1,\rho,\sigma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])=\Big\{u\colon(\underline{\xi},\overline{\xi}]\to E/u\in\mathcal{C}([\underline{\xi},\overline{\xi}]),\ D^{\chi}u,\ ^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}D^{\chi}u\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])\Big\},$$

it is clear to see that $\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}]\subseteq\mathcal{C}^{1,\rho,\sigma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$.

Lemma 2 ([18]). *Let $0<\rho<1$, $0\leq\sigma\leq 1$ and $\gamma=\rho+\sigma-\rho\sigma$. If $\omega(.)\in\mathcal{C}^{\gamma}_{1-\gamma}([\underline{\xi},\overline{\xi}])$, then*

$$\mathfrak{I}^{\gamma,\chi}_{\underline{\xi}^+}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}\omega=\mathfrak{I}^{\rho,\chi}_{\underline{\xi}^+}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}\omega$$

and

$$\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}\mathfrak{I}^{\rho,\chi}_{\underline{\xi}^+}\omega=\mathcal{D}^{\sigma(1-\rho)}_{\underline{\xi}^+}\omega.$$

Lemma 3 ([18]). *Suppose that $f(.,y(.))\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$ for all $y(.)\in\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. If $y(.)\in\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, then, $y(.)$ is a solution of the fractional differential problem:*

$$\begin{cases} ^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}y(\xi)=f(\xi,y(\xi)), & 0<\rho<1,\ 0\leq\sigma\leq 1;\\ \mathfrak{I}^{1-\gamma,\chi}_{\underline{\xi}^+}y(\underline{\xi}^+)=\omega_0, & \gamma=\rho+\sigma-\rho\sigma,\end{cases}$$

if and only if y satisfies the following integral equation:

$$y(\xi)=\frac{\omega_0\Psi_{\gamma-1}(\xi,\underline{\xi})}{\Gamma(\gamma)}+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)f(s,y(s))\mathrm{d}s.$$

Next we give the notion of the noncompactness measure in the sense of Kuratowski and its properties which will be used in the next section, for this purpose, we denote by $\mathrm{Set}_b(E)$ the set of all bounded subsets of Banach space E.

Definition 3 ([9])**.** Let $D \in \mathrm{Set}_b(E)$. The Kuratowski noncompactness measure ϑ of the subset D is defined as follows:

$$\vartheta(\Omega) = \inf\{e > 0 \colon \Omega \text{ admits a finite cover by sets of diameter} \leq e\}.$$

Lemma 4 ([9])**.** *Let $A, B \in \mathrm{Set}_b(E)$, we have the following properties*

(i_1) $\vartheta(A) = 0$ *if and only if A is relatively compact,*
(i_2) $\vartheta(A) = \vartheta(\overline{A})$*, where $\overline{A}$ denotes the closure of A,*
(i_3) $\vartheta(A+B) \leq \vartheta(A) + \vartheta(B)$*,*
(i_4) $A \subset B$ *implies* $\vartheta(A) \leq \vartheta(B)$*,*
(i_5) $\vartheta(a.A) = |a|.\vartheta(A)$ *for all* $a \in \mathbb{R}$*,*
(i_6) $\vartheta(\{a\} \cup A) = \vartheta(A)$ *for all* $a \in E$*,*
(i_7) $\vartheta(A) = \vartheta(Conv(A))$*, where $Conv(A)$ is the smallest convex that contains A.*

Lemma 5 ([9])**.** *If D is a equicontinuous and bounded subset of $\mathcal{C}([\underline{\xi}, \overline{\xi}])$, then $\vartheta(D(.)) \in \mathcal{C}([\underline{\xi}, \overline{\xi}], \mathbb{R}_+)$*

$$\vartheta_C(D) = \max_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(D(\xi)),\ \vartheta\left(\left\{ \int_{\underline{\xi}}^{\overline{\xi}} w(\xi) \mathrm{d}\xi : w \in D \right\}\right) \leq \int_{\underline{\xi}}^{\overline{\xi}} \vartheta(D(\xi)) \mathrm{d}r,$$

where $D(\xi) = \{w(\xi) \colon w \in D\}$ and ϑ_C is the noncompactness measure on the space $\mathcal{C}([\underline{\xi}, \overline{\xi}])$.

Theorem 1 ([2])**.** *Let E be a Banach space and D a closed and convex subset of E such that D is bounded and contains 0, and let $N \colon D \longrightarrow D$ be a continuous mapping. If the following implication:*

$$V = N(V) \cup \{0\} \text{ or } V = \overline{conv}N(V) \implies \gamma(V) = 0,$$

is satisfied for every subset V of D, then N has at least one fixed point.

3. Main results

3.1. *Integral equation*

In the content of Lemma below, we will illustrate the equivalence between the problem at hand (1.1)-(1.2) and the following integral equation

$$\begin{aligned} y(\xi) = {} & \frac{\sum_{i=1}^{n} \zeta_i \Big[\kappa y(\xi_i) + I^{\rho} \hbar\Big(\xi_i, y(\xi_i), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+} y(\xi_i)\Big)\Big]}{\Gamma(\gamma+1) - \gamma \sum_{i=1}^{n} \zeta_i \Psi_{\gamma-1}(\xi_i, \underline{\xi})} \Psi_{\gamma}(\xi, \underline{\xi}) + \kappa \int_{\underline{\xi}}^{\xi} \chi'(s) y(s) \mathrm{d}s \\ & + \frac{1}{\Gamma(\rho+1)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi_{\rho}(\xi, s) \hbar\Big(s, y(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+} y(s)\Big) \mathrm{d}s. \end{aligned} \tag{3.1}$$

Lemma 6. *Let $\gamma = \rho + \sigma - \rho\sigma$ with $0 < \rho < 1$ and $0 \leq \sigma \leq 1$, we assume that the function $\hbar\colon (\underline{\xi}, \overline{\xi}] \times E^2 \to E$ satisfies $\hbar\left(., y(.), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(.)\right) \in \mathcal{C}_{1-\gamma;\chi}([\underline{\xi}, \overline{\xi}])$, for all $y(.) \in \mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi}, \overline{\xi}])$. If $y \in \mathcal{C}^{1,\gamma}_{1-\gamma;\chi}([\underline{\xi}, \overline{\xi}])$. Then, y is a solution of the problem (1.1)-(1.2) if and only if y satisfies the integral equation (3.1).*

Proof. Let $y \in \mathcal{C}^{1,\gamma}_{1-\gamma;\chi}([\underline{\xi}, \overline{\xi}])$ be a solution of the problem (1.1)-(1.2), since $\hbar\left(., y(.), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(.)\right) \in \mathcal{C}_{1-\gamma;\chi}([\underline{\xi}, \overline{\xi}])$, from Lemma 3 we have

$$D^{\chi} y(\xi) = \frac{I_{\underline{\xi}^+}^{1-\gamma,\chi} D^{\chi} y(\underline{\xi}^+)}{\Gamma(\gamma)} \Psi_{\gamma-1}(\xi, \underline{\xi}) + \kappa y(\xi) + I_{\underline{\xi}^+}^{\rho,\chi} \hbar\left(t, y(\xi), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi)\right). \tag{3.2}$$

Next, we substitute ξ by ξ_i into the above equation, we get

$$D^{\chi} y(\xi_i) = \frac{I_{\underline{\xi}^+}^{1-\gamma,\chi} D^{\chi} y(\underline{\xi}^+)}{\Gamma(\gamma)} \Psi_{\gamma-1}(\xi_i, \underline{\xi}) + \kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho,\chi} \hbar\left(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi_i)\right).$$

By utilizing the second condition (1.2), we obtain

$$\begin{aligned} I_{\underline{\xi}^+}^{1-\gamma,\chi} D^{\chi} y(\underline{\xi}^+) = {} & \frac{I_{\underline{\xi}^+}^{1-\gamma,\chi} D^{\chi} y(\underline{\xi}^+)}{\Gamma(\gamma)} \sum_{i=1}^{n} \zeta_i \Psi_{\gamma-1}(\xi_i, \underline{\xi}) \\ & + \sum_{i=1}^{n} \zeta_i \left[\kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho,\chi} \hbar\left(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi_i)\right)\right], \end{aligned}$$

this implies

$$I_{\underline{\xi}^+}^{1-\gamma,\chi} D^{\chi} y(\underline{\xi}^+) = \frac{\Gamma(\gamma) \sum_{i=1}^{n} \zeta_i \left[\kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho,\chi} \hbar\left(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi_i)\right)\right]}{\Gamma(\gamma) - \sum_{i=1}^{n} \zeta_i \Psi_{\gamma-1}(\xi_i, \underline{\xi})}. \tag{3.3}$$

By substituting (3.3) to (3.2), we deduce that

$$\begin{aligned} D^{\chi} y(\xi) = {} & \frac{\sum_{i=1}^{n} \zeta_i \left[\kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho,\chi} \hbar\left(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi_i)\right)\right]}{\Gamma(\gamma) - \sum_{i=1}^{n} \zeta_i \Psi_{\gamma-1}(\xi_i, \underline{\xi})} \Psi_{\gamma-1}(\xi, \underline{\xi}) + \kappa y(\xi) \\ & + \frac{1}{\Gamma(\rho)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi_{\rho-1}(\xi, s) \hbar\left(s, y(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(s)\right) \mathrm{d}s. \end{aligned} \tag{3.4}$$

Next, applying $I_{\underline{\xi}^+}^{\chi}$ to both sides of (3.4), we obtain

$$y(\xi) = \frac{\sum_{i=1}^{n} \zeta_i \left[\kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho,\chi} \hbar\left(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi_i)\right)\right]}{\Gamma(\gamma+1) - \gamma \sum_{i=1}^{n} \zeta_i \Psi_{\gamma-1}(\xi_i, \underline{\xi})} \Psi_{\gamma}(\xi, \underline{\xi}) + \kappa \int_{\underline{\xi}}^{\xi} \chi'(s) y(s) \mathrm{d}s$$

$$+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho}(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}y(s)\Big)\mathrm{d}s.$$

Conversely, let $y\in C_{1-\gamma,\chi}^{1,\gamma}([\underline{\xi},\overline{\xi}])$ be a function verifies equation (3.1), it is clear that $y(0)=0$. By applying D^{χ} to both sides of (3.1), we obtain equation (3.4), using Lemma 3, we can easily establish that the function y satisfies the second condition (1.2). □

3.2. *Existence and compactness*

In this subsection, we will prove that the solution set (denoted **SS**) of the problem (1.1)-(1.2) is nonempty and compact, we necessarily assume the following hypotheses

($\mathbf{H_1}$) Suppose that the function $\hbar\colon (\underline{\xi},\overline{\xi}]\times E^2\to E$ verifies $\hbar(.,u(.),v(.))\in C_{1-\gamma;\chi}^{\sigma(1-\rho)}([\underline{\xi},\overline{\xi}])$, for all $u(.),v(.)\in C([\underline{\xi},\overline{\xi}])$, $\hbar(.,0,0)\in C([\underline{\xi},\overline{\xi}],E)$ and there exists $\alpha,\beta\in\mathbb{R}_+$ such that

($\mathbf{H_{1-1}}$) For all $u,v,\overline{u},\overline{v}\in E$:

$$\|\hbar(\xi,u,v)-\hbar(\xi,\overline{u},\overline{v})\|\leq\alpha\|u-\overline{u}\|+\beta\|v-\overline{v}\|.$$

($\mathbf{H_{1-2}}$) For each nonempty, bounded set $\Omega\subset C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])$, for all $\xi\in(\underline{\xi},\overline{\xi}]$, we have

$$\vartheta\Big(\hbar(\xi,\Omega(\xi),{}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}\Omega(\xi)\Big)\leq\alpha\vartheta\Big(\Omega(\xi))\Big)+\beta\vartheta\Big({}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}\Omega(\xi)\Big),$$

where

$$\Omega(\xi)=\Big\{y(\xi),\,y\in C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\Big\}\text{ and }{}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}\Omega(\xi)=\Big\{{}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}y(\xi),\,y\in C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\Big\}.$$

($\mathbf{H_2}$)

$$\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^{*}n(\Psi_{\gamma}^{*}+\gamma)+\Psi_{1-\gamma}^{*}\Big)+\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi_1^{*}<\frac{\Gamma(\rho+2)}{2},$$

where

$$\mathcal{T}=\frac{1}{\Gamma(\gamma+1)-\gamma\sum_{i=1}^{n}\zeta_i\Psi_{\gamma-1}(\xi_i,\underline{\xi})}\quad\text{and}\quad A_0=\Big(\alpha+\beta\Gamma(\gamma)\Big)\psi_{\rho}^{*}.$$

Define the operator $\Xi\colon C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\}\to C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\}$ by

$$\Xi y(\xi)=\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i)+I_{\underline{\xi}^{+}}^{\rho,\chi}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}y(\xi_i)\Big)\Big]\Psi_{\gamma}(\xi,\underline{\xi})+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y(s)\mathrm{d}s$$
$$+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho}(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma,\chi}y(s)\Big)\mathrm{d}s.$$

and the operator $D^{\chi}\Xi\colon \mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\} \to \mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\}$ by

$$D^{\chi}\Xi y(\xi) = \gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho;\chi}\hbar\Big(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(\xi_i)\Big)\Big]\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa y(\xi)$$
$$+ \frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s, y(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(s)\Big)\mathrm{d}s.$$

In this part, we will present the result concerning the existence of solutions of the problem (1.1)-(1.2). First, we will give some useful lemmas to demonstrate this result.

Lemma 7. *We assume the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_{1-1}})$ *hold. Then*

(1) Ξ *is bounded and continuous.*
(2) $\Xi(B)$ *is equicontinuous for all bounded subset* B *of* $\mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$.

Proof. Let us show condition (1); we begin to prove that Ξ is a bounded operator. Let $y \in \mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$, it is clear to see that $\Xi y \in \mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. Using $(\mathbf{H_1})$ and $(\mathbf{H_{1-1}})$, for all $y \in B_r = \{y \in C^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\colon \|y\|^{1}_{1-\gamma;\chi} < r\}$ and $\xi \in (\underline{\xi},\overline{\xi}]$, we have

$$\|\Xi y(\xi)\| \le |\mathcal{T}|\sum_{i=1}^{n}|\zeta_i|\Big[\kappa\|y(\xi_i)\| + I_{\underline{\xi}^+}^{\rho;\chi}\|\hbar\Big(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(\xi_i)\Big)\|\Big]\Psi_{\gamma}(\xi,\underline{\xi})$$
$$+ \kappa\int_{\underline{\xi}}^{\xi}\chi'(s)\|y(s)\|\mathrm{d}s + \frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho}(\xi,s)\|\hbar\Big(s, y(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(s)\Big)\|\mathrm{d}s,$$
$$\le |\mathcal{T}|\zeta^* n\Psi^*_{\gamma}\Big[\kappa r + \frac{\hbar^*\Psi^*_{\rho}}{\Gamma(\rho+1)} + \frac{r\alpha\Psi^*_{\rho}}{\Gamma(\rho+1)} + \frac{r\beta\Gamma(\gamma)\Psi^*_{\rho}}{\Gamma(\rho+1)}\Big] + \kappa r\Psi^*_1$$
$$+ \frac{\hbar^*\Psi^*_{\rho+1}}{\Gamma(\rho+2)} + \frac{r\alpha\Psi^*_{\rho+1}}{\Gamma(\rho+2)} + \frac{r\beta\Gamma(\gamma)\Psi^*_{\rho+1}}{\Gamma(\rho+2)}.$$

We also have, for each $\xi \in (\underline{\xi},\overline{\xi}]$

$$\|\Psi_{1-\gamma}(\xi,\underline{\xi})D^{\chi}\Xi y(\xi)\| \le \gamma|\mathcal{T}|\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I_{\underline{\xi}^+}^{\rho;\chi}\hbar\Big(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(\xi_i)\Big)\Big]$$
$$+ \kappa\Psi_{1-\gamma}(\xi,\underline{\xi})y(\xi) + \frac{\Psi_{1-\gamma}(\xi,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s, y(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(s)\Big)\mathrm{d}s$$
$$\le (\gamma|\mathcal{T}|\zeta^* n + \Psi^*_{1-\gamma})\Big[\kappa r + \frac{\hbar^*\Psi^*_{\rho}}{\Gamma(\rho+1)} + \frac{r\alpha\Psi^*_{\rho}}{\Gamma(\rho+1)} + \frac{r\beta\Gamma(\gamma)\Psi^*_{\rho}}{\Gamma(\rho+1)}\Big].$$

So,

$$\|\Xi y\|_{\infty} + \|D^{\chi}\Xi y\|_{\gamma;\chi} \le \Big(|\mathcal{T}|\zeta^* n(\gamma + \Psi^*_{\gamma}) + \Psi^*_{1-\gamma}\Big)\Big[\kappa r + \frac{\hbar^*\Psi^*_{\rho}}{\Gamma(\rho+1)} + \frac{r\alpha\Psi^*_{\rho}}{\Gamma(\rho+1)}$$

$$+\frac{r\beta\Gamma(\gamma)\Psi^*_\rho}{\Gamma(\rho+1)}\Big]+\kappa r\Psi^*_1+\frac{\hbar^*\Psi^*_{\rho+1}}{\Gamma(\rho+2)}+\frac{r\alpha\Psi^*_{\rho+1}}{\Gamma(\rho+2)}+\frac{r\beta\Gamma(\gamma)\Psi^*_{\rho+1}}{\Gamma(\rho+2)}.$$

Now we will show that Ξ is continuous. Let $\{y_n\}_{n\in\mathbb{N}}\to y$ in $\mathcal{C}^1_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, from $(\mathbf{H_{1-1}})$ and Lemma 1 we can easily prove that $\Xi y_n(.)\to\Xi y(.)$ in $\mathcal{C}([\underline{\xi},\overline{\xi}])$ and $D^\chi\Xi y_n(.)\to D^\chi\Xi y(.)$ in $\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, that implies $\Xi y_n(.)\to\Xi y(.)$ in $\mathcal{C}^1_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, then Ξ is continuous.

Let us show the second condition (2), it is enough to show that $\Xi(B_r)$ (resp. $D^\chi\Xi(B_r)$) is equicontinuous in $\mathcal{C}([\underline{\xi},\overline{\xi}])$ (resp. in $\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$). Let $y\in B_r$ and $\xi_1,\xi_2\in(\underline{\xi},\overline{\xi}]$ with $\xi_1<\xi_2$, from $(\mathbf{H_{1-1}})$, we have

$$\begin{aligned}
&\|\Xi y(\xi_2)-\Xi y(\xi_1)\|\\
&\le\Big[\kappa r+\frac{\hbar^*\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\alpha\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\beta\Gamma(\gamma)\Psi^*_\rho}{\Gamma(\rho+1)}\Big]\times\Big(\Psi_\gamma(\xi_2,\underline{\xi})-\Psi_\gamma(\xi_1,\underline{\xi})\Big)\\
&\quad+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)[\Psi_\rho(\xi_2,s)-\Psi_\rho(\xi_1,s)]\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\rho+1)}\int_{\xi_1}^{\xi_2}\chi'(s)\Psi_\rho(\xi_2,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s+\kappa\int_{\xi_1}^{\xi_2}\chi'(s)y(s)\mathrm{d}s\\
&\le\Big[\kappa r+\frac{\hbar^*\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\alpha\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\beta\Gamma(\gamma)\Psi^*_\rho}{\Gamma(\rho+1)}\Big]\Big(\Psi_\gamma(\xi_2,\underline{\xi})-\Psi_\gamma(\xi_1,\underline{\xi})\Big)\\
&\quad+\frac{\hbar^*+r(\alpha+\beta\Gamma(\gamma))}{\Gamma(\rho+2)}\Big[\Psi_{\rho+1}(\xi_2,\underline{\xi})-\Psi_{\rho+1}(\xi_1,\underline{\xi})+\Psi_{\rho+1}(\xi_2,\xi_1)\Big]\\
&\quad+\frac{\hbar^*+r(\alpha+\beta\Gamma(\gamma))}{\Gamma(\rho+2)}\Psi_{\rho+1}(\xi_2,\xi_1)+\kappa r\Psi_1(\xi_2,\xi_1).
\end{aligned}$$

As ξ_2 tends to ξ_1, the right-hand side of the last inequality tends to 0. Therefore $\Xi(B_r)$ is equicontinuous in $\mathcal{C}([\underline{\xi},\overline{\xi}])$.

And, we also have

$$\begin{aligned}
&\|\Psi_{1-\gamma}(\xi_2,\underline{\xi})D^\chi\Xi y(\xi_2)-\Psi_{1-\gamma}(\xi_1,\underline{\xi})D^\chi\Xi y(\xi_1)\|\\
&\le\kappa\|\Psi_{1-\gamma}(\xi_2,\underline{\xi})y(\xi_2)-\Psi_{1-\gamma}(\xi_1,\underline{\xi})y(\xi_1)\|\\
&\quad+\Bigg\|\frac{\Psi_{1-\gamma}(\xi_2,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_2}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\\
&\qquad-\frac{\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\rho-1)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Psi_\rho(\xi_1,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\Bigg\|\\
&\le\kappa\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)\|y(\xi_2)\|+\kappa\Psi_{1-\gamma}(\xi_1,\underline{\xi})\|y(\xi_2)-y(\xi_1)\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Big[\Psi_{\rho-1}(\xi_1,s)-\Psi_{\rho-1}(\xi_2,s)\Big]\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(s)\Big)\|\mathrm{d}s\\
&+\frac{\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\alpha)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(s)\Big)\|\mathrm{d}s\\
&+\frac{\Psi_{1-\gamma}(\xi_2,\underline{\xi})}{\Gamma(\rho)}\int_{\xi_1}^{\xi_2}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(s)\Big)\|\mathrm{d}s\\
\leq{}& \kappa\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)r+\frac{r\kappa\Psi_{1-\gamma}^*}{\gamma}\Psi_{\gamma}(\xi_2,\xi_1)\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Big[\Psi_{\rho-1}(\xi_1,s)-\Psi_{\rho-1}(\xi_2,s)\Big]\mathrm{d}s\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\mathrm{d}s\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi_{1-\gamma}(\xi_2,\underline{\xi})}{\Gamma(\rho)}\int_{\xi_1}^{\xi_2}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\mathrm{d}s\\
\leq{}& \kappa\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)r+\frac{r\kappa\Psi_{1-\gamma}^*}{\gamma}\Psi_{\gamma}(\xi_2,\xi_1)\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi_{1-\gamma}^*}{\Gamma(\rho+1)}\Big[\Psi_{\rho}(\xi_2,\underline{\xi})-\Psi_{\rho}(\xi_1,\underline{\xi})+2\Psi_{\rho}(\xi_2,\xi_1)\Big]\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi_{\rho}^*}{\Gamma(\rho)}\Big[\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big].
\end{aligned}
$$

By taking ξ_2 tends to ξ_1, the right-hand side of the last inequality tends to 0, and hence $D^{\chi}\Xi(B_r)$ is equicontinuous in $\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$, thus, $\Xi(B_r)$ is equicontinuous in $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. □

We denote by $\vartheta_{\mathcal{C}},\vartheta_{\gamma}$ and ϑ_{γ}^1 the Kuratowski noncompactness measure defined respectively on $\mathcal{C}([\underline{\xi},\overline{\xi}]))$, $\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$ and $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$.

Lemma 8. *Let B be a bounded subset of $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$, we have*

$$\vartheta_{\gamma}^1(B)\leq\vartheta(B)+\vartheta_{\gamma}(D^{\chi}B)\leq 2\vartheta_{\gamma}^1(B). \tag{3.5}$$

Proof. Let B be a bounded subset of $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$ and let ε be a strictly positive real number. So, there exists a finite partition B_i, $i=1,\dots m$, such that

$$\mathrm{Diam}_{\gamma}^1(B_i)\leq\varepsilon+\vartheta_{\gamma}^1(B),\ \ i=1,\dots m.$$

Then for all y_1, y_2 in B_i and $\xi \in (\underline{\xi}, \overline{\xi}]$, we have

$$\|y_2(\xi) - y_1(\xi)\| \le \varepsilon + \vartheta_\gamma^1(B) \text{ and } \|D^\chi y_2(\xi) - D^\chi y_1(\xi)\| \le \varepsilon + \vartheta_\gamma^1(B),\ i = 1, \dots m.$$

So,

$$\mathrm{Diam}(B_i) \le \varepsilon + \vartheta_\gamma^1(B) \text{ and } \mathrm{Diam}_\gamma(D^\chi B_i) \le \varepsilon + \vartheta_\gamma^1(B),\ i = 1, \dots m.$$

Thus,

$$\vartheta(B) + \vartheta_\gamma(D^\chi B) \le 2\varepsilon + 2\vartheta_\gamma^1(B).$$

Since ε is arbitrary, this means that we arrive at

$$\vartheta(B) + \vartheta_\gamma(D^\chi B) \le 2\vartheta_\gamma^1(B). \tag{3.6}$$

Conversely, we want to prove that $\vartheta_\gamma^1(B) \le \vartheta(B) + \vartheta_\gamma(D^\chi B)$, from the definition of Kuratowski noncompactness measure, we have, for each $\varepsilon > 0$, there are a finite partitions $\{B_i\}_{i=1,\dots,m_1}$ of B and $\{D_j\}_{j=1,\dots,m_2}$ of $D^\chi B$ such that

$$\mathrm{Diam}(B_i) \le \varepsilon + \vartheta(B), \text{ and } \mathrm{Diam}_\gamma(D_j) \le \varepsilon + \vartheta_\gamma(D^\chi B).$$

It is clear that the partition $\{B_i \cap I^\chi_{\underline{\xi}^+} D_j\}_{i,j}$ belongs to $C^1_{1-\gamma,\chi}([\underline{\xi}, \overline{\xi}]))$ and verifies the following inequality:

$$\mathrm{Diam}(B_i \cap I^\chi_{\underline{\xi}^+} D_j) + \mathrm{Diam}_\gamma(D^\chi(B_i \cap I^\chi_{\underline{\xi}^+} D_j)) \le 2\varepsilon + \vartheta(B) + \vartheta_\gamma(D^\chi B).$$

As ε is arbitrary, we obtain

$$\vartheta_\gamma^1(B) \le \vartheta(B) + \vartheta_\gamma(D^\chi B). \tag{3.7}$$

From (3.6)-(3.7), we get

$$\vartheta_\gamma^1(B) \le \vartheta(B) + \vartheta_\gamma(D^\chi B) \le 2\vartheta_\gamma^1(B).$$

□

From Lemma 5 and Lemma 8, we easily show the following inequality

$$\vartheta_\gamma^1(D) \le \sup_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(D(\xi)) + \sup_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(\Psi_{1-\gamma}(\xi, \underline{\xi}) D^\chi D(\xi)) \le 2\vartheta_\gamma^1(D), \tag{3.8}$$

where D is a bounded and equicontinuous subset of $C^1_{1-\gamma,\chi}([\underline{\xi}, \overline{\xi}]))$,

$$D(\xi) = \{y(\xi) : y \in D\} \quad \text{and} \quad D^\chi D(\xi) = \{D^\chi y(\xi) : y \in D\}.$$

Let

$$B_R = \Big\{ y \in C_{1-\gamma,\chi}([\underline{\xi}, \overline{\xi}])) : \|y\|^1_{\gamma,\chi} \le R \Big\}.$$

We are about to present our main result which is as follows.

Theorem 2. *Assume that the hypotheses* $(\mathbf{H_1})-(\mathbf{H_2})$ *are satisfied and that* R *verifies the following inequality*

$$\frac{1}{R} < \frac{\Gamma(\rho+2)-\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma+\gamma)+\Psi^*_{1-\gamma}\Big)}{(\rho+1)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma+\gamma)+\Psi^*_{1-\gamma}\Big)\Psi^*_\rho\hbar^*+\Psi^*_{\rho+1}\hbar^*} - \frac{\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi^*_1}{(\rho+1)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma+\gamma)+\Psi^*_{1-\gamma}\Big)\Psi^*_\rho\hbar^*+\Psi^*_{\rho+1}\hbar^*}. \tag{3.9}$$

Then, the problem (1.1)-(1.2) *has at least one solution in* $\mathcal{C}^{1,\gamma}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. *In addition, the solution set* **SS** *of the problem* (1.1)-(1.2) *is compact in* $\mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$.

Proof. From the definition of Ξ and Lemma 6, it is clear that the solutions of (1.1)-(1.2) is equivalent to the fixed point of Ξ. For this reason, we want to verify that Ξ satisfies the assumptions of Mönch fixed point theorem. First, we will prove that Ξ is well defined from B_R to B_R, indeed, let $y \in B_R$. By using the condition $(\mathbf{H_{1-1}})$ and after some calculations, for each $\xi \in (\underline{\xi},\overline{\xi}]$ and $y \in B_R$, we get

$$\begin{aligned}
&\|\Xi y(\xi)\| + \|\Psi_{1-\gamma}(\xi,\underline{\xi}) D^{\chi}\Xi y(\xi)\| \\
&\leq |\mathcal{T}|\sum_{i=1}^{n}|\zeta_i|\Big[\kappa\|y(\xi_i)\| + I^{\rho;\chi}_{\underline{\xi}^+}\|\hbar\Big(\xi_i, y(\xi_i), {}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\|\Big]\Psi^*_\gamma \\
&\quad + \kappa\int_{\underline{\xi}}^{\xi}\chi'(s)\|y(s)\|\mathrm{d}s + \frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\|\mathrm{d}s \\
&\quad + \gamma|\mathcal{T}|\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I^{\rho;\chi}\hbar\Big(\xi_i, y(\xi_i), {}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\Big] + \kappa\Psi^*_{1-\gamma}y(\xi) \\
&\quad + \frac{\Psi^*_{1-\gamma}}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s \\
&\leq \frac{(\rho+1)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma+\gamma)+\Psi^*_{1-\gamma}\Big)\Psi^*_\rho\hbar^*+\Psi^*_{\rho+1}\hbar^*}{\Gamma(\rho+2)} \\
&\quad + \frac{\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma+\gamma)+\Psi^*_{1-\gamma}\Big)}{\Gamma(\rho+2)}R + \frac{\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi^*_1}{\Gamma(\rho+2)}R.
\end{aligned}$$

From $(\mathbf{H_2})$ and the inequality (3.9), we obtain

$$\forall y \in B_R : \|\Xi y\|^1_{\gamma;\chi} < R.$$

Note that B_R is bounded, convex and closed subset of $\mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$ and Ξ is continuous on B_R. Next, it is enough to show the following implication:

$$V \subset \overline{conv}\{N(V)\cup\{0\}\} \Longrightarrow \vartheta^1_\gamma(V) = 0, \text{ for any } V \subset B_R.$$

Let V be a subset of B_R such that $V \subset \overline{conv}\{N(V)\cup\{0\}\}$. By using Lemmas 4 and 5, we obtain

$$\begin{aligned}
&\vartheta(\Xi V(\xi)) + \vartheta\Big(\psi_{1-\gamma}(\xi,\underline{\xi})D^{\chi}\Xi(V(\xi))\Big)\\
&\leq |\mathcal{T}|\sum_{i=1}^{n}\zeta^{*}\left[\kappa\vartheta\Big(V(\xi_i)\Big) + + I_{\underline{\xi}^{+}}^{\rho;\chi}\vartheta\Big(\hbar\Big(\xi_i, V(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}V(\xi_i)\Big)\Big)\right]\Psi_{\gamma}^{*}\\
&\quad + \kappa\int_{\underline{\xi}}^{\xi}\chi'(s)\vartheta\Big(V(s)\Big)\mathrm{d}s + \frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho}(\xi,s)\vartheta\Big(\hbar\Big(s, V(s), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}V(s)\Big)\Big)\mathrm{d}s\\
&\quad + \gamma|\mathcal{T}|\sum_{i=1}^{n}\zeta^{*}\left[\kappa\vartheta\Big(V(\xi_i)\Big) + I_{\underline{\xi}^{+}}^{\rho;\chi}\vartheta\Big(\hbar\Big(\xi_i, V(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}V(\xi_i)\Big)\Big)\right] + \kappa\Psi_{1-\gamma}^{*}\vartheta\Big(V(\xi)\Big)\\
&\quad + \frac{\Psi_{1-\gamma}^{*}}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\vartheta\Big(\hbar\Big(s, V(s), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}V(s)\Big)\mathrm{d}s\Big).
\end{aligned}$$

From Lemmas 5, 7 and 8 and the hypotheses $(\mathbf{H_{1-2}}) - (\mathbf{H_2})$ and inequality (3.8), we arrive at

$$\begin{aligned}
\vartheta_{\gamma}^{1}(\Xi V) &\leq \sup_{\xi\in[\underline{\xi},\overline{\xi}]}\vartheta(\Xi V(\xi)) + \sup_{\xi\in[\underline{\xi},\overline{\xi}]}\vartheta\Big(\psi_{1_{\gamma}(\xi,\underline{\xi})}D^{\chi}\Xi(V(\xi))\Big)\\
&\leq \frac{2\Big(\kappa\Gamma(\rho+2) + (\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^{*}n(\Psi_{\gamma}^{*}+\gamma) + \Psi_{1-\gamma}^{*}\Big)}{\Gamma(\rho+2)}\vartheta_{\gamma}^{1}(\Xi V)\\
&\quad + \frac{2\Big(A_0 + \kappa\Gamma(\rho+2)\Big)\Psi_{1}^{*}}{\Gamma(\rho+2)}\vartheta_{\gamma}^{1}(\Xi V).
\end{aligned}$$

By the condition $(\mathbf{H_2})$, we get $\vartheta_{\gamma}^{1}(\Xi V) = 0$, that means $\vartheta_{\gamma}^{1}(V) = 0$. From Theorem 1, the operator Ξ has at least one fixed point $y \in B_R$. By using Lemma 6, we conclude that the problem (1.1)-(1.2) has at least one solution. Let us prove that solution set **SS** of (1.1)-(1.2) is included in $\mathcal{C}_{1-\gamma;\chi}^{1,\gamma}([\underline{\xi},\overline{\xi}])$, Let $w \in \{u \in \mathcal{C}_{1-\gamma;\chi}^{1}([\underline{\xi},\overline{\xi}]) : \Xi u = u$ and $D^{\chi}\Xi u = D^{\chi}u\}$, we need to show that $D^{\chi}w \in \mathcal{C}_{1-\gamma;\chi}^{\gamma}([\underline{\xi},\overline{\xi}])$, so, for all $\xi \in (\underline{\xi},\overline{\xi}]$, we have

$$\begin{aligned}
D^{\chi}w(\xi) &= \gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\left[\kappa w(\xi_i) + I_{\underline{\xi}^{+}}^{\rho;\chi}\hbar\Big(\xi_i, w(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}w(\xi_i)\Big)\right]\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa w(\xi)\\
&\quad + \frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s, w(s), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}w(s)\Big)\mathrm{d}s.
\end{aligned}$$

By using ${}^{RL}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma}$ on both sides the last inequality, from Lemmas 1, 2 we obtain

$$(1-\kappa)^{RL}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma}D^{\chi}w(t) = {}^{RL}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma}I_{\underline{\xi}^{+}}^{\rho;\chi}\hbar\Big(s, w(s), {}^{C}\mathcal{D}_{\underline{\xi}^{+}}^{\gamma;\chi}w(s)\Big)$$

$$= {}^{RL}\mathcal{D}^{\sigma(1-\rho)}_{\underline{\xi}^+}\hbar\Big(\xi, w(\xi), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}w(\xi)\Big).$$

So, from $(\mathbf{H_1})$, we have ${}^{RL}\mathcal{D}^{\gamma}_{\underline{\xi}^+}D^{\chi}w(t) \in \mathcal{C}^{\gamma}_{1-\gamma}([\underline{\xi},\overline{\xi}])$, that means $w \in \mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. Finally, the solution set **SS** of problem (1.1)-(1.2) is included in $\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$.

We show now that the solution set **SS** of the problem (1.1)-(1.2) is compact subset o $\mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of the solution set, as $\mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$ is compact space, there exists a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ (still denoted $\{y_n\}_{n\in\mathbb{N}}$) converges to y^*, it is enough to demonstrate that y^* is a solution of (1.1)-(1.2), for each $\xi \in (\underline{\xi},\overline{\xi}]$, we have

$$y_n(\xi) = \mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y_n(\xi_i) + I^{\rho;\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i, y_n(\xi_i), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(\xi_i)\Big)\Big]\Psi_\gamma(\xi,\underline{\xi})$$
$$+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y_n(s)\mathrm{d}s + \frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\hbar\Big(s, y_n(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\mathrm{d}s$$

and

$$D^{\chi}y_n(\xi) = \gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y_n(\xi_i) + I^{\rho;\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i, y_n(\xi_i), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(\xi_i)\Big)\Big]\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa y_n(\xi)$$
$$+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s, y_n(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\mathrm{d}s.$$

From $(\mathbf{H_1})$, we have $\hbar(., y_n(.), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(.))$ converges to $\hbar(., y^*(.), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(.))$ as $n \to +\infty$, let $\xi \in (\underline{\xi},\overline{\xi}]$, form $(\mathbf{H_{1-1}})$, for all $n \in \mathbb{N}$, we have

$$\chi'(s)\Psi_\rho(\xi,s)\|\hbar\Big(s, y_n(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\| \le \Big(\hbar^* + (\alpha+\beta\Gamma(\gamma))M\Big)\chi'(s)\Psi_\rho(\xi,s) \text{ and}$$
$$\chi'(s)\Psi_{\rho-1}(\xi,s)\|\hbar\Big(s, y_n(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\| \le \Big(\hbar^* + (\alpha+\beta\Gamma(\gamma))M\Big)\chi'(s)\Psi_{\rho-1}(\xi,s).$$

Using Lebesgue's dominated convergence theorem, for each $\xi \in (\underline{\xi},\overline{\xi}]$, we obtain

$$y^*(\xi) = \mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y^*(\xi_i) + I^{\rho;\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i, y^*(\xi_i), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(\xi_i)\Big)\Big]\Psi_\gamma(\xi,\underline{\xi})$$
$$+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y^*(s)\mathrm{d}s + \frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\hbar\Big(s, y^*(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(s)\Big)\mathrm{d}s$$

and

$$D^{\chi}y^*(\xi) = \gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y^*(\xi_i) + I^{\rho;\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i, y^*(\xi_i), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(\xi_i)\Big)\Big]\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa y^*(\xi)$$
$$+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s, y^*(s), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(s)\Big)\mathrm{d}s.$$

So, the solution set of Problem (1.1)-(1.2) is a compact subset of $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. □

4. Example

We take $\psi(t) = \frac{4\arctan t}{10\pi}$, $\underline{\xi} = 0, \xi_1 = 0.5$, $\overline{\xi} = 1$, $\sigma = \rho = 0.25$, $\kappa = \frac{1}{40}$, E the Banach space defined by

$$E = \left\{(y_1, y_2, \ldots, y_n, \ldots) \colon \sup_n |y_n| < \infty\right\},$$

with the norm $\|y\| = \sup_n |y_n|$, we define the function $\hbar\colon (0,1] \times E^2 \to E$ by

$$\hbar\Big(\xi, y(\xi), {}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+} y(\xi)\Big) = \Big(\hbar_1\Big(\xi, y_1(\xi), {}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+} y_1(\xi)\Big), \ldots, \hbar_n\Big(\xi, y_n(\xi), {}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+} y_n(\xi)\Big), \ldots\Big),$$

where

$$\hbar_n\Big(\xi, y_n(\xi), {}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+} y_n(\xi)\Big) = \frac{{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+} y_n(\xi)}{40 + nt^2} + \frac{y_n(\xi)}{40 + t^n}, \ \xi \in (0,1].$$

We easily see that $\hbar\colon (0,1] \times E^2 \to E$ is continuous and

$$\|\hbar(\xi, u, v) - \hbar(\xi, \overline{u}, \overline{v})\| \leq \frac{1}{40}\|u - \overline{u}\| + \frac{1}{40}\|v - \overline{v}\|, \text{ for all } \xi \in (0,1] \text{ and } u, v, \overline{u}, \overline{v} \in E.$$

Next, for all Ω a bounded subset of $\mathcal{C}^1_{1-\gamma;\chi}([0,1])$, we have

$$\vartheta\Big(\hbar\Big(\xi, \Omega(\xi), {}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}\Omega(\xi)\Big)\Big) \leq \frac{1}{40}\Big(\vartheta(\Omega(\xi)) + \vartheta\big({}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}\Omega(\xi)\big)\Big), \ \xi \in (0,1].$$

So, $(\mathbf{H_1})$, $(\mathbf{H_{1-1}})$ and $(\mathbf{H_{1-2}})$ are satisfied. A quick calculation gives us

$$\Big(\kappa\Gamma(\rho+2) + (\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma + \gamma) + \Psi^*_{1-\gamma}\Big) + \Big(A_0 + \kappa\Gamma(\rho+2)\Big)\Psi^*_1 < \frac{\Gamma(\rho+2)}{2}.$$

So, $(\mathbf{H_2})$ holds. Therefore, Theorem 2 ensures that the solution set of Problem (1.1)-(1.2) is nonempty and compact.

References

[1] M. I. Abbas and M. A. Ragusa, "On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function." *Symmetry*, vol. 13(2), no. 264, 2021, doi: 10.3390/sym13020264.

[2] R. P. Agarwal, M. Meehan, and D. O´ Regan, *Fixed Point Theory and Applications, Cambridge Tracts in Mathematics.* Cambridge University Press, 2001. doi: 10.1017/CBO9780511543005.

[3] H. M. Ahmed and M. A. Ragusa, "Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential." *Bull. Malays. Math. Sci. Soc.*, vol. 45, no. 6, pp. 3239–3253, 2022, doi: 10.1007/s40840-022-01377-y.

[4] M. G. A. Alshehri, A. Hyder, H. Budak, and M. A. Barakat, "Some new improvements for fractional Hermite-Hadamard inequalities by Jensen-Mercer inequalities." *J. Funct. Spaces*, vol. 2024, 2024, doi: 10.1155/2024/6691058.

[5] J. Andres and L. Górniewicz, *Topological Principles for Boundary Value Problems.* Dordrecht, 2003.

[6] E. Azroul, N. Kamali, M. A. Ragusa, and M. Shimi, “Variational methods for a $p(x,.)-$fractional bi-nonlocal problem of elliptic type.” *Rend. Circ. Mat. Palermo*, vol. 74, no. 47, 2025, doi: 10.1007/s12215-024-01156-7.

[7] K. Balachandran and J. Y. Park, “Nonlocal Cauchy problem for abstract fractional semi-linear evolution equations.” *Nonlinear Anal.*, vol. 71, no. 10, pp. 4471–4475, 2009, doi: 10.1016/j.na.2009.03.005.

[8] K. Balachandran and J. J. Trujillo, “The nonlocal Cauchy problem for nonlinear fractional integro-differential equations in Banach spaces.” *Nonlinear Anal.*, vol. 72, no. 2, pp. 4587–4593, 2010, doi: 10.1016/j.na.2010.02.035.

[9] J. Banaŝ and K. Goebel, *Measures of noncompactness in Banach spaces.* New York: Dekker, 1980. doi: 10.1112/blms/13.6.583b.

[10] M. Beddani and H. Beddani, “Compactness of boundary value problems for impulsive integro-differential equation.” *Filomat*, vol. 37, no. 20, pp. 6855–6866, 2023, doi: 10.2298/FIL2320855B.

[11] M. Beddani, H. Beddani, and M. Fečkan, “Qualitative study for impulsive pantograph fractional integro-differential equation via ψ-Hilfer derivative.” *Miskolc Math. Notes.*, vol. 24, no. 2, pp. 635–651, 2023, doi: 10.18514/MMN.2023.4032.

[12] M. Beddani and B. Hedia, “Existence result for a fractional differential equation involving a sequential derivative.” *Moroccan J. of Pure and Appl. Anal.*, vol. 8, no. 1, pp. 67–77, 2022, doi: 10.2478/mjpaa-2022-0006.

[13] L. Byszewski, *Existence and uniqueness of mild and classical solutions of semilinear functional differential evolution nonlocal Cauchy problem. Selected Problems of Mathematics.* Krakow: Cracow University of Technology, 1995.

[14] T. G. Chakuvinga and F. S. Topal, “Positive solutions for integral boundary value problems of nonlinear fractional differential equations.” *Miskolc Math. Notes*, vol. 25, no. 1, pp. 173–188, 2024, doi: 10.18514/MMN.2024.4233.

[15] E. Guariglia, “Riemann zeta fractional derivative - functional equation and link with primes.” *Adv. Differ. Equ.*, vol. 2019(1), no. 261, 2019, doi: 10.1186/s13662-019-2202-5.

[16] E. Guariglia, “Fractional calculus of the Lerch zeta function.” *Adv. Differ. Equ.*, vol. 19(1), no. 109, 2022, doi: 10.1007/s00009-021-01971-7.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Amsterdam: Elsevier B. V., 2006.

[18] K. D. Kucche and A. D. Mali, “On the nonlinear (k,ψ)-Hilfer fractional differential equations.” *Chaos, Solitons and Fractals*, vol. 152, p. 111335, 2022, doi: 10.1016/j.chaos.2021.111335.

[19] C. Li, X. Dao, and P. Guo, “Fractional derivatives in complex planes.” *Nonlinear Anal.*, vol. 71(5-6), pp. 1857–1869, 2009, doi: 10.1016/j.na.2009.01.021.

[20] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering.* New York: Academic Press, 1999.

[21] M. A. Ragusa, “Commutators of fractional integral operators on Vanishing-Morrey spaces.” *J Global Optim.*, vol. 40(1-3), pp. 361–368, 2008, doi: 10.1007/s10898-007-9176-7.

[22] M. A. Ragusa and A. Razani, “Weak solutions for a system of quasilinear elliptic equations.” *Contrib. Math.*, vol. 1, pp. 11–16, 2020, doi: 10.47443/cm.2020.0008.

[23] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications.* Yverdon: Gordon and Breach, 1993.

[24] S. Sitho, S. K.Ntouyas, A. Samadi, and J. Tariboon, “Boundary value problems for ψ-Hilfer type sequential fractional differential equations and inclusions with integral multi-point boundary conditions.” *Mathematics.*, vol. 9, no. 9, pp. 101–119, 2021, doi: 10.3390/math9091001.

[25] J. Vanterler da, C. Sousa, and E. Capelas de Oliveira, “On the ψ-Hilfer fractional derivative.” *Commun. Nonlinear Sci. Numer. Simul.*, vol. 60, pp. 72–91, 2018, doi: 10.1016/J.CNSNS.2018.01.005.

[26] J. Wang and Y. Zhang, "Nonlocal initial value problems for differential equations with Hilfer fractional derivative." *Appl. Math. Comput.*, vol. 266, pp. 850—-859, 2015, doi: 10.1016/j.amc.2015.05.144.

Authors' addresses

Belqassim Azzouz
Relizane University, Department of Mathematics, Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Ben Bella, Algeria
E-mail address: azzouzbel27@gmail.com

Moustafa Beddani
(**Corresponding author**) ENS of Mostaganem, Department of Mathematics, Mostaganem, Algeria
E-mail address: beddani2004@yahoo.fr

Hamid Beddani
Laboratory of Complex Systems, Higher School of Electrical and Energy Engineering, Oran, Algeria
E-mail address: beddanihamid@gmail.com