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EXISTENCE OF SOLUTIONS FOR A GENERAL CLASS OF NONLINEAR FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this study, we aim to investigate the existence results for a general class of fractional nonlinear integral equations with order $\alpha \in (0,1)$ in a continuous function space $\big(C[a,b], \|\cdot\|\big)$. We use the Schauder fixed point theorem as a tool with providing a compact integral operator from a subset of the Banach space $\big(C[a,b], \|\cdot\|\big)$ into itself. Furthermore, we present an example to illustrate and support the work.

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1. Introduction

In many applications, nonlinear fractional integral equations play a considerable role in mathematical physics and engineering due to their ability to model complex phenomena in real-world problems, particularly in fields such as fluid mechanics, engineering, and chemical reactions [1, 2, 7, 9]. Nonlinear fractional and weakly singular integral equations naturally arise in a wide range of applications across many other fields, including physics, biology (involving population dynamics), chemical, mechanical engineering, thermal explosions, nuclear physics, chemical kinetics, and control theory [1–3, 7]. We consider the following general nonlinear fractional integral equation:

$$u(\xi) = g(\xi) + \int_{t_0}^{\xi} (\xi-\rho)^{\alpha-1}\Psi\Big(\rho, u(\rho), \mathrm{D}^{\gamma}u(\rho)\Big)\mathrm{d}\rho, \tag{1.1}$$

where $\alpha \in (0,1)$, g is a given continuous function on $[t_0,T]$, $\Psi\colon [t_0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function satisfying specified regularity conditions, and u is unknown solution of (1.1), while D^{γ} is the Riemann-Liouville fractional integral operator of order $\gamma > 0$. It is important to emphasize that the operator D^{γ} in (1.1) can be switched to other differential or integral operators with changing the relative continuous space $\big(C[a,b], \|\cdot\|\big)$.

The kernels in numerous real modeled integral equations are not smooth so this makes challenging to determine the exact and numerical solutions in such type of problems. Existence and uniqueness of solutions for nonlinear modeling for example nonlinear singular integral equations are crucial because it is impossible to accomplish analytic solutions to almost mathematical models of real-world problems. Existence and uniqueness results are the theoretical principle for effective numerical methods to establish approximate solutions.

Our goal in this work is to use the Schauder fixed point theorem as a main tool to achieve an existence of solution u to the proposed problem (1.1) in a subset of the Banach space $\big(C[t_0,T],\|\cdot\|\big)$.

In regard to previous study on existence and uniqueness of solutions to different type of singular integral equations, we recommend [2,4–7,9] as references related to the subject in this work.

In 2023, existence and properties of travelling-wave solutions for a family of singular Volterra integral equations:

$$\mathbb{X}(t)=\mathbb{F}(t)+\int_0^t \frac{\mathbb{G}(\tau)}{\mathbb{X}^{\beta}(\tau)}\mathrm{d}\tau, \quad \text{where } \beta\in(0,\infty) \text{ and } t>0.$$

have been studied by Garriz [4]. Matoog and Abdou [7] have worked on the existence and uniqueness of solutions for the nonlinear integral equation of the following problem:

$$\frac{\partial}{\partial s}(\omega\Theta(s,u)-f(s,u))=\lambda\xi(s)\int_0^1 k(u,v)\Theta(s,v,\Theta(t,v))\mathrm{d}v$$

in the Banach space $L^2[0,1]\times C[0,T]$. Aghajani, Banaś, and Jalilian [1] demonstrated the existence of solutions for the following singular integral equations using the fixed point theorem of the Darbo type:

$$\mathbb{F}(x)=\mathbb{F}_1(x,\mathbb{F}(x),\mathbb{F}(a(x)))+G\mathbb{F}(x)\int_0^x k(x,s)/(x-s)^{1-\alpha}Q\mathbb{F}(s)\mathrm{d}s.$$

In 2023, Alhazmi, Mahdy, and Mohamed [2] studied stability and error analysis for the symmetric singular kernel mixed integral equations with discussing existence and uniqueness of solution for the following problem:

$$\Phi(\tau,\xi)\Theta(\tau,\xi)=f(\tau,\xi)+\lambda\int_0^{\xi}\int_{-1}^{1}(\tau-y)^{-2}F(\xi,s)\Theta(y,s)\mathrm{d}y\mathrm{d}s,$$

subject to the boundary conditions; $\Theta(1,\xi)=\Theta(-1,\xi)=0$, for all $\xi\in[0,a]$ and $a<1$.

2. Preliminary notation

In this section, we recall some fundamental concepts and definitions, which are used as supplementary in this paper.

Definition 1. Let $\gamma > 0$ and let R be a continuous function on $[a_0, \infty)$. Then

$$\mathrm{D}^{\gamma} R(\zeta) = \frac{1}{\Gamma(\gamma)} \int_{a_0}^{\zeta} (\zeta - \xi)^{\gamma - 1} R(\xi) \mathrm{d}\xi, \tag{2.1}$$

is said to be the Riemann-Liouville fractional integral operator of order γ.

Definition 2. [[8]] Let $-\infty < t_0 < b < \infty$, and let $\big(X[t_0,b], \|\cdot\|\big)$ be a Banach space. Let W be a subset of $X[t_0,b]$ with the following conditions:

- W is bounded set,
- W is equicontinuous. That is, for every positive ε, there corresponds a positive δ such that

$$\big|w(t) - w(\tau)\big| < \varepsilon, \text{when } |t - \tau| < \delta, \text{ and for all } w \in W. \tag{2.2}$$

Then W is a relatively compact subset of $X[t_0,b]$.

Definition 3 ([8, Definition 11])**.** Let $\big(X_1, \|\cdot\|\big)$ and $\big(X_2, \|\cdot\|\big)$ be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$). The operator

$$\mathrm{A}\colon W \subseteq X_1 \to X_2,$$

is said to be compact if and only if

1) A is continuous;
2) A maps bounded sets into relatively compact sets.

Theorem 1 ([8])**.** *Let $-\infty < t_0 < b < \infty$, and let $\big(X[t_0,b], \|\cdot\|\big)$ be a Banach space. Let $W \neq \varnothing$ be closed, bounded, and convex subset of $X[t_0,b]$. Then any compact operator $\mathrm{B}\colon W \to W$ has at least one fixed point. This is Schauder's fixed point theorem.*

3. Existence of solutions

The main findings of the existence of a solution for general nonlinear singular integral operators are discussed and illustrated in this section. Let $I = [t_0, T] \times \mathbb{R} \times \mathbb{R}$ and assume that the following hypotheses are true for Ψ:

(H1) $\Psi\colon I \to \mathbb{R}$ is a measurable function and uniformly bounded by some positive constant M on each compact subset of I;

(H2) For each compact interval $[-\theta, \theta] \subset \mathbb{R}$, there corresponds a positive constant L such that:

$$\big|\Psi(\omega, \phi_1, \varphi_1) - \Psi(\omega, \phi_2, \varphi_2)\big| \le L\big(|\phi_1 - \varphi_1| + |\phi_2 - \varphi_2|\big), \text{ for all } \omega \in [t_0, T],$$

and all $\phi_1, \phi_2, \varphi_1, \varphi_2 \in [-\theta, \theta]$.

Let y be a function in $C[t_0,T]$ and let its supremum norm be denoted and defined by:

$$\|y\|_{\sup} = \sup_{\tau \in [t_0,T]} \big\{|y(\tau)|\big\},$$

for simplicity in this paper, we use $\|\cdot\|$ sometimes for $\|\cdot\|_{\sup}$.

Lemma 1. *The fractional integral operator* $\mathrm{D}^{\gamma}\mathfrak{g}$ *is continuous.*

Proof. Let $\mathfrak{g}_n$ be a sequence of continuous functions on $[t_0,T]$ that converges to a function $\mathfrak{g}$. That is, for each $\varepsilon > 0$, there is $N > 0$ such that $\|\mathfrak{g}_n - \mathfrak{g}\| < \frac{\varepsilon\gamma\Gamma(\gamma)}{(T-t_0)^{\gamma}}$ for all $n > N$.

$$\begin{aligned}\left\|\mathrm{D}^{\gamma}\mathfrak{g}_n(\xi)-\mathrm{D}^{\gamma}\mathfrak{g}(\xi)\right\| &\le \frac{1}{\Gamma(\gamma)}\sup_{\xi\in[t_0,T]}\int_{t_0}^{\xi}(\xi-\tau)^{\gamma-1}\left|\mathfrak{g}_n(\tau)-\mathfrak{g}(\tau)\right|\mathrm{d}\tau\\ &\le \frac{1}{\Gamma(\gamma)}\|\mathfrak{g}_n-\mathfrak{g}\|\sup_{\xi\in[t_0,T]}\int_{t_0}^{\xi}(\xi-\tau)^{\gamma-1}\mathrm{d}\tau\\ &= \frac{1}{\Gamma(\gamma)}\|\mathfrak{g}_n-\mathfrak{g}\|\,\frac{(T-t_0)^{\gamma}}{\gamma}\\ &< \varepsilon\end{aligned}$$

for all $n > N$. Thus, $\mathrm{D}^{\gamma}\mathfrak{g}$ is continuous on $[t_0,T]$. □

Lemma 2. *Let* $a,b \in [0,\infty)$ *and* $\theta \in (0,1)$. *Then the following inequality holds*

$$b^{\theta}-a^{\theta} \le \left|b-a\right|^{\theta}.$$

Proof. Since, $0 < \theta < 1$ and for all $\mu,\nu \ge 0$. Then

$$1 \le \left(\frac{\mu}{\mu+\nu}\right)^{\theta}+\left(\frac{\nu}{\mu+\nu}\right)^{\theta},$$

implies that

$$(\mu+\nu)^{\theta} \le \mu^{\theta}+\nu^{\theta}.$$

Let $\mu = b-a$ and $\nu = a$ implies that

$$b^{\theta} \le (b-a)^{\theta}+a^{\theta} \quad\Rightarrow\quad b^{\theta}-a^{\theta} \le \left(b-a\right)^{\theta} \le \left|b-a^{\theta}\right|.$$

□

For any fix $f \in C[t_0,T]$ and define an operator Λ as:

$$\Lambda u(\eta) = f(\eta)+\int_{t_0}^{\eta}(\eta-\rho)^{\alpha-1}\Psi(\rho,u(\rho),\mathrm{D}^{\gamma}u(\rho))\mathrm{d}\rho,$$

for all $u \in C[t_0,T]$ and $\eta \in [t_0,T]$.

Theorem 2. *If* $u \in C[t_0,T]$ *and* $\alpha \in (0,1)$. *Then* Λu *belongs to* $C[t_0,T]$.

Proof. Let u_k be a convergent sequence of functions in $C[t_0,T]$ that converges to u. That is, for any $\varepsilon > 0$ there is a sufficiently large $N > 0$ such that for all $k > N$ then,

$$\|u_k - u\| = \sup_{\eta\in[t_0,T]} |u_k(\eta) - u(\eta)| < \frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}},$$

and it follows from Lemma 1 that

$$\|\mathrm{D}^{\gamma}u_k - \mathrm{D}^{\gamma}u\| = \sup_{\eta\in[t_0,T]} |\mathrm{D}^{\gamma}u_k(\eta) - \mathrm{D}^{\gamma}u(\eta)| < \frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}}.$$

Implies from hypothesis $(H2)$ that

$$\begin{aligned}
&\|\Lambda u_k(\eta) - \Lambda u(\eta)\| \\
&= \sup_{\eta\in[t_0,T]} \left| \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1} \Big(\Psi(\rho, u_k(\rho), \mathrm{D}^{\gamma}u_k(\rho)) - \Psi(\rho, u(\rho), \mathrm{D}^{\gamma}u(\rho)) \Big) \mathrm{d}\rho \right| \\
&\le \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1} \Big| \Psi(\rho, u_k(\rho), \mathrm{D}^{\gamma}u_k(\rho)) - \Psi(\rho, u(\rho), \mathrm{D}^{\gamma}u(\rho)) \Big| \mathrm{d}\rho \\
&\le L \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1} \Big(|u_k(\rho) - u(\rho)| + |\mathrm{D}^{\gamma}u_k(\rho) - \mathrm{D}^{\gamma}u(\rho)| \Big) \mathrm{d}\rho \\
&\le L \Big(\|u_k - u\| + \|\mathrm{D}^{\gamma}u_k - \mathrm{D}^{\gamma}u\| \Big) \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1} \mathrm{d}\rho \\
&< L \left(\frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}} + \frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}} \right) \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1} \mathrm{d}\rho \\
&= \frac{\alpha\varepsilon}{(T-t_0)^{\alpha}} \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1} \mathrm{d}\rho \\
&\le \frac{\alpha\varepsilon}{(T-t_0)^{\alpha}} \frac{(T-t_0)^{\alpha}}{\alpha} = \varepsilon.
\end{aligned}$$

Therefore, Λu is continuous on $[t_0, T]$. □

Let $\alpha \in (0,1)$ and for any fix $v \in C[t_0,T]$, there corresponds a constant $K > 0$ such that $K = \max_{\tau\in[t_0,T]} |v(\tau)|$. We define a subspace $Y[t_0,T] = \{y \in C[t_0,T]$ such that $\|y\| \le r\}$, where r is some positive constant. It is easy to observe that $Y[t_0,T]$ is closed, bounded, and convex subset of the Banach space $(C[t_0,T], \|\cdot\|)$. For simplicity, in the rest of the paper we use Y sometimes instead of $Y[t_0,T]$.

Now, we want to show that $\Lambda(Y) \subset C[t_0,T]$ is relatively compact. To do this, we state the following Lemmas and Theorem.

Lemma 3. $\Lambda(Y)$ *is a bounded set.*

Proof. For any continuous function f in $[t_0,T]$ and any $u \in Y$. It follows from $(H1)$ that; there is a constant $M>0$ so that:

$$\begin{aligned}
\|\Lambda u\| &= \sup_{\tau\in[t_0,T]} \left| f(\tau) + \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1}\Psi(\rho,u(\rho),D^{\gamma}u(\rho))d\rho \right| \\
&\le \sup_{\tau\in[t_0,T]} |f(\tau)| + \sup_{\tau\in[t_0,T]} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} |\Psi(\rho,u(\rho),D^{\gamma}u(\rho))| d\rho \\
&\le K + M \sup_{\tau\in[t_0,T]} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} d\rho \\
&\le K + M \sup_{\tau} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} d\rho \\
&\le K + M \sup_{\tau\in[t_0,T]} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} d\rho \\
&\le K + M \frac{(T-t_0)^{\alpha}}{\alpha} \\
&= r,
\end{aligned}$$

where $r = K + \frac{M(T-t_0)^{\alpha}}{\alpha}$ is a positive constant. Hence, $\Lambda(Y)$ is bounded. □

Lemma 4. *If $u \in Y$. Then $\Lambda u \in Y$.*

Proof. The proof is followed by using Theorem 2 and Lemma 3. □

Theorem 3. *The operator $\Lambda\colon Y \to Y$ is equicontinuous.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of functions in Y and let $\alpha \in (0,1)$. Without loss of generality, let $t_0 \le \tau < \xi \le T$, and since from hypothesis $(H1)$; Ψ is uniformly bounded function by a positive number M. Assume f belongs to Y then for any $\varepsilon > 0$, there is $\delta = \left(\frac{\alpha\varepsilon}{2M}\right)^{1/\alpha}$, whenever $|\xi-\tau| < \delta$ implies $|f(\xi)-f(\tau)| < \frac{\varepsilon}{2}$ for all $\tau,\xi \in [t_0,T]$. Then,

$$\begin{aligned}
|\Lambda u_k(\xi) - \Lambda u_k(\tau)| \le\ & |f(\xi)-f(\tau)| + \left| \int_{t_0}^{\xi} (\xi-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right. \\
& \left. - \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right| \\
<\ & \frac{\varepsilon}{2} + \left| \int_{t_0}^{\xi} (\xi-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right. \\
& \left. - \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right| \\
\le\ & \frac{\varepsilon}{2} + \int_{t_0}^{\tau} \left((\xi-\rho)^{\alpha-1} - (\tau-\rho)^{\alpha-1} \right) |\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))| d\rho
\end{aligned}$$

$$+\int_{\tau}^{\xi}(\xi-\rho)^{\alpha-1}\left|\Psi(\rho,u_k(\rho),\mathrm{D}^{\gamma}u_k(\rho))\right|\mathrm{d}\rho.$$

It follows from boundedness of Ψ in $(H1)$ that

$$\begin{aligned}
&\left|\Lambda u_k(\xi)-\Lambda u_k(\tau)\right|\\
&\leq \frac{\varepsilon}{2}+M\int_{t_0}^{\tau}\left((\xi-\rho)^{\alpha-1}-(\tau-\rho)^{\alpha-1}\right)\mathrm{d}\rho+M\int_{\tau}^{\xi}(\xi-\rho)^{\alpha-1}\mathrm{d}\rho.\\
&=\frac{\varepsilon}{2}+\frac{M}{\alpha}\left\{-(\xi-\tau)^{\alpha}+(\xi-t_0)^{\alpha}-(\tau-t_0)^{\alpha}+(\xi-\tau)^{\alpha}\right\}\\
&=\frac{\varepsilon}{2}+\frac{M}{\alpha}\left\{(\xi-t_0)^{\alpha}-(\tau-t_0)^{\alpha}\right\}.
\end{aligned}$$

Since $|\xi-\tau|<\delta$ and applying Lemma 2 yields to

$$\begin{aligned}
\left|\Lambda u_k(\xi)-\Lambda u_k(\tau)\right| &\leq \frac{\varepsilon}{2}+\frac{M}{\alpha}\left|\xi-\tau\right|^{\alpha}\\
&<\frac{\varepsilon}{2}+\frac{M}{\alpha}\delta^{\alpha}\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&=\varepsilon,
\end{aligned}$$

for all $\tau,\xi\in[t_0,T]$. Therefore, $\Lambda(Y)$ is equicontinuous. □

Theorem 4. *Let $0\leq t_0<T<\infty$, let Ψ satisfy hypotheses $(H1)-(H2)$, and f belongs to Y. Then there is at least a solution u belongs to Y for problem (1.1).*

Proof. We note that $u\mapsto\Lambda u$ maps Y into itself for all u and all f in Y. Clearly, Y is bounded, closed, and convex subset of the Banach space $\left(C[t_0,T],\|\cdot\|\right)$. In addition, it follows from Theorem 2, Lemma 3, and Theorem 3 that Λ is continuous, $\Lambda(Y)$ is bounded and equicontinuous, respectively. Hence, by the Arzela-Ascoli Theorem, $\Lambda(Y)$ is a relatively compact set in $C[t_0,T]$. As a result, $\Lambda\colon Y\to Y$ is a compact operator because Y is closed. In conclusion, the Schauder fixed point theorem guarantees that problem (1.1) has a fixed point. That is, there is some $u^*\in Y$ so that $u^*=\Lambda(u^*)$. □

4. Application

In this section, we provide an example to support our work. It is clear that determining the true solution to such kind of nonlinear fractional problem (1.1) especially with non smooth kernel is either quite difficult or impossible unless for a simple case. For this matter, we consider the following example.

Let $\alpha = \gamma = 1/2$ and we consider the space $Y = \{u \in C[0,1] \colon \|u\| \leq r\}$. Then the following problem

$$u(\xi) = g(\xi) + \int_0^{\xi} (\xi - \rho)^{-1/2} \Psi\Big(\rho, u(\rho), \mathrm{D}^{\gamma} u(\rho)\Big) \, \mathrm{d}\rho, \tag{4.1}$$

where $g(\xi) = \xi^{1/2} - \frac{2}{3}\sqrt{\pi}\xi^{3/2} - \frac{1}{2}\pi\xi$ with $\Psi\Big(\rho, u(\rho), \mathrm{D}^{\gamma} u(\rho)\Big) = u(\rho) + \mathrm{D}^{1/2} u(\rho)$. We observe that Ψ satisfy the hypotheses $(H1) - (H2)$ and its easy to see that $g \in Y[0,1]$ where $r = K + M\frac{(T-t_0)^{\alpha}}{\alpha} \approx 5.5$. It follows from Theorem 4 that problem (4.1) has a solution u in $Y[0,1]$. In this example, we used the Adomian decomposition method with noise terms phenomenon to get the exact solution and the true solution is $u(\xi) = \xi^{1/2}$ in $Y[0,1]$.

CONCLUSIONS

In this work, we demonstrated that the integral operator (1.1) is compact and maps a closed, bounded, and convex subset of the Banach space $\big(C[t_0, T], \|\cdot\|\big)$ into itself. Consequently, Schauder's fixed point theorem guarantees the existence of at least one solution. Moreover, the illustrative example further verifies the applicability and effectiveness of the theoretical results. These findings contribute to the broader study of fractional integral equations and provide a framework that can be extended in future work to more complex models arising in applied mathematics and related fields.

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