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HERMITE-HADAMARD TYPE INEQUALITIES FOR (M_{ϕ},M_{ψ})-CONVEX FUNCTIONS

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Abstract. In this paper, the authors introduce the generalization of Hermite-Hadamard inequality by using (M_{φ},M_{ψ})-convex functions and getting some other theorems with (M_{φ},M_{ψ})-convex functions. Some natural applications to special means of real numbers are also given.

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1. INTRODUCTION AND PRELIMINARIES

It is well known in mathematical analysis that a function $\mathfrak{f}\colon I\subseteq\mathbb{R}\to\mathbb{R}$, $I\neq\emptyset$ is said to be convex on I if the inequality

$$\mathfrak{f}(\lambda x+(1-\lambda)y)\leq\lambda\mathfrak{f}(x)+(1-\lambda)\mathfrak{f}(y) \tag{1.1}$$

holds for all $x,y\in I$ and $\lambda\in[0,1]$ [25] .

Definition 1 ([4, 12])**.** A function $\mathfrak{f}\colon I\to[0,\infty)$ is said to be AG-convex or log-convex or multiplicatively convex if $\log\mathfrak{f}$ is convex, or equivalently if for all $x,y\in I$ and $\lambda\in[0,1]$ one has the inequality:

$$\mathfrak{f}(\lambda x+(1-\lambda)y)\leq\mathfrak{f}(x)^{\lambda}\mathfrak{f}(y)^{1-\lambda}. \tag{1.2}$$

Definition 2 ([16])**.** Let $\mathbb{I}\subseteq\mathbb{R}\setminus\{0\}$ be an interval. Then a real-valued function $\mathfrak{f}\colon I\to\mathbb{R}$ is said to be harmonically convex if

$$\mathfrak{f}\left(\frac{xy}{\lambda x+(1-\lambda)y}\right)\leq\lambda\mathfrak{f}(y)+(1-\lambda)\mathfrak{f}(x) \tag{1.3}$$

holds for all $x,y\in\mathbb{I}=[c,d]$ and $\lambda\in[0,1]$.

Definition 3 ([4, 12])**.** A function $\mathfrak{f}\colon \mathbb{I}\subseteq\mathbb{R}\to\mathbb{J}\subseteq\mathbb{R}\setminus\{0\}$ is called AH-convex on the convex set C if the following inequality holds

$$\mathfrak{f}(\lambda x+(1-\lambda)y)\leq\frac{\mathfrak{f}(x)\mathfrak{f}(y)}{(1-\lambda)\mathfrak{f}(y)+\lambda\mathfrak{f}(x)} \tag{1.4}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Definition 4 ([4])**.** Let $I\subset(0,\infty)$ be an interval; a real-valued function $\mathfrak{f}\colon I\to\mathbb{R}$ is said to be GH-convex on I if

$$\mathfrak{f}(x^{1-\lambda}y^{\lambda})\leq\frac{\mathfrak{f}(x)\mathfrak{f}(y)}{(1-\lambda)\mathfrak{f}(y)+\lambda\mathfrak{f}(x)} \tag{1.5}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 5 ([4, 12])**.** Let $I\subset(0,\infty)$ be an interval; a real-valued function $\mathfrak{f}\colon I\to\mathbb{J}\subseteq\mathbb{R}\setminus\{0\}$ is said to be GA-convex on I if

$$\mathfrak{f}(x^{1-\lambda}y^{\lambda})\leq(1-\lambda)\mathfrak{f}(x)+\lambda\mathfrak{f}(y) \tag{1.6}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 6 ([4, 12])**.** The function $\mathfrak{f}\colon I\subset(0,\infty)\to(0,\infty)$ is called GG-convex on the interval I of real numbers $\mathbb{R}$ if

$$\mathfrak{f}(x^{1-\lambda}y^{\lambda})\leq\mathfrak{f}(x)^{(1-\lambda)}\cdot\mathfrak{f}(y)^{\lambda} \tag{1.7}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 7 ([4, 12])**.** We say that the function $\mathfrak{f}\colon I\subset\mathbb{R}\setminus\{0\}\to(0,\infty)$ is HG-convex or harmonically convex if

$$\mathfrak{f}\left(\frac{xy}{(1-\lambda)y+\lambda x}\right)\leq\mathfrak{f}(x)^{(1-\lambda)}\mathfrak{f}(y)^{\lambda} \tag{1.8}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 8 ([4, 12])**.** We say that the function $\mathfrak{f}\colon I\subset\mathbb{R}\setminus\{0\}\to(0,\infty)$ is HH-convex or harmonically convex if

$$\mathfrak{f}\left(\frac{xy}{(1-\lambda)y+\lambda x}\right)\leq\frac{\mathfrak{f}(x)\mathfrak{f}(y)}{(1-\lambda)\mathfrak{f}(y)+\lambda\mathfrak{f}(x)} \tag{1.9}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 9 ([18, 28])**.** Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\setminus\{0\}$. A function $\mathfrak{f}\colon I\to\mathbb{R}$ is said to be p-convex, if

$$\mathfrak{f}\left([\lambda x^{p}+(1-\lambda)y^{p}]^{\frac{1}{p}}\right)\leq\lambda\mathfrak{f}(x)+(1-\lambda)\mathfrak{f}(y) \tag{1.10}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 10 ([27]). Let $I \subset (0,\infty)$ be a interval, $\varphi\colon I \to \mathbb{R}$ be a continuous and strictly monotonic function. $\mathfrak{f}\colon I \to \mathbb{R}$ is said to be $M_\varphi A$ convex, if

$$\mathfrak{f}\left(\varphi^{-1}(\lambda\varphi(x)+(1-\lambda)\varphi(y))\right) \leq \lambda\mathfrak{f}(x)+(1-\lambda)\mathfrak{f}(y) \tag{1.11}$$

for all $x,y \in I$ and $\lambda \in [0,1]$.

A number of inequalities have been written for convex functions but the most famous is the Hermite-Hadamard inequality which is stated as follows (see, e.g., [22]):

If $\mathfrak{f}\colon [c,d] \to \mathbb{R}$ is a convex function, then the following inequality is known as Hermite-Hadamard inequality:

$$\mathfrak{f}\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \mathfrak{f}(x)\mathrm{d}x \leq \frac{\mathfrak{f}(c)+\mathfrak{f}(d)}{2}. \tag{1.12}$$

Note that some of classical inequalities for means can be derived from (1.12) for appropriate particular selections of the mapping $\mathfrak{f}$. Both inequalities hold in the reversed direction if $\mathfrak{f}$ is concave. For some results which generalize, improve and extend the inequality (1.12) we refer the reader to the recent papers (see [2,3,5–7,14,15,17,19–21,23,24]).

A convex (or concave) function $\mathfrak{f}$ is bounded on every compact subinterval $[u,v]$ of its interval of definition. If $\mathfrak{f}\colon I \subset R \to R$ is convex, then $\mathfrak{f}$ is continuous on interior I^o of I [26].

Let $A(c,d;\lambda) = \lambda c + (1-\lambda)d$, $G(c,d;\lambda) = c^\lambda d^{1-\lambda}$, $H(c,d;\lambda) = cd/(\lambda c + (1-\lambda)d)$ and $M_p(c,d;\lambda) = (\lambda c^p + (1-\lambda)d^p)^{1/p}$ be the weighted arithmetic, geometric, harmonic, power of order p means of two positive real numbers c and d with $c \neq d$ for $\lambda \in [0,1]$, respectively. The most used class of means is quasi-arithmetic mean, which is associated to a continuous and strictly monotonic function $\varphi\colon I \to R$ by the formula

$$M_\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \text{ for } x,y \in I.$$

Weighted quasi-arithmetic mean is given by the formula

$$M_\varphi(x,y;\lambda) = \varphi^{-1}\left(\lambda\varphi(x)+(1-\lambda)\varphi(y)\right), \text{ for } x,y \in I,\ \lambda \in [0,1].$$

Here $\lambda \in (0,1)$ and $x < y$ always implies $x < M_\varphi(x,y;\lambda) < y$. The function is called Kolmogoroff-Naguma function of M. Of special interest are the power means M_p on $\mathbb{R}_+$, defined by

$$\varphi_p(x) := \begin{cases} x^p, & p \neq 0, \\ \ln x, & p = 0. \end{cases}$$

For $p = 1$, we get the arithmetic mean $A = M_1$, for $p = 0$, we get the geometric mean $G = M_0$ and for $p = -1$, we get the harmonic mean $H = M_{-1}$.

Definition 11 ([1])**.** For any two quasi-arithmetic means M,N (with Kolmogoroff-Naguma function φ,ψ defined on I,J, respectively), a function $\mathfrak{f}\colon I\to J$ can be called (M_{φ},M_{ψ})-convex if it satisfies

$$\mathfrak{f}(M_{\varphi}(x,y;\lambda))\le M_{\psi}(\mathfrak{f}(x),\mathfrak{f}(y);\lambda) \tag{1.13}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. If the inequality in (1.13) is reversed, then $\mathfrak{f}$ said to be (M_{φ},M_{ψ})-concave.

If $\psi\colon \mathbb{R}\to\mathbb{R}$, $\psi(x)=x$, (i.e., $M_{\psi}(\mathfrak{f}(x),\mathfrak{f}(y);\lambda)=A(c,d;\lambda)$), then we just say that $\mathfrak{f}$ is $M_{\varphi}A$-convex. Let $\mathfrak{f}$ be $M_{\varphi}A$-convex.

(1) If we take $\varphi\colon I\subset\mathbb{R}\to\mathbb{R}$, $\varphi(x)=x$, then $M_{\varphi}A$-convexity deduces usual convexity.
(2) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=\ln x$, then $M_{\varphi}A$-convexity deduces GA-convexity.
(3) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=x^{-1}$, then $M_{\varphi}A$-convexity deduces harmonically convexity.
(4) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=x^{p}$, then $M_{\varphi}A$-convexity deduces p-convexity.

If $\psi\colon (0,\infty)\to\mathbb{R}$, $\psi(x)=\ln x$, (i.e., $M_{\psi}(\mathfrak{f}(x),\mathfrak{f}(y);\lambda)=G(c,d;\lambda)$), then we just say that $\mathfrak{f}$ is $M_{\varphi}G$-convex. Let $\mathfrak{f}$ be $M_{\varphi}G$-convex.

(1) If we take $\varphi\colon I\subset\mathbb{R}\to\mathbb{R}$, $\varphi(x)=x$, then $M_{\varphi}G$-convexity deduces logarithmic convexity.
(2) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=\ln x$, then $M_{\varphi}G$-convexity deduces GG-convexity.
(3) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=x^{-1}$, then $M_{\varphi}G$-convexity deduces harmonically G-convexity.
(4) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=x^{p}$, then $M_{\varphi}G$-convexity deduces pG-convexity.

If $\psi\colon I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $\psi(x)=x^{-1}$, (i.e., $M_{\psi}(\mathfrak{f}(x),\mathfrak{f}(y);\lambda)=H(c,d;\lambda)$), then we just say that $\mathfrak{f}$ is $M_{\varphi}H$-convex. Let $\mathfrak{f}$ be $M_{\varphi}H$-convex.

(1) If we take $\varphi\colon I\subset\mathbb{R}\to\mathbb{R}$, $\varphi(x)=x$, then $M_{\varphi}H$-convexity deduces AH-convexity.
(2) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=\ln x$, then $M_{\varphi}H$-convexity deduces GH-convexity.
(3) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=x^{-1}$, then $M_{\varphi}H$-convexity deduces HH-convexity.
(4) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R}$, $\varphi(x)=x^{p}$, then $M_{\varphi}H$-convexity deduces pH-convexity.

If $\psi\colon (0,\infty)\to\mathbb{R}$, $\psi(x)=x^{p}$, then we just say that $\mathfrak{f}$ is $M_{\varphi}p$-convex. Let $\mathfrak{f}$ be $M_{\varphi}p$-convex.

(1) If we take $\varphi\colon I\subset\mathbb{R}\to\mathbb{R}$, $\varphi(x)=x$, then $M_{\varphi}p$-convexity deduces p-convexity.

(2) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R},\ \varphi(x)=\ln x$, then $M_\varphi p$-convexity deduces Gp-convexity.
(3) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R},\ \varphi(x)=x^{-1}$, then $M_\varphi p$-convexity deduces harmonically p-convexity.
(4) If we take $\varphi\colon I\subset(0,\infty)\to\mathbb{R},\ \varphi(x)=x^{p}$, then $M_\varphi p$-convexity deduces pp-convexity.

Lemma 1 ([1]). *Let φ and ψ be two continuous and strictly monotonic functions on intervals I and J respectively and let $\mathfrak{f}\colon I\to J$ is a function.*

If ψ is strictly increasing , then $\mathfrak{f}$ is (M_φ,M_ψ)-convex (concave) if and only if $\psi\circ\mathfrak{f}\circ\varphi^{-1}$ is convex (concave) on $\varphi(I)$ in the usual sense.

If ψ is strictly decreasing , then $\mathfrak{f}$ is (M_φ,M_ψ)-convex (concave) if and only if $\psi\circ\mathfrak{f}\circ\varphi^{-1}$ is concave (convex) on $\varphi(I)$ in the usual sense.

The main purpose of this paper is to introduce the generalization of Hermite-Hadamard inequality by using (M_φ,M_ψ)-convex functions and getting some other theorems with (M_φ,M_ψ)-convex functions.

2. MAIN RESULTS

Theorem 1. *Let φ and ψ are two continuous and strictly monotonic functions on intervals I and J respectively and let $\mathfrak{f},\mathfrak{g}\colon I\to J$ are two functions. If $\mathfrak{f}$ and $\mathfrak{g}$ (M_φ,M_ψ)-convex functions, then $\mathfrak{f}\oplus_\psi\mathfrak{g}$ is a (M_φ,M_ψ)-convex function, where $(\mathfrak{f}\oplus_\psi\mathfrak{g})(x):=f(x)\oplus_\psi g(x):=\psi^{-1}(\psi(\mathfrak{f}(x))+\psi(\mathfrak{g}(x)))$, $x\in I$.*

Proof. Firstly, let ψ be strictly increasing, then ψ^{-1} is also strictly increasing. Since $\mathfrak{f}$ and $\mathfrak{g}$ are (M_φ,M_ψ)-convex functions, we have

$$\mathfrak{f}(M_\varphi(x,y;\lambda))\le M_\psi(\mathfrak{f}(x),\mathfrak{f}(y);\lambda) \tag{2.1}$$

and

$$\mathfrak{g}(M_\varphi(x,y;\lambda))\le M_\psi(\mathfrak{g}(x),\mathfrak{g}(y);\lambda), \tag{2.2}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. Then, since ψ and ψ^{-1} are strictly increasing, with (2.1) and (2.2), we have

$$\begin{aligned}(\mathfrak{f}\oplus_\psi\mathfrak{g})(M_\varphi(x,y;\lambda)) &= \psi^{-1}(\psi(\mathfrak{f}(M_\varphi(x,y;\lambda)))+\psi(\mathfrak{g}(M_\varphi(x,y;\lambda))))\\ &\le \psi^{-1}(\psi(M_\psi(\mathfrak{f}(x),\mathfrak{f}(y);\lambda))+\psi(M_\psi(\mathfrak{g}(x),\mathfrak{g}(y);\lambda)))\\ &= \psi^{-1}(\lambda(\psi(\mathfrak{f}(x))+\psi(\mathfrak{g}(x)))+(1-\lambda)(\psi(\mathfrak{f}(y))+\psi(\mathfrak{g}(y))))\\ &= \psi^{-1}(\lambda\psi((\mathfrak{f}\oplus_\psi\mathfrak{g})(x))+(1-\lambda)\psi((\mathfrak{f}\oplus_\psi\mathfrak{g})(y)))\\ &= M_\psi((\mathfrak{f}\oplus_\psi\mathfrak{g})(x),(\mathfrak{f}\oplus_\psi\mathfrak{g})(y);\lambda),\end{aligned}$$

which completes the proof. □

Remark 1. In the above theorem, it reduces to geometric and arithmetic inequalities when special choices are made.

Theorem 2. *Let* φ *and* ψ *be two continuous and strictly monotonic functions on intervals* I *and* J *respectively. If* $\mathfrak{f}$ (M_φ, M_ψ)*-convex function on* $[u,v] \subset I$*, then* $\mathfrak{f}$ *is bounded on* $[u,v]$.

Proof. Since $\mathfrak{f}$ is (M_φ, M_ψ)-convex function on $[u,v]$, if φ is strictly increasing (φ is strictly decreasing), then $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex (concave) on $\varphi([u,v])$. Thus $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is bounded on $\varphi([u,v])$. Therefore , with the continuity of ψ^{-1}, $\mathfrak{f} \circ \varphi^{-1}$ becomes bounded on $\varphi([u,v])$ which gives us that $\mathfrak{f}$ is bounded on $[u,v]$. This completes the proof. ☐

Theorem 3. *Let* φ *and* ψ *be two continuous and strictly monotonic functions on intervals* I *and* J *respectively. If* $\mathfrak{f}$ (M_φ, M_ψ)*-convex function on* I*, then* $\mathfrak{f}$ *is a continuous function on* I^o.

Proof. Since $\mathfrak{f}$ is (M_φ, M_ψ)-convex function, if ψ is strictly increasing (ψ is strictly decreasing), then $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex(concave) on $\varphi(I)$. Thus $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is continuous on $\varphi(I^o) = (\varphi(I))^o$ (this equality is satisfied because φ is a homeomorphism). Therefore, with continuity of ψ^{-1}, $\mathfrak{f} \circ \varphi^{-1}$ becomes continuous on $\varphi(I^o)$ which gives us that $\mathfrak{f}$ is continuous on I^o. This completes the proof. ☐

Theorem 4. *Let* φ *and* ψ *be two continuous and strictly monotonic functions on* $(0,\infty)$ *and let* $\mathfrak{f}\colon (0,\infty) \to \mathbb{R}$ *is a function. If* $c,d \in (0,\infty)$ *with* $c<d$ *and* $\mathfrak{f}$ *is* (M_φ, M_ψ)*-convex then the following inequalities hold:*

$$\begin{aligned} \mathfrak{f}\left(\varphi^{-1}\left(\frac{\varphi(c)+\varphi(d)}{2}\right)\right) &\le \psi^{-1}\left[\frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)} (\psi \circ \mathfrak{f} \circ \varphi^{-1})(x)\mathrm{d}x\right] \\ &\le \psi^{-1}\left(\frac{\psi(\mathfrak{f}(c))+\psi(\mathfrak{f}(d))}{2}\right). \end{aligned} \tag{2.3}$$

The above inequalities are sharp.

Proof. Firstly, let ψ be strictly increasing. Since $\mathfrak{f}\colon (0,\infty) \to \mathbb{R}$ is a (M_φ, M_ψ)-convex function, $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex on $\varphi((0,\infty))$. So, by the inequalities (1.12) we have

$$\begin{aligned} (\psi \circ \mathfrak{f} \circ \varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right) &\le \frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)} (\psi \circ \mathfrak{f} \circ \varphi^{-1})(x)\mathrm{d}x \\ &\le \frac{(\psi \circ \mathfrak{f} \circ \varphi^{-1})(\varphi(c))+(\psi \circ \mathfrak{f} \circ \varphi^{-1})(\varphi(d))}{2}, \end{aligned}$$

i.e.

$$\begin{aligned} (\psi \circ \mathfrak{f} \circ \varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right) &\le \frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)} (\psi \circ \mathfrak{f} \circ \varphi^{-1})(x)\mathrm{d}x \\ &\le \frac{(\psi \circ \mathfrak{f})(c)+(\psi \circ \mathfrak{f})(d)}{2}. \end{aligned}$$

Since ψ^{-1} is strictly increasing, we have (2.3).

Secondly, let ψ be strictly decreasing. Since $\mathfrak{f}\colon (0,\infty)\to\mathbb{R}$ is a (M_φ,M_ψ)-convex function, $\psi\circ\mathfrak{f}\circ\varphi^{-1}$ is concave on $\varphi((0,\infty))$. So, by the inequalities (1.12) we have

$$\begin{aligned}(\psi\circ\mathfrak{f}\circ\varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right) &\geq \frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)}(\psi\circ\mathfrak{f}\circ\varphi^{-1})(x)\mathrm{d}x\\ &\geq \frac{(\psi\circ\mathfrak{f}\circ\varphi^{-1})(\varphi(c))+(\psi\circ\mathfrak{f}\circ\varphi^{-1})(\varphi(d))}{2},\end{aligned}$$

i.e.

$$\begin{aligned}(\psi\circ\mathfrak{f}\circ\varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right) &\geq \frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)}(\psi\circ\mathfrak{f}\circ\varphi^{-1})(x)\mathrm{d}x\\ &\geq \frac{(\psi\circ\mathfrak{f})(c)+(\psi\circ\mathfrak{f})(d)}{2}.\end{aligned}$$

Since ψ^{-1} is strictly decreasing, we have (2.3). □

Remark 2. (1) If we choose $\varphi(x)=x$ and $\psi(x)=x$ in Theorem 4, our result deduces to (1.12).

(2) If we choose $\varphi(x)=\ln x$ and $\psi(x)=x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for GA-convex functions in [9].

(3) If we choose $\varphi(x)=x^{-1}$ and $\psi(x)=x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for Harmonic functions in [16].

(4) If we choose $\varphi(x)=x^p$ and $\psi(x)=x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for p-convex functions in [13, 18].

(5) If we choose $\varphi(x)=x$ and $\psi(x)=\ln x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for Logaritmic convex functions in [12].

(6) If we choose $\varphi(x)=\ln x$ and $\psi(x)=\ln x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for GG-convex functions in [10].

(7) If we choose $\varphi(x)=x^{-1}$ and $\psi(x)=\ln x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for HG-convex functions in [8].

(8) If we choose $\varphi(x)=\ln x$ and $\psi(x)=x^{-1}$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for GH-convex functions in [11].

3. CONCLUSION

The aim of this study is to make a generalized proof with this function that can reach more specific results for the algebraic properties of many convexities in the literature or for the Hermite-Hadamard inequality. This function will give us more general results and its special cases will be reduced to the literature.

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