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SOME APPROXIMATIONS ON A SYSTEM OF MULTI-QUADRATIC-QUARTIC FUNCTIONAL EQUATIONS

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Abstract. In the current work, we define multi-quadratic-quartic mappings as a system of k quadratic and $n-k$ quartic functional equations and then present them as an equation. In continuation, we establish the (Hyers-Ulam, Rassias and Găvruţa) stability and hyperstability of the mentioned mappings, by applying the direct (Hyers) method in the setting of Banach spaces. Using a characterization result, we illustrate an example for the case that a multi-quadratic-quartic mapping in the singularity case can not be stable.

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1. INTRODUCTION

Let $(\mathcal{G},+)$ be a commutative group, W be a linear space, and $n \geq 2$ be a natural number. Throughout this paper, for a set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n-\text{times}}$ by X^n. A multivariable mapping $f\colon \mathcal{G}^n \longrightarrow W$ is called *multi-quadratic* [17] if f satisfies the quadratic functional equation

$$\mathcal{Q}(g_1+g_2)+\mathcal{Q}(g_1-g_2)=2\mathcal{Q}(g_1)+2\mathcal{Q}(g_2) \tag{1.1}$$

in each component (see also [11]). Moreover, f is defined as *multi-quartic* if it satisfies the quartic functional equation

$$\begin{aligned}\mathfrak{Q}(2g_1+g_2)+&\mathfrak{Q}(2g_1-g_2)\\ &=4\mathfrak{Q}(g_1+g_2)+4\mathfrak{Q}(g_1-g_2)+24\mathfrak{Q}(g_1)-6\mathfrak{Q}(g_2).\end{aligned} \tag{1.2}$$

in each variable. For more details we refer to [1] and [16].

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Zhao et al. [21] proved that a mapping $f\colon \mathcal{G}^n \longrightarrow W$ is multi-quadratic if and only if the relation

$$\sum_{s\in\{-1,1\}^n} f(\vartheta_1 + s\vartheta_2) = 2^n \sum_{j_1,\ldots,j_n\in\{1,2\}} f(\vartheta_{j_1 1},\ldots,\vartheta_{j_n n})$$

is valid, where $\vartheta_j = (\vartheta_{j1},\ldots,\vartheta_{jn}) \in \mathcal{G}^n$ with $j \in \{1,2\}$. Then, various versions of multi-quadratic mappings (with unification each of them as an equation) were studied in [5, 8, 12, 19]. Moreover, Abbasbeygi et al. [1] presented a characterization of multi-quartic mappings. In fact, they illustrated that every multi-quartic mapping can be described as a equation and vice versa; see also [7] for a different class.

Let us recall that the stability theory has been pioneered by Ulam [20] concerning a question stability of of homomorphisms on groups. Hyers [15] reacted positively to the mentioned query for more groups, assuming that Banach spaces are the groups and homomorphisms are the linear mappings. Recall that a functional equation $\mathfrak{F}$ is said to be *stable* if any approximate solution φ of $\mathfrak{F}$ is near to an exact solution. If φ is an exact solution of $\mathfrak{F}$, then $\mathfrak{F}$ is *hyperstable* [9]. Next, Aoki [3] (resp. Th. M. Rassias [18]) solved the Ulam problem for additive mappings (resp. linear mappings) by an unbounded Cauchy difference. After that, many Hyers-Ulam stability problems for various functional equations and mappings were introduced and investigated by authors; see for instance [11, 13, 14] and other resources. Some stability results for multi-quadratic and multi-quartic mappings in various spaces are available in [7, 10, 17, 19].

Motivated by the discussion above, in this study, we define multi-quadratic-quartic mappings and investigate their structure. In other words, we provide a characterization of such mappings. In fact, we show that every multi-quadratic-quartic mapping can be presented as an equation. we bring some Hyers-Ulam, Rassias and Găvruţa stability results for multi-quadratic-quartic mappings in Banach spaces through the direct method. Finally, we bring an example to show that a multi-quadratic mapping is non-stable in the case of singularity.

2. Presentation of multi-quadratic-quartic mappings

Throughout this paper, for any $t \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}, n \in \mathbb{N}$, $s = (s_1,\cdots,s_n) \in \{-1,1\}^n$ and $v = (v_1,\cdots,v_n) \in V^n$, where V is a linear space. We write $tv := (tv_1,\ldots,tv_n)$ and $sv := (s_1v_1,\ldots,s_nv_n)$.

Definition 1. Given $n \in \mathbb{N}$ and $k \in \{0,\cdots,n\}$. Suppose that V and W are linear spaces. A multivariable mapping $f\colon V^n \longrightarrow W$ is called k-quadratic and $n-k$-quartic (briefly, multi-quadratic-quartic) if $v \mapsto f^{\#}_j(v)$ satisfies (1.1) for all $j \in \{1,\ldots,k\}$ and fulfills (1.2) for all $j \in \{k+1,\ldots,n\}$, where $f^{\#}(v) = f(u_1,\ldots,u_{j-1},v,u_{j+1},\ldots,u_n)$, in which $u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n$ are fixed and arbitrary elements in V.

It is clear that for $k=n$ (resp. $k=0$), we arrive at the multi-quadratic (resp. the multi-quartic) mappings. It is easily verified that the mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ defined through $f(u_1,\cdots,u_n)=\prod_{j=1}^{k}\prod_{i=k+1}^{n} u_j^2 u_i^4$ is multi-quadratic-quartic.

In this section, we assume that V and W are vector spaces over the field $\mathbb{Q}$. Moreover, we consider $v^{[n]}=(v_1,\ldots,v_n)\in V^n$ with $\left(v^{[k]},v^{[n-k]}\right)\in V^k\times V^{n-k}$, where $v^{[k]}:=(v_1,\ldots,v_k)$ and $v^{[n-k]}:=(v_{k+1},\ldots,v_n)$. Put $v_i^{[k]}=(v_{i1},\ldots,v_{ik})\in V^k$ and $v_i^{[n-k]}=(v_{i,k+1}\ldots,v_{in})\in V^{n-k}$ where $i\in\{1,2\}$. Furthermore, for $v_1^{[n]},v_2^{[n]}\in V^n$, we set

$$\mathcal{A}=\left\{\mathfrak{A}_n=(A_{k+1},\ldots,A_n)|\,A_j\in\{v_{1j}\pm v_{2j},v_{1j},v_{2j}\}\right\},$$

where $j\in\{k+1,\ldots,n\}$. Let $p_i\in\mathbb{N}_0$ with $0\le p_i\le n$ and $i\in\{1,2\}$. Consider $\mathcal{A}_{(p_1,p_2)}^{n-k}$ of $\mathcal{A}$ as follows:

$$\mathcal{A}_{(p_1,p_2)}^{n-k}:=\left\{\mathfrak{A}_n\in\mathcal{A}|\ \mathrm{Card}\{A_j: A_j=v_{ij}\}=p_i,\ i\in\{1,2\},j\in\{k+1,\ldots,n\}\right\}.$$

where Card$X$ is the cardinality of the set X.

From now on, we use the following notations:

$$f\left(\mathcal{A}_{(p_1,p_2)}^{n-k}\right):=\sum_{\mathfrak{A}_n\in\mathcal{A}_{(p_1,p_2)}^{n-k}} f(\mathfrak{A}_n)$$

and

$$f\left(u_1,\ldots,u_k,\mathcal{A}_{(p_1,p_2)}^{n-k}\right):=\sum_{\mathfrak{A}_n\in\mathcal{A}_{(p_1,p_2)}^{n-k}} f\left(u_1,\ldots,u_k,\mathfrak{A}_n\right). \tag{2.1}$$

Proposition 1. *If a mapping $f\colon V^n\longrightarrow W$ is k-quadratic and $n-k$-quartic mapping, then f satisfies the equation*

$$\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]}+sv_2^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)$$
$$=2^k\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{j_1,\ldots,j_k\in\{1,2\}} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(v_{j_11},\ldots,v_{j_kk},\mathcal{A}_{(p_1,p_2)}^{n-k}\right), \tag{2.2}$$

for all $v_i^{[k]}=(v_{i1},\ldots,v_{ik})\in V^k$, $v_i^{[n-k]}=(v_{i,k+1}\ldots,v_{in})\in V^{n-k}$ with $i\in\{1,2\}$, where $f\left(v_{j_11},\ldots,v_{j_kk},\mathcal{A}_{(p_1,p_2)}^{n-k}\right)$ is defined in (2.1).

Proof. It is known that for $k\in\{0,n\}$, our result concludes from [21, Theorem 3] and [1, Proposition 2.2] and hence it is assumed that $k\in\{1,\ldots,n-1\}$. Let $v^{[n-k]}\in V^{n-k}$ be a fixed and arbitrary element. Define the mapping $\Phi_{v^{[n-k]}}: V^k\longrightarrow W$ through $\Phi_{v^{[n-k]}}\left(v^{[k]}\right):=f\left(v^{[k]},v^{[n-k]}\right)$ for $v^{[k]}\in V^k$. By hypothesis, $\Phi_{v^{[n-k]}}$ is k-quadratic, and [21, Theorem 3] necessitates that

$$\sum_{s\in\{-1,1\}^k}\Phi_{v^{[n-k]}}\left(v_1^{[k]}+sv_2^{[k]}\right)$$

$$= 2^k \sum_{j_1,\ldots,j_k\in\{1,2\}} \Phi_{v^{[n-k]}}\left(v_{j_1 1},\cdots,v_{j_k k}\right), \qquad \left(v_1^{[k]},v_2^{[k]}\in V^k\right).$$

By the definition of Φ, the above equality implies that

$$\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^k, v^{[n-k]}\right) = 2^k \sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\cdots,v_{j_k k},v^{[n-k]}\right) \tag{2.3}$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $v^{[n-k]}\in V^{n-k}$. Similar to the above, for any $v^{[k]}\in V^k$, define the mapping $h_{v^{[k]}}\colon V^{n-k}\longrightarrow W$ via $\Psi_{v^{[k]}}\left(v^{[n-k]}\right):=f\left(v^{[k]},v^{[n-k]}\right)$, $v^{[n-k]}\in V^{n-k}$ which is $n-k$-quartic and so we figure out from [1, Proposition 2.2] that

$$\sum_{t\in\{-1,1\}^{n-k}} \Psi_{v^{[k]}}\left(2v_1^{[n-k]}+tv_2^{[n-k]}\right) = \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2}\Psi_{v^{[k]}}\left(\mathcal{A}^{n-k}_{(p_1,p_2)}\right) \tag{2.4}$$

for all $v_1^{[n-k]},v_2^{[n-k]}\in V^{n-k}$. It follows from (2.4) that

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(v^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right) = \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(v^{[k]},\mathcal{A}^{n-k}_{(p_1,p_2)}\right) \tag{2.5}$$

for all $v_1^{[n-k]},v_2^{[n-k]}\in V^{n-k}$ and $v^{[k]}\in V^k$. Plugging (2.3) and (2.5), we get

$$\begin{aligned}
&\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]}+sv_2^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)\\
&=\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(v_1^{[k]}+sv_2^{[k]},\mathcal{A}^{n-k}_{(p_1,p_2)}\right)\\
&=2^k\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2}\sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\ldots,v_{j_k k},\mathcal{A}^{n-k}_{(p_1,p_2)}\right)
\end{aligned}$$

for all $v_i^{[k]}=(v_{i1},\ldots,v_{ik})\in V^k$ and $v_i^{[n-k]}=(v_{ik+1}\ldots,v_{in})\in V^{n-k}$. □

Definition 2. We say a mapping $f\colon V^n\longrightarrow W$ satisfies (has) the *quartic condition* in the jth variable if the mapping $f_j^\bullet\colon V\longrightarrow W$ defined by

$$f_j^\bullet(u)=f(u_1,\ldots,u_{j-1},u,u_{j+1},\ldots,u_n)$$

that has the property $f_j^\bullet(2u)=2^4 f_j^\bullet(u)$ for all $j\in\{1,\ldots,n\}$, where $u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n$ are fixed and arbitrary elements in V.

In the following result, we show that if every mapping f satisfies (2.2), then it is a multi-quadratic-quartic under the condition which is given in Definition 2.

Proposition 2. *Given a mapping $f\colon V^n \longrightarrow W$. Suppose that f satisfies equation (2.2) and moreover has the assumption $(H1)$ as follows:*

$(H1)$**:** *f has the quartic condition in the last $n-k$ variables.*

Then, f is a multi-quadratic-quartic mapping.

Proof. We first note that similar to proof of [4, Lemma 2.5], one can show that $(H2)$ is true, where

$(H2)$**:** $f\left(v^{[n]}\right)=0$ for any $v^{[n]}\in V^n$ provided that at least one component of $v^{[n]}$ is zero.

By putting $v_2^{[n-k]} = (\overbrace{0,\ldots,0}^{n-k-\text{times}})$ in (2.2) and using the hypothesis, the left side of (2.2) will be

$$
\begin{aligned}
2^{n-k}\times 2^{4(n-k)} &\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^{[k]},v_1^{[n-k]}\right)\\
&= 2^{5(n-k)}\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^{[k]},v_1^{[n-k]}\right),
\end{aligned}
\tag{2.6}
$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $v_1^{[n-k]}\in V^{n-k}$. Moreover, under the above replacement, the right side of (2.2) converts to

$$
\begin{aligned}
&2^k\sum_{p_1=0}^{n-k} 2^{n-k-p_1}4^{n-k-p_1}24^{p_1}\sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\ldots,v_{j_k k},v_1^{[n-k]}\right)\\
&=2^k\sum_{p_1=0}^{n-k} 8^{n-k-p_1}24^{p_1}\sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\ldots,v_{j_k k},v_1^{[n-k]}\right)\\
&=2^{5n-4k}\sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\ldots,v_{j_k k},v_1^{[n-k]}\right),
\end{aligned}
\tag{2.7}
$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $x_1^{[n-k]}\in V^{n-k}$. It follows from (2.6) and (2.7) that

$$
\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^{[k]},v_1^{[n-k]}\right)=2^k\sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\ldots,v_{j_k k},v_1^{[n-k]}\right),
$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $v_1^{[n-k]}\in V^{n-k}$. By [21, Theorem 3], we find that f is quadratic in each of the k first components. In addition, replacing $v_2^{[k]}$ by $(0,\ldots,0)$ in (2.2), we get

$$
2^k\sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)
$$

$$= 2^k \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(v_1^{[k]}, \mathcal{A}_{(p_1,p_2)}^{n-k}\right),$$

for all $v_1^{[k]} \in V^k$ and $v_1^{[n-k]}, v_2^{[n-k]} \in V^{n-k}$. It follows from [1, Proposition 2.2] that f is multi-quartic in the $n-k$ last variables and now it completes the proof. □

3. Stability of (2.2)

For a given mapping $f: V^n \longrightarrow W$, for simplicity, we use the difference notation

$$\mathfrak{D}_q f\left(v_1^{[n]}, v_2^{[n]}\right) := \sum_{s\in\{-1,1\}^k} \sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]} + s v_2^{[k]}, 2v_1^{[n-k]} + t v_2^{[n-k]}\right)$$

$$- 2^k \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \sum_{j_1,\dots,j_k\in\{1,2\}} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(v_{j_1 1}, \dots, v_{j_k k}, \mathcal{A}_{(p_1,p_2)}^{n-k}\right),$$

for all $v_i^{[k]} = (v_{i1}, \dots, v_{ik}) \in V^k$, $v_i^{[n-k]} = (v_{i,k+1} \dots, v_{in}) \in V^{n-k}$ and $i \in \{1,2\}$, where $f\left(v_{j_1 1}, \dots, v_{j_k k}, \mathcal{A}_{(p_1,p_2)}^{n-k}\right)$ is defined in (2.1). From now on, we assume that all multivariable mappings f have the condition $(H2)$.

Theorem 1. *Let V and W be a linear space and a Banach space, respectively. Let also $f: V^n \longrightarrow W$ be a mapping in which there exists a function $\psi: V^n \times V^n \longrightarrow [-\alpha, \infty)$ so that*

$$\widetilde{\psi}\left(v_1^{[n]}, v_2^{[n]}\right) := \sum_{j=0}^{\infty} \frac{1}{2^{(5n-3k)\beta j}} \psi\left(2^{\frac{\beta-1}{2}+\beta j} v_1^{[n]}, 2^{\frac{\beta-1}{2}+\beta j} v_2^{[n]}\right) < \infty, \tag{3.1}$$

$$\left\|\mathfrak{D}_q f(v_1, v_2)\right\| \le \alpha \left(\frac{\beta+1}{2}\right) + \psi\left(v_1^{[n]}, v_2^{[n]}\right), \tag{3.2}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$, where $\alpha \in [0,\infty)$ and $\beta \in \{-1,1\}$. Then, there exists a solution $Q: V^n \longrightarrow W$ of (2.2) such that

$$\left\|f\left(v^{[n]}\right) - Q\left(v^{[n]}\right)\right\| \le \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\frac{a^{(5n-3k)\beta}\alpha}{(a^{(5n-3k)\beta}-1)} \left(\frac{\beta+1}{2}\right) + \widetilde{\psi}\left(v_1^{[n]}, v_1^{[k]}, \mathbf{0}\right)\right], \tag{3.3}$$

for all $v_1^{[n]} := v^{[n]} \in V^n$, where $\mathbf{0} = (\overbrace{0,\dots,0}^{n-k-times})$. Moreover, if Q has assumption $(H1)$, then it is a unique multi-quadratic-quartic mapping satisfying (3.3).

Proof. Putting $v_1^{[k]} = v_2^{[k]}$ and $v_2^{[n-k]} = \mathbf{0}$ in (3.2), we have

$$\left\|f\left(2v_1^{[n]}\right) - Sf\left(v_1^{[n]}\right)\right\| \le \alpha\left(\frac{\beta+1}{2}\right) + \psi\left(v_1^{[n]}, v_1^{[k]}, \mathbf{0}\right), \tag{3.4}$$

for all $v_1^{[n]} \in V^n$ in which

$$S = 2^{2k} \sum_{p_1=0}^{n-k} \begin{pmatrix} n \\ p_1 \end{pmatrix} 2^{n-k-p_1} 4^{n-k-p_1} 24^{p_1} f\left(v_1^{[n]}\right).$$

It is not hard to show that $S = 2^{5(n-k)}$. For the rest of proof, we set $v_1^{[n]}$ by $v^{[n]}$ unless otherwise stated explicitly. It concludes from relation (3.4) that

$$\left\| f\left(2v^{[n]}\right) - 2^{5n-3k} f\left(v^{[n]}\right) \right\| \leq \alpha\left(\frac{\beta+1}{2}\right) + \psi\left(v^{[n]}, v^{[k]}, \mathbf{0}\right),$$

and so the equation above can be rewritten as

$$\left\| \frac{f\left(a^{\beta} v^{[n]}\right)}{2^{(5n-3k)\beta}} - f\left(v^{[n]}\right) \right\| \leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\alpha\left(\frac{\beta+1}{2}\right) + \phi\left(2^{\frac{\beta-1}{2}} v^{[n]}, 2^{\frac{\beta-1}{2}} v^{[k]}, \mathbf{0}\right)\right], \tag{3.5}$$

for all $v^{[n]} \in V^n$. Replacing v by $2^{\beta} v$ in (3.5), one can obtain

$$\left\| \frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}} - f\left(v^{[n]}\right) \right\|$$
$$\leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\frac{\beta+1}{2} \sum_{j=0}^{m-1} \frac{\alpha}{2^{(5n-3k)\beta j}} + \sum_{j=0}^{m-1} \frac{\psi\left(2^{\frac{\beta-1}{2}+j\beta} v^{[n]}, 2^{\frac{\beta-1}{2}+j\beta} v^{[k]}, \mathbf{0}\right)}{2^{(5n-3k)\beta j}}\right], \tag{3.6}$$

for all $v^{[n]} \in V^n$. On the other hand, we can use induction to find

$$\left\| \frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}} - \frac{f\left(2^{\beta l} v^{[n]}\right)}{2^{(5n-3k)l\beta}} \right\|$$
$$\leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\frac{\beta+1}{2} \sum_{j=l}^{m-1} \frac{\alpha}{2^{(5n-3k)\beta j}} + \sum_{j=l}^{m-1} \frac{\psi\left(2^{\frac{\beta-1}{2}+j\beta} v^{[n]}, 2^{\frac{\beta-1}{2}+j\beta} v^{[k]}, \mathbf{0}\right)}{2^{(5n-3k)\beta j}}\right], \tag{3.7}$$

for all $v^{[n]} \in V^n$, and $m > l \geq 0$. By applying (3.1) and (3.7), we deduce that the sequence $\left\{\frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}}\right\}$ is Cauchy. The completeness of W necessitates that there exists a mapping $Q\colon V^n \longrightarrow W$ so that

$$\lim_{m\to\infty} \frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}} = Q\left(v^{[n]}\right), \qquad \left(v^{[n]} \in V^n\right). \tag{3.8}$$

By letting $m \to \infty$ in (3.6) and using (3.8), the validity of inequality (3.3) is now shown. By switching $v_1^{[n]}, v_2^{[n]}$ into $2^m v_1^{[n]}, 2^m v_2^{[n]}$, respectively in (3.2) and dividing to

$2^{(5n-3k)m\beta}$, we get

$$\frac{1}{2^{(5n-3k)m\beta}}\left\|\mathfrak{D}_q f\left(2^{\beta m}v_1^{[n]},2^{\beta m}v_2^{[n]}\right)\right\| \leq \frac{\alpha}{2^{(5n-3k)m\beta}}\left(\frac{\beta+1}{2}\right)+\frac{\psi\left(2^{\beta m}v_1^{[n]},2^{\beta m}v_2^{[n]}\right)}{2^{(5n-3k)m\beta}}.$$

Taking the limit as $m\to\infty$ in the last relation, and applying (3.1) and (3.8) we obtain

$$\mathfrak{D}_q Q\left(v_1^{[n]},v_2^{[n]}\right)=0, \qquad \left(v_1^{[n]},v_2^{[n]}\in V^n\right)$$

and so Q is a solution of (2.2). Assume now that Q has $(H1)$, then it is a multi-quadratic-quartic mapping by Proposition 2. Let now $Q'\colon V^n\longrightarrow W$ be another multi-quartic-quadratic mapping with (3.3). Then, we find

$$\begin{aligned}
&\left\|Q\left(v^{[n]}\right)-Q'\left(v^{[n]}\right)\right\| \\
&=\frac{1}{2^{(5n-3k)m\beta}}\left\|C\left(2^{\beta m}v^{[n]}\right)-Q'\left(2^{\beta m}v^{[n]}\right)\right\| \\
&\leq\frac{1}{2^{(5n-3k)m\beta}}\left(\left\|Q\left(2^{\beta m}v^{[n]}\right)-f\left(2^{\beta m}v^{[n]}\right)\right\|+\left\|f\left(2^{\beta m}v^{[n]}\right)-Q'\left(2^{\beta m}v^{[n]}\right)\right\|\right) \\
&\leq\frac{2}{2^{(5n-3k)m\beta}}\left[\frac{2^{(5n-3k)\beta}\alpha}{2^{(5n-3k)\beta}-1}\left(\frac{\beta+1}{2}\right)+\widetilde{\psi}\left(2^{\beta m}v^{[n]},2^{\beta m}v^{[k]},\mathbf{0}\right)\right] \\
&=\frac{2}{2^{(5n-3k)m\beta}}\Bigg[\frac{2^{(5n-3k)\beta}\alpha}{2^{(5n-3k)\beta}-1}\left(\frac{\beta+1}{2}\right) \\
&\qquad+\sum_{j=0}^{\infty}\frac{1}{2^{(5n-3k)\beta j}}\psi\left(2^{\frac{\beta-1}{2}+(j+m)\beta}v^{[n]},2^{\frac{\beta-1}{2}+(j+m)\beta}v^{[k]},\mathbf{0}\right)\Bigg] \\
&=\frac{2}{2^{(5n-3k)m\beta}}\Bigg[\frac{2^{(5n-3k)\beta}\alpha}{2^{(5n-3k)\beta}-1}\left(\frac{\beta+1}{2}\right) \\
&\qquad+2^{(5n-3k)m\beta}\sum_{j=m}^{\infty}\frac{1}{2^{(5n-3k)\beta j}}\psi\left(2^{\frac{\beta-1}{2}+j\beta}v^{[n]},2^{\frac{\beta-1}{2}+j\beta}v^{[k]},\mathbf{0}\right)\Bigg],
\end{aligned}$$

for all $v^{[n]}\in V^n$. Taking $m\to\infty$ in the above inequality, we have $Q=Q'$ and hence it is shown the uniqueness of solution. □

In the next results, it is assumed that V is a normed space and W is a Banach space. The upcoming (Rassias) corollary is a consequence of Theorem 1, which investigates the Hyers-Ulam stability of (2.2).

Corollary 1. *Given* $\alpha,\delta,r\in\mathbb{R}$ *with* $r\neq 5n-3k$ *and* $\delta,\alpha\in[0,\infty)$. *Moreover,* $r_{ij}>0$ *with* $\sum_{i=1}^{2}\sum_{j=1}^{n}r_{ij}\neq 5n-3k$. *Let a mapping* $f\colon V^n\longrightarrow W$ *satisfies*

$$\|\mathfrak{D}_q f\left(v_1^{[n]},v_2^{[n]}\right)\|\leq \alpha+\delta\sum_{i=1}^{2}\sum_{j=1}^{n}\|v_{ij}\|^r+\prod_{i=1}^{2}\prod_{j=1}^{n}\|v_{ij}\|^{r_{ij}},$$

for all $v_1^{[n]},v_2^{[n]}\in V^n$. *Then, there exists a mapping* $Q\colon V^n\longrightarrow W$ *as a solution of* (2.2) *such that*

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\\ \leq\begin{cases}\frac{\alpha}{2^{5n-3k}-1}+\frac{\delta}{2^{5n-3k}-2^r}\left(\sum_{j=1}^{n}\|v_{1j}\|^r+\sum_{j=1}^{k}\|v_{1j}\|^r\right) & r\in(0,5n-3k),\\ \frac{\delta}{2^r-2^{5n-3k}}\left(\sum_{j=1}^{n}\|v_{1j}\|^r+\sum_{j=1}^{k}\|v_{1j}\|^r\right) & r\in(5n-3k,\infty),\end{cases}\tag{3.9}$$

for all $v^{[n]}:=v_1^{[n]}\in V^n$. *Furthermore, if* Q *has* $(H1)$*, then it is a unique multi-quadratic-quartic mapping fulfilling* (3.9).

Proof. Setting

$$\psi\left(v_1^{[n]},v_2^{[n]}\right)=\delta\sum_{i=1}^{2}\sum_{j=1}^{n}\|v_{ij}\|^r+\prod_{i=1}^{2}\prod_{j=1}^{n}\|v_{ij}\|^r$$

in Theorem 1, we reach to the desired results. □

Remark 1. Considering $\beta=1$ and putting $\psi:=0$ in Theorem 1, we conclude that there exists a mapping $Q\colon V^n\longrightarrow W$ as a solution of (2.2) such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\leq\frac{\alpha}{2^{5n-3k}-1},$$

for all $v^{[n]}\in V^n$. In addition, if Q has $(H1)$, then it is a unique multi-quadrati-quartic mapping satisfies the last inequality. In the case that $n=k$, with the above assumptions, there exists a multi-quadratic mapping $Q\colon V^n\longrightarrow W$ such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\leq\frac{\alpha}{2^{2n}-1},$$

for all $v^{[n]}\in V^n$. Furthermore, in the case that $k=0$ and Q has the quartic condition in each component, there exists a multi-quartic mapping $Q\colon V^n\longrightarrow W$ such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\leq\frac{\alpha}{2^{5n}-1},$$

for all $v^{[n]}\in V^n$. Furthermore, if $\alpha=\delta=0$ in Theorem 1, then f satisfies (2.2). Furthermore, if it has assumption $(H1)$, then it is a multi-quadratic-quartic mapping.

The next merged proposition was proved in [2] and [6]. We use it to constract a counterexample.

Proposition 3. *Given a function* $\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}$. *If it is continuous n-quadratic* (*resp. n-quartic*), *then* ρ *has the form*

$$\rho(r_1,\dots,r_n) = c\prod_{j=1}^{n} r_j^2 \left(\text{resp. } \rho(r_1,\dots,r_n) = c\prod_{j=1}^{n} r_j^4\right),$$

where c is a constant in $\mathbb{R}$.

Applying the above proposition, we have the following characterization.

Theorem 2. *Let* $\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous k-quadratic and* $n-k$*-quartic function. Then, it has the form*

$$\rho(r_1,\dots,r_n) = c\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4,$$

for all $r_1,\dots,r_n \in \mathbb{R}$, *where* $c \in \mathbb{R}$ *is a constant.*

Proof. We firstly identify $r = (r_1,\dots,r_n) \in \mathbb{R}^n$ with $\left(r^{[k]}, r^{[n-k]}\right) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, where $r^{[k]} := (r_1,\dots,r_k)$ and $r^{[n-k]} := (r_{k+1},\dots,r_n)$. For any $r^{[n-k]} \in \mathbb{R}^{n-k}$, consider the mapping $\mathcal{T}_{r^{[n-k]}} \colon \mathbb{R}^k \longrightarrow \mathbb{R}$ defined by

$$\mathcal{T}_{r^{[n-k]}}(r_1,\dots,r_k) := \rho\left(r_1,\dots,r_k, r^{[n-k]}\right).$$

By assumption, $\mathcal{T}_{r^{[n-k]}}$ is k-quadratic. It follows from Proposition 3 that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\mathcal{T}_{r^{[n-k]}}(r_1,\dots,r_k) = \rho\left(r_1,\dots,r_k, r^{[n-k]}\right) = c_1\prod_{j=1}^{k} r_j^2. \tag{3.10}$$

Note that c_1 depends on $r^{[n-k]}$. In fact,

$$c_1 = T\left(r^{[n-k]}\right). \tag{3.11}$$

Putting $r_1 = \cdots = r_k = 1$ in (3.10) and applying (3.11), we get

$$c_1 = T(r^{[n-k]}) = \rho\left(1,\dots,1, r^{[n-k]}\right). \tag{3.12}$$

Once again, for any $r^{[k]} \in \mathbb{R}^k$, define the mapping $\mathcal{S}_{r^{[k]}} \colon \mathbb{R}^{n-k} \longrightarrow \mathbb{R}$ through

$$\mathcal{S}_{r^{[k]}}(r_{k+1},\dots,r_n) := \rho(1,\dots,1,r_{k+1},\dots,r_n).$$

Since $\mathcal{S}_{r^{[k]}}$ is $n-k$-quartic, by Proposition 3 there exists a constant $c_2 \in \mathbb{R}$ such that

$$\mathcal{S}_{r^{[k]}}(r_{k+1},\dots,r_n) = \rho\left(1,\dots,1, r^{[n-k]}\right) = c_2\prod_{l=k+1}^{n} r_l^4. \tag{3.13}$$

It is obvious that c_2 depends on $r^{[k]}$ and hence

$$c_2 = S\left(r^{[k]}\right). \tag{3.14}$$

Letting $r_{k+1} = \cdots = r_n = 1$ in (3.13) and using (3.14), we get

$$c_2 = \rho(\overbrace{1,\dots,1}^{k-\text{times}}, \overbrace{1,\dots,1}^{n-k-\text{times}}). \tag{3.15}$$

The result now follows from (3.10), (3.12), (3.13) and (3.15). □

In the following, we show the assumption $r \neq 5n-3k$ is necessary and can not be eliminated in Corollary 1 when $\alpha = 0$. This means that a multi-quadratic-quartic mapping can be non-stable example; see [6, Example 1] and [13]). In view of the proof of Theorem 2, we observe that by removing the continuity condition of ρ, the result remains valid when the domain of ρ is replaced by $\mathbb{Q}^n$, where $c = \rho(\overbrace{1,\dots,1}^{n-\text{times}})$.

Example 1. Let $\delta > 0$ and $n \in \mathbb{N}$. Consider the function $\mathbf{1}\colon \mathbb{Q}^n \longrightarrow \mathbb{R}$ whose range is the constant 1. Put $\mu = \frac{2^{5n-3k}-1}{2^{2(5n-3k)}S}\delta$, where S is grater than or equal to

$$\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}} \mathbf{1}\left(v_1^{[k]} + sv_2^{[k]}, 2v_1^{[n-k]} + tv_2^{[n-k]}\right)$$
$$+2^k \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{j_1,\dots,j_k\in\{1,2\}} 4^{n-k-p_1-p_2}24^{p_1}6^{p_2}\mathbf{1}\left(v_{j_1 1},\dots,v_{j_k k},\mathcal{A}^{n-k}_{(p_1,p_2)}\right);$$

see the definition of $\mathfrak{D}_q f\left(v_1^{[n]}, v_2^{[n]}\right)$. Consider the function $\psi\colon \mathbb{Q}^n \longrightarrow \mathbb{R}$ defined via

$$\psi(r_1,\dots,r_n) = \begin{cases} \mu\prod_{j=1}^{k}\prod_{t=k+1}^{n} r_j^2 r_t^4 & r_j \text{ with } |r_j| < 1, \\ \mu & \text{otherwise.}\end{cases}$$

According the function above, consider the function $\rho\colon \mathbb{Q}^n \longrightarrow \mathbb{R}$ defined by

$$\rho(r_1,\dots,r_n) = \sum_{l=0}^{\infty} \frac{\psi(2^l r_1,\dots,2^l r_n)}{2^{(5n-3k)l}}, \qquad (r_j \in \mathbb{R}).$$

It is clear that ρ is a non-negative and even function in all components. Moreover, ψ is bounded by μ, and so ρ is bounded by $\frac{2^{5n-3k}}{2^{5n-3k}-1}\mu$, and additionally

$$\left|\mathfrak{D}_q\rho\left(v_1^{[n]}, v_2^{[n]}\right)\right| \leq \frac{2^{5n-3k}}{2^{5n-3k}-1}\mu S, \tag{3.16}$$

where $v_i^{[n]} = (v_{i1},\dots,v_{in}) \in \mathbb{Q}^n$ for $i \in \{1,2\}$. We show that

$$\left|\mathfrak{D}_q\rho\left(v_1, v_2\right)\right| \leq \delta\sum_{i=1}^{2}\sum_{j=1}^{n} |v_{ij}|^{5n-3k}, \tag{3.17}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$. Clearly, for $v_1^{[n]} = v_2^{[n]} = \mathbf{0} := (\overbrace{0,\ldots,0}^{n-\text{times}})$, inequality (3.17) is true. Let $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$ with

$$\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} < \frac{1}{2^{5n-3k}}. \tag{3.18}$$

It concludes from (3.18) that there is $M \in \mathbb{N}$ such that

$$\frac{1}{2^{(5n-3k)(M+1)}} < \sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} < \frac{1}{2^{(5n-3k)M}}, \tag{3.19}$$

and hence $|v_{ij}|^{5n-3k} < \sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} < \frac{1}{2^{(5n-3k)M}}$. Now, the last relation necessitates that $2^M|v_{ij}| < 1$ for all $i = 1,2$ and $j = 1,\ldots,n$. Hence, $2^{M-1}|v_{ij}| < 1$. Let $u_1, u_2 \in \{v_{ij} | i = 1,2,\ j = 1,\ldots,n\}$. Then

$$\{2^{M-1}|u_1 - u_2|, 2^{M-1}|u_1 + 2u_2|\} \subseteq (-1,1)$$

for all $l \in \{0,1,2,\ldots,M-1\}$. Since ψ is multi-quadratic-quartic function on $(-1,1)^n$, we find $\mathfrak{D}_q\psi\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right) = 0$ for all $l \in \{0,1,2,\ldots,M-1\}$. It follows from the last equality and (3.19) that

$$\begin{aligned}
\frac{\left|\mathfrak{D}_q\rho\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right)\right|}{\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}} &\le \sum_{l=M}^{\infty}\frac{\left|\mathfrak{D}_q\psi\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right)\right|}{2^{(5n-3k)l}\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}}\\
&\le \sum_{l=0}^{\infty}\frac{\mu S}{2^{(5n-3k)(l+N)}\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}}\\
&\le \mu S\sum_{l=0}^{\infty}\frac{1}{2^{(5n-3k)l}} \le \mu S 2^{5n-3k}\frac{2^{5n-3k}}{2^{5n-3k}-1} = \mu S\frac{2^{2(5n-3k)}}{2^{5n-3k}-1} = \delta,
\end{aligned}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$. Hence, (3.17) holds for the case (3.18). Assume that $\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} \ge \frac{1}{2^{5n-3k}}$. A direct consequence of (3.16) shows that

$$\frac{\left|\mathfrak{D}_q\rho\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right)\right|}{\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}} \le 2^{2n}\frac{2^{5n-3k}}{2^{5n-3k}-1}\mu S = \delta.$$

Thus, ρ satisfies in (3.17) for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$. Now, suppose the assertion is false, that there exist a multi-quadratic-quartic mapping $Q\colon \mathbb{Q}^n \longrightarrow \mathbb{R}$ and $\eta > 0$ such that

$$|\rho(r_1,\ldots,r_n) - Q(r_1,\ldots,r_n)| \le \eta\left(\sum_{j=1}^{n}|r_j|^{5n-3k} + \sum_{l=1}^{k}|r_l|^{5n-3k}\right).$$

Since $5n-3k$ is a fixed number, one can find a number $\lambda \in [0,\infty)$ as large enough such that

$$\eta\left(\sum_{j=1}^{n} r_j^{5n-3k} + \sum_{l=1}^{k} r_l^{5n-3k}\right) \leq \lambda \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4,$$

and so

$$|\rho(r_1,\ldots,r_n) - Q(r_1,\ldots,r_n)| < \lambda \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4.$$

On the other hand, Theorem 2 and the paragraph before this example imply that there is a constant $\alpha \in \mathbb{R}$ such that $Q(r_1,\ldots,r_n) = \alpha \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4$ and hence

$$\rho(r_1,\ldots,r_n) \leq (|\alpha| + \lambda) \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4 \tag{3.20}$$

holds. In addition, take $M \in \mathbb{N}$ such that $M\mu > |\alpha| + \lambda$. Take $r = (r_1,\ldots,r_n)$ in $\mathbb{Q}^n$ in which $r_j \in \left(0, \frac{1}{2^{M-1}}\right)$ for all $j \in \{1,\ldots,n\}$, then $2^l r_j \in (0,1)$ for all $l = 0,1,\ldots,M-1$. Therefore

$$\begin{aligned}\rho(r_1,\ldots,r_n) &= \sum_{l=0}^{\infty} \frac{\psi\left(2^l r_1,\ldots,2^l r_2\right)}{2^{(5n-3k)l}} \geq \mu \sum_{l=0}^{M-1} \frac{2^{(5n-3k)l} \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4}{2^{(5n-3k)l}} \\ &= M\mu \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4 > (|\alpha| + \lambda) \prod_{j=1}^{k} \prod_{l=k+1}^{n} r_j^2 r_l^4.\end{aligned}$$

The relation above contradicts inequality (3.20).

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