Miskolc Mathematical Notes
Vol. 27 (2026), No. 1, pp. 425–440

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2026.5186

IMPULSIVE BASSET FRACTIONAL DIFFERENTIAL EQUATION WITH NONLINEAR BOUNDARY CONDITION

WEIBING WANG AND PIAO LIU

Received 01 November, 2024

Abstract. In this paper, we consider the existence of solutions for a class of impulsive Basset fractional differential equation with nonlinear boundary condition. Using the comparison principle established and Schauder's fixed point theorem, we show that the problem has at least a solution between the upper and lower solution under appropriate conditions. Meanwhile, the existence of extreme solutions is obtained by means of quasilinearization technique. Finally, two examples are presented to illustrate the applicability of our main results.

2010 *Mathematics Subject Classification:* 34A08; 34A12

Keywords: Basset fractional differential equation, impulse, upper and lower solutions

1. INTRODUCTION

In the paper, we consider the following nonlinear Basset fractional differential equation with impulses

$$\begin{cases} x'(t)+MD^{\alpha}x(t)=f(t,x(t)), & t\in(0,1],\ t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k)), & k=1,2,\dots,p,\\ g(x(0), \qquad x(1))=0, & \end{cases} \tag{1.1}$$

where $0<\alpha<1, M\in\mathbb{R}$, $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=1$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ denotes the jump of $x(t)$ at $t=t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively, and $D^{\alpha}x={}_0D_t^{\alpha}x$ is the Caputo fractional derivative.

Fractional integrals and derivatives are vital for modeling phenomena across engineering, physics, and biology [8, 11, 18, 20]. As such, the widespread application of fractional differential equations has led to significant and growing research interest, as seen in [5, 9, 13, 15, 25, 26] and related works.

The first author was supported by Hunan Provincial Natural Science Foundation of China (No 2022JJ30236).

Significant attention has been directed towards fractional differential equations that blend classical and fractional derivatives for modeling specialized physical phenomena. A foundational example is the Basset fractional differential equation, which originated from Basset's study of a sphere under gravity [4]. He introduced a special hydraulic force, now known as the "Basset force," which Mainardi [16] later interpreted as being proportional to the fractional derivative of order $1/2$ of the particle's relative velocity. Consequently, this model incorporates both a first-order derivative and a fractional derivative. Staněk [22] considered the general Basset fractional equation

$$\begin{cases} u'(t) = AD^{\alpha}u(t) + f(t,u(t)), \\ u(0) = u(T), \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$. Under appropriate conditions, the author showed the existence of solution for (1.2) by using the Leray-Schauder degree method.

In modelling the motion of a rigid body immersed in Newtonian fluid, Torvik and Bagley [23] introduced the following fractional differential equation

$$Au''(t) + BD^{\frac{3}{2}}u(t) + Cu(t) = f(t),$$

where A, B, C are real numbers, f is the known function, which is referred to as Bagley-Torvik equation by later literature.

Fazli, Sun, Aghchi and Nieto [6] studied the following fractional differential equation with the nonlinear conditions

$$\begin{cases} u^{(m)}(t) + MD^{\delta}u(t) = f(t,u(t)), & 0 < t \leq T, \\ g_k(u^{(k)}(t_0), u^{(k)}(t_1), \dots, u^{(k)}(t_r)) = 0, \end{cases}$$

where $m-1 < \delta < m$, $0 = t_0 < t_1 < \cdots < t_r = T$, $k = 0,1,\dots,m-1$. The authors obtained the existence of extremal solutions by establishing one comparison theorem and applying the monotone iterative method. The other results about those equations, we refer the reader to [1–3, 14, 17, 19] and the references therein.

Impulsive perturbation originates from external disturbances in the process of time evolution and is commonly present in practical problems in modern technology. Impulsive fractional differential equation has also attracted the interest of many researchers, see [7, 12, 21]. In [10], Guo and Jiang studied the fractional impulsive problem

$$\begin{cases} D^q u(t) = f(t,u(t)), & t \in [0,T]/\{t_1,t_2,\dots,t_m\},\ 0 < q < 1, \\ \Delta u(t_k) = I_k(u(t_k)), & k = 1,2,\dots,m, \\ au(0) + bu(T) = c, \end{cases} \tag{1.3}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and $f\colon J \times \mathbb{R} \to \mathbb{R}$ is continuous. They obtained some existence results by using fixed point method and generalized singular Gronwall's inequality. In [24], Yang and Chen studied the following impulsive fractional differential equation

$$\begin{cases} D^{\alpha}u(t)=f(t,u(t)), & t\in[0,1]/\{t_1,t_2,\ldots,t_m\},\\ \Delta u(t_k)=I_k(u(t_k)), \quad \Delta D^{\beta}u(t_k)=J_k(u(t_k)), & k=1,2,\ldots,m,\\ u(0)+K_1D^{\beta}u(1)=\theta_1, & \\ D^{\beta}u(0)+K_2u(1)=\theta_2, & \end{cases} \tag{1.4}$$

where $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}, \alpha\in(1,2], \beta\in(0,1]$, $K_1,K_2,\theta_1,\theta_2$ are constants. By applying Krasnoselskii's fixed point theorem and contraction mapping principle, the authors showed the existence of solution of (1.4) under appropriate conditions.

The model (1.1) investigated in this work possesses several distinctive features when compared to existing formulations. Unlike (1.2), it incorporates impulsive perturbations and a nonlinear boundary condition. Furthermore, it differs from (1.3) by including a first-order derivative term and a more complex boundary structure. To the best of our knowledge, the solvability of impulsive Basset equations remains an unexplored area. Therefore, this paper aims to establish the existence of solutions to (1.1) employing the method of lower and upper solutions.

The paper is organized as follows. In section 2, we establish a comparison principle related to the problem (1.1). In section 3, the concept of lower and upper solution of (1.1) is introduced. By using fixed point theorem and the approach of quasilinearization, we obtain the existence results of (extreme) solution for (1.1).

2. PRELIMINARIES

Let $J_0=[0,t_1), J_1=(t_1,t_2),\ldots,J_{p-1}=(t_{p-1},t_p), J_p=(t_p,1], J=[0.1]$,

$$PC(J)=\big\{x\colon J\to\mathbb{R}\mid x\in C(J_k),\ k=0,1,\ldots,p,\ x(t_i^+),x(t_i^-)\text{exist,}, \\ x(t_i^-)=x(t_i),\ i=1,2,\ldots,p\big\},$$

then $PC(J)$ is Banach spaces with the norm $\|x\|=\sup_{t\in J}|x(t)|$. A function $x\in\Lambda:=\{u\in PC(J)\cap C^1(J^*)\colon u'\in L^1(0,1)\}$ is called a solution of (1.1) if it satisfies (1.1), where $J^*=J/\{t_1,t_2,\ldots,t_p\}$.

Definition 1 ([11, (3.1) of Chapter 3])**.** Two-parameter Mittag-Leffler function

$$E_{a,b}(\xi)=\sum_{i=0}^{\infty}\frac{\xi^i}{\Gamma(ia+b)}, \qquad \xi\in\mathbb{R},\ a>0,\ b\in\mathbb{R}.$$

Definition 2 ([11, Definition 2.1])**.** Let $x\in L^1(a,b)$, its Riemann-Liouville fractional integral ${}_aI_t^{\gamma}x$ of order $\gamma>0$ is defined as

$${}_aI_t^{\gamma}x(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}x(s)\mathrm{d}s.$$

Definition 3 ([11, Definition 2.3])**.** Let $x \in L^1(a,b)$, its Caputo fractional derivative ${}_aD_t^{\gamma}x$ of order $n-1<\gamma\leq n$ is defined by

$$ {}_aD_t^{\gamma}x(t) = ({}_aI_t^{n-\gamma}x^{(n)})(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-1-\gamma}x^{(n)}(s)\mathrm{d}s, $$

provided that the right-hand side integral exists and is finite.

The Caputo fractional derivative ${}_aD_t^{\gamma}x$ of x can also be defined by

$$ {}_aD_t^{\gamma}x(t) = D^n({}_aI_t^{n-\gamma})\left(x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k\right), \qquad n-1<\gamma\leq n, $$

provided that the right-hand side integral exists and is finite, see [6].

Lemma 1. *Assume that $x\in\Lambda$, then its Caputo fractional derivative of order α exists and*

$$ D^{\alpha}x(t) \in C(J^*)\cap L^1(0,1). $$

Proof. Without losing generality, we assume that $p=1$. Since ${}_0I_t^{1-\alpha}$ is bounded on $L^p(0,1)$ for any $1\leq p\leq\infty$ (see Theorem 2.2(i) of [11]), $x'\in L^1(0,1)$ and

$$ D^{\alpha}x(t) = ({}_0I_t^{1-\alpha}x')(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)\mathrm{d}s, $$

$D^{\alpha}x$ exists almost everywhere and $D^{\alpha}x\in L^1(0,1)$.Using Theorem 2.2(iii) of [11], from that fact that $x'\in C[0,t_1)$, we have

$$ D^{\alpha}x(t)\in C[0,\theta] $$

for any $[0,\theta]\subset[0,t_1)$. For $t_1<t\leq 1$,

$$ \begin{aligned} D^{\alpha}x(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}x'(s)\mathrm{d}s + \frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t-s)^{-\alpha}x'(s)\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}x'(s)\mathrm{d}s + ({}_{t_1}I_t^{1-\alpha}x')(t) := h_1(t)+h_2(t), \end{aligned} $$

where

$$ h_1(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}x'(s)\mathrm{d}s,\ \ h_2(t) = ({}_{t_1}I_t^{1-\alpha}x')(t) ={}_{t_1}D_t^{\alpha}x(t). $$

Since $(t-s)^{-\alpha}x'(s)$ is continuous in $t\in(t_1,1]$ and integrable in $s\in(0,t_1)$, $h_1\in C(t_1,1]$. Similar to the case $x\in C[0,t_1)$, $h_2\in C(t_1,1]$. Hence, $D^{\alpha}x(t)\in C(J^*)$. ☐

Consider the linear equation

$$ \begin{cases} x'(t)+MD^{\alpha}x(t)=h(t), & t\neq t_k, \\ \Delta x(t_k)=d_k, & k=1,2,\ldots,p, \\ x(0)=x_0, \end{cases} \tag{2.1} $$

where $h\in C(J^*)\cap L^1(0,1)$ and $M, d_k(1\leq k\leq p), x_0\in\mathbb{R}$.

Lemma 2. *The function $\tilde{x} \in \Lambda$ is the solution of if and only if*

$$\widetilde{x}(t) = \int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right) h(s)\mathrm{d}s + x_0 + \sum_{0<t_k<t} d_k.$$

Proof. Assume that x_1, x_2 are two solutions of (2.1) and $y = x_1 - x_2$, then

$$\begin{cases} y'(t) + MD^{\alpha}y(t) = 0, & t \neq t_k, \\ \Delta y(t_k) = 0, & k = 1,2,\dots,p, \\ y(0) = 0. \end{cases}$$

Since $y \in C^1[0,t_1)$, by employing Laplace transform, one can obtain that $y \equiv 0$ in $[0,t_1)$ and thus $y(t_1) = 0$ since y is left continuous. For $t \in (t_1,t_2)$, we have

$$y'(t) + MD^{\alpha}y(t) = y'(t) + M_0 D_t^{\alpha} y(t) = y'(t) + M_{t_1} D_t^{\alpha} y(t) = 0.$$

Therefore $y \equiv 0$ in $(t_1,t_2]$. Similarly, $y \equiv 0$ in $[t_i,t_{i+1}], i = 2,\dots,p$. Hence, the solution of (2.1) is unique.

Let $g(t) = \int_0^t E_{1-\alpha,1}(-M(t-s)^{1-\alpha})h(s)\mathrm{d}s$ and

$$\psi(t) = h(t) - M(1-\alpha)\int_0^t E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s,$$

where $E^{(1)}_{1-\alpha,1}(t) = (E_{1-\alpha,1}(t))'$, which is continuous in $\mathbb{R}$. It follows from $h \in C(J^*) \cap L^1(0,1)$ that $g \in C[0,1]$. Hence,

$$\tilde{x}(0) = x_0, \quad \tilde{x}(t_k^-) = \tilde{x}(t_k), \quad \Delta\tilde{x}(t_k) = d_k, \quad k = 1,2,\dots,p.$$

For $0 < y < x < 1$, we have

$$\begin{aligned}
&\int_y^x \psi(t)\mathrm{d}t = \int_y^x h(t)\mathrm{d}t - M(1-\alpha)\int_y^x \mathrm{d}r \int_0^r E^{(1)}_{1-\alpha,1}(-M(r-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s \\
&= \int_y^x h(t)\mathrm{d}t - M(1-\alpha)[\int_0^y h(s)\mathrm{d}s \int_y^x E^{(1)}_{1-\alpha,1}(-M(r-s)^{1-\alpha})(r-s)^{-\alpha}\mathrm{d}r \\
&\quad + \int_y^x h(s)\mathrm{d}s \int_s^x E^{(1)}_{1-\alpha,1}(-M(r-s)^{1-\alpha})(r-s)^{-\alpha}\mathrm{d}r] \\
&= \int_y^x h(t)\mathrm{d}t + \int_0^y [E_{1-\alpha,1}(-M(x-s)^{1-\alpha}) - E_{1-\alpha,1}(-M(y-s)^{1-\alpha})]h(s)\mathrm{d}s \\
&\quad + \int_y^x [E_{1-\alpha,1}(-M(x-s)^{1-\alpha}) - E_{1-\alpha,1}(0)]h(s)\mathrm{d}s \\
&= g(x) - g(y),
\end{aligned}$$

which implies that $g'(t) = \psi(t)$.

We show that $\psi \in C(J^*)$. For simplicity, we assume that $p = 1$. Let $H(t,s) = E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)$. If $0 < t < t_1$,

$$\int_0^t H(t,s)\mathrm{d}s = \int_0^t E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)\mathrm{d}s.$$

Since $E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)$ is continuous in t and integrable in $s\in[0,t]$, $\int_0^t H(t,s)\mathrm{d}s\in C[0,t_1)$. If $t_1<t<1$, taking $t-t_1<\tau<t$, we have

$$\int_0^t H(t,s)\mathrm{d}s=\int_0^{\tau} H(t,s)\mathrm{d}s+\int_0^{t-\tau} E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)\mathrm{d}s:=g_1(t)+g_2(t).$$

Noting that $E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)$ is continuous in t and integrable in $s\in[0,\tau]$, $E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)$ is continuous in t and integrable in $s\in[0,t-\tau]$, we get that $g_1,g_2\in C(t_1,1]$. Hence, $\psi\in C(J^*)$ and $g\in C^1(J^*)$.

It follows from the fact $|E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})|\le E^{(1)}_{1-\alpha,1}(M)$ in region $\{(t,s)|0\le s\le t\le 1\}$ that there exists $C>0$ such that

$$\begin{aligned}\left|\int_0^t E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s\right| &\le C\int_0^t (t-s)^{-\alpha}|h(s)|\mathrm{d}s\\ &=C\Gamma(1-\alpha){}_0I_t^{1-\alpha}|h(t)|.\end{aligned}$$

Since ${}_0I_t^{1-\alpha}$ is bounded on $L^1(0,1)$, we have

$$\int_0^t E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s\in L^1(0,1),$$

which implies that $\psi\in L^1(0,1)$ and thus $g'\in L^1(0,1)$. It follows from Lemma 1 that $D^{\alpha}g(t)$ exists. In addition, for $0<t,l<1$,

$$\begin{aligned}&M[{}_0I_t^{1-\alpha}g(t)-{}_0I_l^{1-\alpha}g(l)]\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\int_0^r E_{1-\alpha,1}(-M(r-s)^{1-\alpha})h(s)\mathrm{d}s\mathrm{d}r\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l (l-r)^{-\alpha}\int_0^r E_{1-\alpha,1}(-M(r-s)^{1-\alpha})h(s)\mathrm{d}s\mathrm{d}r\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t h(s)\mathrm{d}s\int_s^t (t-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t h(s)\mathrm{d}s\int_s^t (t-r)^{-\alpha}\sum_{i=0}^{\infty}\frac{(-M)^i(r-s)^{(1-\alpha)i}}{\Gamma((1-\alpha)i+1)}\mathrm{d}r\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t h(s)\sum_{i=0}^{\infty}\frac{(-M)^i\int_s^t(t-r)^{-\alpha}(r-s)^{(1-\alpha)i}dr}{\Gamma((1-\alpha)i+1)}\mathrm{d}s\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&= M\int_0^t h(s)\sum_{i=0}^{\infty}\frac{(-M)^i(t-s)^{(1-\alpha)(i+1)}}{\Gamma((1-\alpha)(i+1)+1)}\mathrm{d}s\\
&\quad -M\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&= -\int_0^t [E_{1-\alpha,1}(-M(t-s)^{1-\alpha})-1]h(s)\mathrm{d}s\\
&\quad +\int_0^l [E_{1-\alpha,1}(-M(l-s)^{1-\alpha})-1]h(s)\mathrm{d}s\\
&= g(l)-g(t)+\int_l^t h(s)\mathrm{d}s,
\end{aligned}
$$

where we use the formula

$$\int_s^t (t-r)^{-\alpha}(r-s)^{(1-\alpha)i}\mathrm{d}r = \frac{(t-s)^{(1-\alpha)(i+1)}\Gamma(1-\alpha)\Gamma((1-\alpha)i+1)}{\Gamma((1-\alpha)(i+1)+1)}.$$

Hence,

$$\int_l^t g'(s)\mathrm{d}s + M\int_l^t ({}_0I_s^{1-\alpha}g(s))'\mathrm{d}s = \int_l^t h(s)\mathrm{d}s,$$

$$\begin{cases} g'(t)+MD^{\alpha}g(t)=h(t), & a.e,\\ g(0)=0. \end{cases} \tag{2.2}$$

It follows from (2.2) and the fact $\tilde{x}'=g'\in C(J^*)$ for $t\neq t_k$ and $D^{\alpha}\tilde{x}(t)={}_0I_t^{1-\alpha}\tilde{x}'(t)$ that $\tilde{x}$ is the solution of (2.1). The proof is completed. □

Remark 1. If $h\in C(J^*)\cap L^1(0,1)$ and $a_k,c_k,x_0\in\mathbb{R}$, then the problem

$$\begin{cases} x'(t)+MD^{\alpha}x(t)=h(t), & t\neq t_k,\\ \Delta x(t_k)=a_kx(t_k)+c_k, & k=1,2,\dots,p,\\ x(0)=x_0 \end{cases} \tag{2.3}$$

has a unique solution

$$
\begin{aligned}
x(t) &= \int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right)h(s)\mathrm{d}s\\
&\quad + x_0\prod_{0<t_k<t}(1+a_k)+\sum_{0<t_k<t}c_k\prod_{t_k<t_j<t}(1+a_j)\\
&\quad + \sum_{0<t_k<t}a_k\prod_{t_k<t_j<t}(1+a_j)\int_0^{t_k}E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right)h(s)\mathrm{d}s.
\end{aligned} \tag{2.4}
$$

Lemma 3. *If $a\in[0,1]$, $b\geq a$, then*

$$E_{a,b}(x)\geq 0,\quad \forall x\in\mathbb{R}.$$

Proof. By Corollary 3.2 of [11], $E_{a,b}(-x)$ is completely monotone on $\mathbb{R}_+$, which implies that $E_{a,b}(-x) \geq 0$ for $x > 0$. In addition, it follows from the definition of $E_{a,b}$ that $E_{a,b}(x) \geq 0$ for $x \geq 0$. □

Using (2.4) and Lemma 3, we have

Lemma 4. *If $h \geq 0$, $a_k \geq -1, c_k \geq 0 (1 \leq k \leq p)$ and $x_0 \geq 0$, then the solution x of (2.3) satisfies*

$$x \geq 0, \quad t \in J.$$

3. MAIN RESULTS

Definition 4. The function $u \in \Lambda$ is said to be the lower solution of (1.1) if

$$\begin{cases} u'(t) + MD^{\alpha}u(t) \leq f(t,u(t)), & t \neq t_k, \\ \Delta u(t_k) \leq I_k(u(t_k)), & k = 1,2,\ldots,p, \\ g(u(0),u(1)) \leq 0 \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reverted.

We list the following assumptions.

- (H_1) (1.1) has the lower solution u_0, the upper solution v_0 and $u_0 \leq v_0$ for $t \in J$.
- (H_2) $f\colon J \times [\gamma_1,\gamma_2] \to \mathbb{R}$ is continuous, here $\gamma_1 = \min\{\min_{t\in J} u_0, \min_{t\in J} v_0\}$ and $\gamma_2 = \max\{\max_{t\in J} u_0, \max_{t\in J} v_0\}$. $I_k(u)$ is continuous in $u \in [u_0(t_k), v_0(t_k)]$ for $k = 1,2,\ldots,p$.
- (H_3) $g\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and $g(\cdot,v)$ is nonincreasing in $v \in [\gamma_1,\gamma_2]$.
- (H_4) For $u_0 \leq v \leq u \leq v_0$ and $t \in J$, $f(t,u) \geq f(t,v)$. There exist $b_k \leq 1 (k = 1,2,\ldots,p)$ such that $I_k(\xi) + b_k\xi \geq I_k(\zeta) + b_k\zeta$ for $u_0(t_k) \leq \zeta \leq \xi \leq v_0(t_k)$ and $k = 1,2,\ldots,p$.
- (H_5) $g\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exist constants $\lambda > 0, \mu \geq 0$ such that

$$g(\bar{y}_1,\bar{y}_2) - g(y_1,y_2) \leq \lambda(\bar{y}_1 - y_1) - \mu(\bar{y}_2 - y_2)$$

for $\gamma_1 \leq y_i \leq \bar{y}_i \leq \gamma_2, i = 1,2$.

Theorem 1. *Assume that $(H_1) - (H_4)$ are satisfied, then (1.1) has one solution $x \in [u_0, v_0] = \{u \in PC(J) : u_0 \leq u \leq v_0, t \in J\}$.*

Proof. Let $n(t,x) = \max\{u_0(t), \min\{x, v_0(t)\}\}, F(t,x) = f(t,n(t,x))$ and $I_k^*(x) = b_k n(t_k,x) + I_k(n(t_k,x))$. Consider the equation

$$\begin{cases} x'(t) + MD^{\alpha}x(t) = F(t,x(t)), & t \neq t_k, \\ \Delta x(t_k) = -b_k x(t_k) + I_k^*(x(t_k)), & k = 1,2,\ldots,p, \\ x(0) = n(0, x(0) - g(x(0),x(1))). \end{cases} \tag{3.1}$$

According to Remark 1, the solution of (3.1) satisfies

$$\begin{aligned}x(t) = &\int_0^t E_{1-\alpha,1}(-M(t-s)^{1-\alpha})F(s,x(s))\mathrm{d}s + \sum_{0<t_k<t} I_k^*(x(t_k)) \prod_{t_k<t_j<t}(1-b_j)\\ &- \sum_{0<t_k<t} b_k \prod_{t_k<t_j<t}(1-b_j)\int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right)F(s,x(s))\mathrm{d}s\\ &+ n(0,x(0)-g(x(0),x(1)))\prod_{0<t_k<t}(1-b_k) =: (Ax)(t).\end{aligned}$$

Let the right-hand part of the above equality be Ax so that we define the operator A in $PC(J)$. Clearly, the fixed point of A in $PC(J)$ is the solution of (3.1).

By the continuity of f, g, I_k and the definition of n, there is $L > 0$ such that for any $x \in PC(J)$,

$$|F(t,x)| < L, \quad |n(0,x(0)-g(x(0),x(1)))| < L, \quad |I_k^*(x(t_k))| < L,$$

which imply that there exists $D > 0$ such that $||Ax|| \le D$ for any $x \in PC(J)$.

Let $\Omega = \{u \in PC(J) \colon ||u|| \le D\}$, then $A\colon \Omega \to \Omega$. It is obvious that $A\colon \Omega \to \Omega$ is continuous. Let $t_*, t^* \in (t_k, t_{k+1}]$ and $t_* < t^*$, then for $x \in \Omega$,

$$\begin{aligned}|(Ax)(t^*) - (Ax)(t_*)| \le L &\int_0^{t_*} |E_{1-\alpha,1}(-M(t^*-s)^{1-\alpha}) - E_{1-\alpha,1}(-M(t_*-s)^{1-\alpha})|\mathrm{d}s\\ &+ L\int_{t_*}^{t^*} |E_{1-\alpha,1}(-M(t^*-s)^{1-\alpha})|\mathrm{d}s,\end{aligned}$$

which implies that $|(Ax)(t^*) - (Ax)(t_*)| \to 0$ if $|t^* - t_*| \to 0$ and thus $A\colon \Omega \to \Omega$ is completely continuous. It follows from Schauder's fixed point theorem that there exists $x \in \Omega$ such that $Ax = x$. Moreover, $x \in \Lambda$ since x is the solution of (3.1).

Next, we show that $u_0 \le x \le v_0$. Let $y = x - u_0$, from the definition of lower solution and (H_4), we have

$$\begin{aligned}y' + MD^\alpha y &= f(t,n(t,x(t))) - u_0'(t) - MD^\alpha u_0(t) \ge f(t,n(t,x(t))) - f(t,u_0(t)) \ge 0,\\ \Delta y(t_k) &= -b_k x(t_k) + I_k^*(x(t_k)) - \Delta u_0(t_k)\\ &\ge -b_k x(t_k) + I_k(u_0(t_k)) + b_k u_0(t_k) - \Delta u_0(t_k) \ge -b_k y(t_k),\\ y(0) &= x(0) - u_0(0) \ge 0.\end{aligned}$$

Clearly, $f(t,n(t,x(t))), u_0', D^\alpha u_0 \in C(J^*) \cap L^1(0,1)$. Using Lemma 4, we obtain that $x \ge u_0$ for all $t \in J$. Similarly, $x \le v_0$ for all $t \in J$. Hence,

$$\begin{cases} x'(t) + MD^\alpha x(t) = f(t,x(t)), & t \ne t_k,\\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\dots,p.\end{cases}$$

Finally, we show that $g(x(0),x(1)) = 0$. We only need to show that $u_0(0) \le x(0) - g(x(0),x(1)) \le v_0(0)$. If $x(0) - g(x(0),x(1)) < u_0(0)$, then $x(0) = n(0,x(0) -$

$g(x(0),x(1))) = u_0(0)$ and thus $g(x(0),x(1)) > 0$. From the definition of lower solution and (H_3), we have

$$g(u_0(0),u_0(1)) \leq 0 < g(x(0),x(1)) = g(u_0(0),x(1)) \leq g(u_0(0),u_0(1)) \leq 0,$$

which is a contradiction. Hence $x(0) - g(x(0),x(1)) \geq u_0(0)$. Similarly, $x(0) - g(x(0),x(1)) \leq v_0(0)$. Hence, $x(0) = n(0,x(0)-g(x(0),x(1))) = x(0)-g(x(0),x(1))$ and thus $g(x(0),x(1)) = 0$. x is a solution of (1.1) and $x \in [u_0,v_0]$. $\square$

Theorem 2. *Assume that* $(H_1)-(H_2)$ *and* $(H_4)-(H_5)$ *are satisfied, then there exist sequences* $\{u_i\},\{v_i\} \subseteq \Lambda$ *such that* $\lim_{i\to\infty} u_i = u^*, \lim_{i\to\infty} v_i = v^*$ *and* $u^*,v^* \in [u_0,v_0]$ *are minimal and maximal solutions of* (1.1)*, respectively.*

Proof. The proof is divided into four steps.

Step 1: Constructing sequences $\{u_i\},\{v_i\}$. Consider the following linear equation

$$\begin{cases} W'_{i+1}(t) + MD^{\alpha}W_{i+1}(t) = f(t,W_i(t)), & t \in J,\ t \neq t_k, \\ \Delta W_{i+1}(t_k) = -b_k W_{i+1}(t_k) + I_k^*(W_i(t_k)), & k = 1,2,\dots,p, \\ W_{i+1}(0) = W_i(0) - \frac{1}{\lambda} g(W_i(0),W_i(1)), \end{cases} \tag{3.2}$$

where $W_0 = u_0$ or $W_0 = v_0$. From Remark 1, (3.2) has a unique solution

$$\begin{aligned} W_{i+1}(t) = &\int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right) f(t,W_i(t))\mathrm{d}s \\ &+ (W_i(0) - \lambda^{-1} g(W_i(0),W_i(1))) \prod_{0<t_k<t} (1-b_k) \\ &- \sum_{0<t_k<t} b_k \prod_{t_k<t_j<t} (1-b_j) \int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right) f(t,W_i(t))\mathrm{d}s \\ &+ \sum_{0<t_k<t} I_k^*(W_i(t_k)) \prod_{t_k<t_j<t} (1-b_j). \end{aligned} \tag{3.3}$$

Setting $W_i = u_i$ if $W_0 = u_0$, $W_i = v_i$ if $W_0 = v_0$, we obtain two sequences $\{u_i\}$ and $\{v_i\}$ and $u_i, v_i \in \Lambda$ for $i = 1,2,\dots$.

Step 2: Monotone property of sequences $\{u_i\},\{v_i\}$:

$$u_0 \leq u_1 \leq u_2 \leq \dots \leq u_i \leq u_{i+1} \leq v_{i+1} \leq v_i \leq \dots \leq v_1 \leq v_0.$$

Let $z = u_1 - u_0$, we obtain that

$$\begin{aligned} z'(t) + MD^{\alpha}z(t) &\geq f(t,u_0(t)) - f(t,u_0(t)) = 0, \\ \Delta z(t_k) &= -b_k u_1(t_k) + I_k^*(u_0(t_k)) - \Delta u_0(t_k) \\ &\geq -b_k(u_1(t_k) - u_0(t_k)) + I_k(u_0(t_k)) - \Delta u_0(t_k) \\ &\geq -b_k(u_1(t_k) - u_0(t_k)), \\ z(0) &= -\frac{1}{\lambda} g(u_0(0),u_0(1)) \geq 0. \end{aligned}$$

It follows from Lemma 4 that $z \geq 0$, so $u_0 \leq u_1$ for all $t \in J$. Similarly, one can prove that $v_1 \leq v_0$ for all $t \in J$. Now, let $\omega = v_1 - u_1$, using (H_4) and (H_5), we obtained

$$\begin{aligned}
\omega'(t) + MD^{\alpha}\omega(t) &\geq 0,\\
\Delta\omega(t_k) &= -b_k v_1(t_k) + I_k^*(v_0(t_k)) + b_k u_1(t_k) - I_k^*(u_0(t_k))\\
&\geq -b_k(v_1(t_k) - u_1(t_k)) + b_k(v_0(t_k) - u_0(t_k)) + I_k(v_0(t_k)) - I_k(u_0(t_k))\\
&\geq -b_k(v_1(t_k) - u_1(t_k)) = -b_k w(t_k),\\
\omega(0) &\geq \frac{\mu}{\lambda}(v_0(1) - u_0(1)) \geq 0.
\end{aligned}$$

Hence, $v_1 \geq u_1$. From (H_4) and (H_5), we obtain that

$$\begin{aligned}
u_1'(t) + MD^{\alpha}u(t) &\leq f(t, u_1(t)),\\
\Delta u_1(t_k) &= -b_k u_1(t_k) + I_k^*(u_0(t_k)) = -b_k(u_1(t_k) - u_0(t_k)) + I_k(u_0(t_k))\\
&\leq I_k(u_1(t_k)),\\
g(u_1(0), u_1(1)) &\leq g(u_0(0), u_0(1)) + \lambda(u_1(0) - u_0(0)) - \mu(u_1(1) - u_0(1))\\
&= -\mu(u_1(1) - u_0(1)) \leq 0.
\end{aligned}$$

Therefore, u_1 is the lower solution of (1.1). Similarly, v_1 is the upper solution of (1.1). Using the similar argument, we can show that $u_i \leq u_{i+1} \leq v_{i+1} \leq v_i$ for $i \geq 1$.

Step 3: According to Step 2, the sequences $\{u_i\}, \{v_i\}$ are monotonic and bounded. Therefore, the pointwise limits exist and we assume that

$$\lim_{i\to\infty} u_i = u^*, \quad \lim_{i\to\infty} v_i = v^*,$$

where $u^*, v^* \in [u_0, v_0]$.

From (H_2), (H_5) and $\lim_{i\to\infty} W_i = W \in [u_0, v_0]$, where $W = u^*$ or v^*, let $i \to \infty$ in (3.3) and applying the dominated convergence theorem, we obtain that

$$\begin{aligned}
W(t) = &\int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right) f(t, W(t))\mathrm{d}s + \sum_{0<t_k<t} I_k^*(W(t_k)) \prod_{t_k<t_j<t}(1-b_j)\\
&- \sum_{0<t_k<t} b_k \prod_{t_k<t_j<t}(1-b_j) \int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right) f(t, W(t))\mathrm{d}s\\
&+ (W(0) - \lambda^{-1}g(W(0), W(1))) \prod_{0<t_k<t}(1-b_k).
\end{aligned}$$

Through simple calculation, we have

$$\begin{cases}
W'(t) + MD^{\alpha}W(t) = f(t, W(t)), & t \neq t_k,\\
\Delta W(t_k) = -b_k W(t_k) + I_k^*(W(t_k)) = I_k(W(t_k)), & k = 1, 2, \ldots, p,\\
W(0) = W(0) - \lambda^{-1}g(W(0), W(1)).
\end{cases}$$

Therefore, u^*, v^* are solutions of (1.1).

Step 4: u^*, v^* are the extremal solutions of (1.1) in $[u_0, v_0]$. Assume that $u \in [u_0, v_0]$ is a solution of (1.1), we suppose that $u_i \leq u \leq v_i$ for some $i \in \mathbb{N}$. By (H_4), we have

$$f(t, u_i(t) \leq f(t, u(t)) \leq f(t, v_i(t)),$$

$$\begin{aligned}\Delta(u(t_k) - u_{i+1}(t_k)) &= I_k(u(t_k)) + b_k u_{i+1} - I_k^*(u_i(t_k)) \\ &\leq b_k(u_{i+1}(t_k) - u_i(t_k)) + I_k(u(t_k)) - I_k u_i(t_k) \\ &\leq -b_k(u(t_k) - u_{i+1}(t_k)).\end{aligned}$$

Similarly, $\Delta(v_{i+1}(t_k) - u(t_k)) \geq -b_k(v_{i+1}(t_k) - u(t_k))$.

Using (H_5), we have

$$\begin{aligned}u_{i+1}(0) &= u_i(0) - \frac{1}{\lambda} g(u_i(0), u_i(1)) \\ &= u_i(0) + \frac{1}{\lambda} g(u(0), u(1)) - \frac{1}{\lambda} g(u_i(0), u_i(1)) \\ &\leq u(0) - \frac{\mu}{\lambda}(u(1) - u_i(1)) \leq u(0).\end{aligned}$$

Similarly, $u(0) \leq v_{i+1}(0)$. It follows from Lemma 4 that $u_{i+1} \leq u \leq v_{i+1}$. Therefore,

$$u_j \leq u \leq v_j, \quad j = i, i+1, i+2, \ldots. \tag{3.4}$$

Taking limit in (3.4) as $j \to \infty$, we get that $u^* \leq u \leq v^*$. Therefore, u^*, v^* are the extremal solutions of (1.1) in $[u_0, v_0]$.

□

Example 1. Consider the equation

$$\begin{cases} x'(t) + \kappa D^{\frac{1}{2}} x(t) = t(1 + x^{\beta}(t)), & t \neq \frac{1}{2}, \\ \Delta x(0.5) = 0.1 - \sin x(0.5), \\ x^4(0) \sin x(0) - x^2(1) - x(1) = 0, \end{cases} \tag{3.5}$$

where $\beta > 0$ and κ is a positive parameter.

We claim that for any $l \in \mathbb{N}$, (3.5) has at least l solutions for $\kappa \geq 2(2\pi l + 2)^{\beta}$. In fact,

$$f(t,s) = t(1 + s^{\beta}), \quad I_1(s) = 0.1 - \sin s, \quad g(u,v) = u^4 \sin u - v^2 - v.$$

Let $U_j = 2j\pi$, $V_j = 2j\pi + 1 + t$, $j = 1, \ldots, l$, then

$$\begin{cases} U_j'(t) + \kappa D^{\frac{1}{2}} U_j(t) = 0 \leq t(1 + U_j^{\beta}(t)), & t \neq 0.5, \\ \Delta U_j(0.5) = 0 < 0.1 - \sin U_j(0.5) = 0.1, \\ U_j^4(0) \sin U_j(0) - U_j^2(1) - U_j(1) = -(2j\pi)^2 - 2j\pi < 0, \end{cases}$$

$$\begin{cases} V_j'(t)+\kappa D^{\frac{1}{2}}V_j(t)=1+\frac{2\kappa\sqrt{t}}{\sqrt{\pi}}\geq t(1+V_j^{\beta}(t)), & t\neq 0.5,\\ \Delta V_j(0.5)=0>0.1-\sin V_j(0.5)=0.1-\sin(1.5), & \\ V_j^4(0)\sin V_j(0)-V_j^2(1)-V_j(1) & \\ =(2j\pi+1)^4\sin(1.5)-(2j\pi+2)^2-2j\pi-2>0, & \end{cases}$$

which imply that U_j and V_j are the lower and upper solutions of (3.5), respectively. Hence, (H_1) holds. Obviously, $f\colon J\times\mathbb{R}\to\mathbb{R}$, $I_1\colon\mathbb{R}\to\mathbb{R}$ and $g\colon\mathbb{R}^2\to\mathbb{R}$ are continuous. In addition, $f(\cdot,s)$ is nondecreasing in $(0,+\infty)$ and $g(\cdot,v)$ is nonincreasing in $(0,+\infty)$. Moreover, there exists $b_1=1$ such that

$$I_1(\xi)+b_1\xi\geq I_1(\zeta)+b_1\zeta$$

for $\xi\geq\zeta$. Therefore, $(H_2)-(H_4)$ holds. It follows from Theorem 1 that (3.5) has solutions $x_j\in[U_j,V_j](j=1,\dots,l)$.

Example 2. Consider the equation

$$\begin{cases} x'(t)-\frac{1}{2}D^{\frac{1}{2}}x(t)=\frac{t}{4}\left[\frac{2+2x(t)}{2+x(t)}-\frac{1}{40}x^2(t)\right], & t\neq t_1,t_2,\\ \Delta x(t_k)=\frac{1}{4+k}\ln(1+x^2(t_k)), & k=1,2,\\ 100x(0)+x^3(1)-15x(1)=0, & \end{cases} \tag{3.6}$$

where $0<t_1<t_2<1$.

In fact,

$$f(t,s)=\frac{t}{4}\left[\frac{2+2s}{2+s}-\frac{s^2}{40}\right],\quad I_k(s)=\frac{1}{4+k}\ln(1+s^2),\quad g(u,v)=100u+v^3-15v.$$

Let

$$U(t)=0,\quad V(t)=\begin{cases}\frac{1}{4}+t, & 0\leq t\leq t_1,\\ \frac{1}{2}+t, & t_1<t\leq t_2,\\ 1+t, & t_2<t\leq 1.\end{cases}$$

Clearly, U is a lower solution of (3.6). In addition,

$$V'(t)-\frac{1}{2}D^{\frac{1}{2}}V(t)=1-\frac{\sqrt{t}}{\sqrt{\pi}}>\frac{t}{3}\geq f(t,V(t)),$$

$$\Delta V(t_1)=\frac{1}{4}>\frac{1}{5}\ln(1+(0.25+t_1)^2),\quad \Delta V(t_2)=\frac{1}{2}>\frac{1}{6}\ln(1+(0.5+t_2)^2),$$
$$g(V(0),V(1))=3.$$

So, V is a upper solution of (3.6) and (H_1) holds. Moreover, $f\colon J\times\mathbb{R}\to\mathbb{R}$ and $I_1,I_2\colon\mathbb{R}\to\mathbb{R}$ are continuous. For f, we have

$$f_s(t,s)=\frac{t}{4}\left[\frac{40-s(2+s)^2}{20(2+s)^2}\right]\geq 0,\qquad \forall s\in[0,2],\ t\in[0,1].$$

Hence, $f(t,s)$ is nondecreasing in $s \in [0,2]$. There exist $b_1 = b_2 = 0$ such that for $0 \leq \zeta \leq \xi \leq 2$, $k = 1,2$,

$$I_k(\xi) + b_k\xi = \frac{1}{4+k}\ln(1+\xi^2) \geq I_k(\zeta) + b_k\zeta = \frac{1}{4+k}\ln(1+\zeta^2).$$

Hence, (H_2) and (H_4) are satisfied.

In addition, $g\colon \mathbb{R}^2 \to \mathbb{R}$ are continuous and

$$\begin{aligned} g(\overline{y_1},\overline{y_2}) - g(y_1,y_2) &= 100(\overline{y_1} - y_1) + (\overline{y_2}^2 + y_2\overline{y_2} + y_2^2 - 15)(\overline{y_2} - y_2) \\ &\leq 100(\overline{y_1} - y_1) - 3(\overline{y_2} - y_2) \end{aligned}$$

for $0 \leq y_i \leq \overline{y_i} \leq 2, i = 1,2$. Therefore, (H_5) holds. It follows from Theorem 2 that there exist monotone iterative sequences $\{u_j\}, \{v_j\}$ which converge to the extremal solutions u^*, v^* of (3.6),respectively.

Remark 2. Even if $I_k \equiv 0$ and $g(u,v) = u - v$, our results are also new because one of the prerequisites of [22] is that $A > 0$ in (1.2), which is equivalent to the condition $M < 0$ in (1.1). Our conditions are different from those in [22].

REFERENCES

[1] A. Armand, T. Allahviranloo, and Z. Gouyandeh, “General solution of Basset equation with Caputo generalized Hukuhara derivative.” *J. Appl. Anal. Comput.*, vol. 6, no. 1, pp. 119–130, 2016, doi: 10.11948/2016010.

[2] H. Baghani, M. Feckan, J. Farokhi-Ostad, and J. Alzabut, “New existence and uniqueness result for fractional Bagley-Torvik differential equation.” *Miskolc Math. Notes*, vol. 23, no. 2, pp. 537–549, 2022, doi: 10.18514/mmn.2022.3702.

[3] H. Baghani, J. Alzabut, and J. J. Nieto, “Further results on the existence of solutions for generalized fractional Basset-Boussinesq-Oseen equation,” *Iran. J. Sci. Technol. Trans. A Sci.*, vol. 44, no. 5, pp. 1461–1467, 2020, doi: 10.1007/s40995-020-00942-z.

[4] A. B. Basset, “On the descent of a sphere in a viscous liquid,” *Quart. J.*, vol. 41, no. 3, pp. 369–381, 1910.

[5] I. Cabrera, J. Hernando, and K. Sadarangani, “Existence and uniqueness of solutions for a boundary value problem of fractional type with nonlocal integral boundary conditions in Hölder spaces.” *Mediterr. J. Math.*, vol. 15, no. 3, p. 98, 2018, doi: 10.1007/s00009-018-1142-8.

[6] H. Fazli, H. G. Sun, S. Aghchi, and J. J. Nieto, “On a class of nonlinear nonlocal fractional differential equations.” *Carpathian J. Math.*, vol. 37, no. 3, pp. 441–448, 2021, doi: 10.37193/cjm.2021.03.07.

[7] Z. H. Gao, T. L. Hu, and H. H. Pang, “Existence and uniqueness theorems for a fractional differential equation with impulsive effect under band-like integral boundary conditions.” *Adv. Math. Phys.*, vol. 2020, no. 1, p. 6360128, 2020, doi: 10.1155/2020/6360128.

[8] M. Gomoyunov, “Sensitivity analysis of value functional of fractional optimal control problem with application to feedback construction of near optimal controls.” *Appl. Math. Optim.*, vol. 88, no. 2, p. 41, 2023, doi: 10.1007/s00245-023-10022-4.

[9] A. Guezane-Lakoud and R. Rodríguez-López, “Positive solutions to mixed fractional p-Laplacian boundary value problems.” *J. Appl. Anal.*, vol. 29, no. 1, pp. 49–58, 2023, doi: 10.1515/jaa-2021-2085.

[10] T. L. Guo and W. Jiang, "Impulsive problems for fractional differential equations with boundary value conditions." *Comput. Math. Appl.*, vol. 64, no. 10, pp. 3281–3291, 2012, doi: 10.1016/j.camwa.2012.02.006.

[11] B. Jin, *Fractional differential equations: An approach via fractional derivatives.* Switzerland: Springer, pp.3–15, 2021. doi: 10.1007/978-3-030-76043-4.

[12] K. Kaliraj, M. Manjula, and C. Ravichandran, "New existence results on nonlocal neutral fractional differential equation in concepts of Caputo derivative with impulsive conditions." *Chaos Solitons Fractals.*, vol. 161, p. 112284, 2022, doi: 10.1016/j.chaos.2022.112284.

[13] C. Leal, "Existence of weighted bounded solutions for nonlinear discrete-time fractional equations." *Appl. Anal.*, vol. 99, no. 10, pp. 1780–1794, 2020, doi: 10.1080/00036811.2018.1546001.

[14] Z. Liu and W. Wang, "Positive solutions for multipoint boundary value problem of fractional differential equation with parameter," *Carpathian J. Math.*, vol. 41, no. 2, pp. 441–453, 2025, doi: 10.37193/CJM.2025.02.11.

[15] F. F. Luo, P. Liu, and W. Wang, "Extreme solution for fractional differential equation with nonlinear boundary condition." *Carpathian J. Math.*, vol. 40, no. 3, pp. 681–690, 2024, doi: 10.37193/CJM.2024.03.09.

[16] F. Mainardi, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," *Chaos, Solitons Fractals*, vol. 7, no. 9, pp. 1461–1477, 1996, doi: 10.1016/0960-0779(95)00125-5.

[17] S. J. C. Mary and A. Tamilselvan, "Second order spline method for fractional Bagley-Torvik equation with variable coefficients and Robin boundary conditions." *J. Math. Model.*, vol. 11, no. 1, pp. 117–132, 2023, doi: 10.22124/jmm.2022.23040.2047.

[18] A. Ouannas, A. Khennaoui, G. Grassi, and S. Bendoukha, "On chaos in the fractional-order Grassi-Miller map and its control." *J. Comput. Appl. Math.*, vol. 358, pp. 293–305, 2019, doi: 10.1016/j.cam.2019.03.031.

[19] D. H. Pang, W. Jiang, J. Du, and A. U. K. Niazi, "Analytical solution of the generalized Bagley-Torvik equation." *Adv. Difference Equ.*, vol. 2019, p. 207, 2019, doi: 10.1186/s13662-019-2082-8.

[20] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, ser. Math. Sci. Eng. San Diego, CA: Academic Press, 1999, vol. 198.

[21] K. Shah, I. Ahmad, J. J. Nieto, G. U. Rahman, and T. Abdeljawad, "Qualitative investigation of nonlinear fractional coupled pantograph impulsive differential equations." *Qual. Theory Dyn. Syst.*, vol. 21, no. 4, p. 131, 2022, doi: 10.1007/s12346-022-00665-z.

[22] S. Staněk, "Periodic problem for the generalized Basset fractional differential equation." *Fract. Calc. Appl. Anal.*, vol. 18, no. 5, pp. 1277–1290, 2015, doi: 10.1515/fca-2015-0073.

[23] P. J. Torvik and R. L. Bagley, "On the appearance of the fractional derivative in the behavior of real materials," *J. Appl. Mech.*, vol. 51, no. 6, pp. 294–298, 1984, doi: 10.1115/1.3167615.

[24] J. Yang and G. P. Chen, "Existence of solutions for impulsive hybrid boundary value problems to fractional differential systems." *AIMS Math.*, vol. 6, no. 8, pp. 8895–8911, 2021, doi: 10.3934/math.2021516.

[25] T. Yu, K. Deng, and M. K. Luo, "Existence and uniqueness of solutions of initial value problems for nonlinear Langevin equation involving two fractional orders." *Commun. Nonlinear Sci. Numer. Simul.*, vol. 19, no. 6, pp. 1661–1668, 2014, doi: 10.1016/j.cnsns.2013.09.035.

[26] X. Zuo and W. Wang, "Existence of solutions for fractional differential equation with periodic boundary condition." *AIMS Math.*, vol. 7, no. 4, pp. 6619–6633, 2022, doi: 10.3934/math.2022369.

Authors' addresses

Weibing Wang
(**Corresponding author**) Hunan University of Science and Technology, Department of Mathematics, Xiangtan, Hunan 411201, P.R. China
E-mail address: `wwbing2013@126.com`

Piao Liu
Hunan University of Science and Technology, Department of Mathematics, Xiangtan, Hunan 411201, P.R. China
E-mail address: `2282486005@qq.com`