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WEAKLY REGULAR RESIDUATED MAPPINGS

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Abstract. For residual functions, two regularity notions are known in the literature, one is weak regularity introduced by Blyth and Janowitz, and the other is a closely related condition introduced by Nicholson et al., which we call N-regularity. In our paper, we provide several characterizations of weakly regular residuated maps defined in complete lattices and of N-regular residuated maps in complemented modular lattices. The relations between the mentioned notions and some related conditions (such as Jordan conditions J1 and J2 on residuated maps) are also investigated. We provide some significant examples of weakly regular residuated maps and of N-regular residuated maps.

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1. INTRODUCTION

Let A and B be two partially ordered sets. The mapping $f\colon A \to B$ is called *residuated* [1] if there exists a mapping $f^*\colon B \to A$ such that for all $x \in A$ and all $y \in B$

$$f(x) \leq y \iff x \leq f^*(y).$$

In this case the mapping f^*, called the *residual* of f, is uniquely determined. In fact, $f^*(y) = \bigvee\{z \in A \mid f(z) \leq y\}$. The set of all residuated mappings from A to B is denoted by $\mathrm{Res}(A,B)$. Let now L be a complete lattice. We use the notations $\mathrm{Res}(L) = \mathrm{Res}(L,L)$ and $\mathrm{Res}^*(L) = \{f^* | f \in \mathrm{Res}(L)\}$. It is well-known that $\mathrm{Res}(L)$ and $\mathrm{Res}^*(L)$ consist of complete join-endomorphisms and complete meet-endomorphisms of L, respectively, and they are complete lattices with respect to the pointwise defined order relation [5].

Residuated maps are applied in several areas of Algebra, Logic, Formal Languages, etc. The study of their lattice was initiated by G.N. Reney and E.T. Schreiner [7] (some basic results are going back to G. Grätzer and E.T. Schmidt [4]). For instance, they independently proved that for a completely distributive lattice L the atoms of $\mathrm{Res}(L)$ are mappings with 2-element range (see our Example 3(d)). Their results were extended by Z. Shmuely [8]. In their seminal book [2] S. Blyth and M.F. Janowitz, initiated a systematic study of residuated maps and their monoid [2]. The concept of weakly regular residuated maps was introduced also in [2], and it turned out to be an extremely useful tool in characterizing modular lattices. Weakly regular residuated maps occur in several fundamental algebraic areas, for instance, linear transformations of a vector space always induce weakly regular residuated maps on its subspace lattice, as it is shown in Example 1(c). K. Kaarli and S. Radeleczki in [5] characterized those regular residuated maps which are atoms in Res(L); from their result it follows that in several important lattice classes the atoms of Res(L) are exactly the weakly regular maps. Another notion of weak regularity was introduced by G. E. Nicholson at al. in [6] for modular lattices. In a pioneering paper, J. Szigeti [9] introduced the Jordan conditions J1, J2 strongly related with weak regularity, and using them he proved the existence of the so-called n-nilpotent Jordan base for the join-endomorphisms of a submodule lattice. These examples show the importance to examine the relationship between the mentioned natural concepts. In our paper we provide several characterizations of weakly regular residuated maps. Then, by using these results, the relations between the above conditions are investigated.

2. Preliminaries

A *complete join-endomorphis* of L is a mapping $f\colon L \to L$ satisfying

$$f\left(\bigvee_{t\in T} x_t\right) = \bigvee_{t\in T} f(x_t)$$

for any system of elements $x_t \in L$, $t \in T$. Notice that this definition yields also $f(0) = 0$. A *complete meet-endomorphism* of L is defined dually. It is worth noting that any complete join-endomorphism of a complete lattice L is a residuated map on L, and any complete meet-endomorphism is a residual map on L. A complete $\vee$-semilattice congruence of a complete lattice L is an equivalence relation $\theta \subseteq L \times L$ preserved by arbitrary joins. We note that for any $f \in \mathrm{Res}(L)$, the equivalence

$$\mathrm{Ker} f = \{(x,y) \in L^2 \mid f(x) = f(y)\}$$

is a complete $\vee$-semilattice congruence, and $[a]_{\mathrm{Ker} f}$ stands for the equivalence class of the element $a \in L$. For any elements $a, b \in L$ we will use the notations $\downarrow a = \{x \in L | x \leq a\}$, $\uparrow b = \{x \in L | x \geq b\}$. The set $\downarrow a$ is called the principal ideal belonging to a, and $\uparrow b$ is called the principal filter of b. Since $[0]_{\mathrm{Ker} f}$ is a principal ideal, it has a greatest element k_f, and hence $[0]_{\mathrm{Ker} f} = \downarrow k_f$. It is clear, that for any $x, y \in L$, $x \vee k_f = y \vee k_f$ implies $f(x) = f(y)$, i.e. $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f}$.

Definition 1. A mapping $f \in \mathrm{Res}(L)$ is called *range-closed* if $f(L) = [0, f(1)]$ and *dually range-closed* if $f^*(L) = [f^*(0), 1]$. A mapping which is both range-closed and dually range-closed is called *weakly regular*. [2]

In view of [1] a residuated mapping f defined on a complete lattice L is dually range-closed, if and only if $x \vee f^*(0) = f^*(f(x))$, for all $x \in L$. Observe that $f^*(0) = \bigvee\{x \in L \mid f(x) \leq 0\} = \bigvee[0]_{\mathrm{Ker} f} = k_f$ and $f^*(f(x)) = \bigvee\{z \in L \mid f(z) \leq f(x)\}$

Lemma 1. *The following assertions are equivalent:*

(i) $f \in \mathrm{Res}(L)$ *is dually range-closed.*

(ii) *For any* $x, y \in L$,

$$[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f} \Leftrightarrow x \vee k_f = y \vee k_f \tag{2.1}$$

Proof.

(i)⇒(ii): Assume that (i) holds. Let $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f}$ for some $x, y \in L$. Then $f(x) = f(y)$ implies $f^*(f(x)) = f^*(f(y))$. Now, in view of [1] we obtain $x \vee k_f = x \vee f^*(0) = f^*(f(x)) = f^*(f(y)) = y \vee f^*(0) = y \vee k_f$.

As $x \vee k_f = y \vee k_f$ obviously yields $f(x) = f(y)$, i.e. $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f}$, our result means that (ii) is satisfied.

(ii)⇒(i): Suppose that (ii) holds and let us prove that $x \vee f^*(0) = f^*(f(x))$. Clearly, $f(x \vee f^*(0)) = f(x \vee k_f) = f(x)$, hence $x \vee f^*(0) \leq f^*(f(x))$. Conversely, $f(f^*(f(x))) = f(x)$ implies $[f^*(f(x))]_{\mathrm{Ker} f} = [x]_{\mathrm{Ker} f}$, and now, by using (ii) we get $f^*(f(x)) \vee k_f = x \vee k_f = x \vee f^*(0)$. This yields $f^*(f(x)) \leq x \vee f^*(0)$, and hence $x \vee f^*(0) = f^*(f(x))$. In view of [1] this implies (i).

□

Corollary 1. *A map* $f \in \mathrm{Res}(L)$ *is weakly regular if and only if* $f(L) = [0, f(1)]$ *and we have* $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f} \Leftrightarrow x \vee k_f = y \vee k_f$, *for any* $x, y \in L$.

An other characterization of weakly regular residuated mappings can be found in [2] (see also [5]) In view of [2] Theorem 13.2 and the remark after it we obtain:

Proposition 1. *Let L be a complete lattice and* $f : L \to L$. *Then the following are equivalent:*

(i) $f \in \mathrm{Res}(L)$ *is weakly regular;*

(ii) *There are* $a, b \in L$ *and a lattice isomorphism* $i : \uparrow a \to \downarrow b$ *such that for all* $x \in L$ $f(x) = i(x \vee a)$ *holds.*

Remark 1. Clearly, we have $i(1) = b$, $i(a) = 0$ and $f(x) = i(x)$, for all $x \geq a$ in Proposition 1(ii). Observe, that $f^*(y) = i^{-1}(y)$, for any $y \leq b$, and $a = k_f$ also hold. Indeed, $f^*(y) = \bigvee\{z \in L \mid i(z \vee a) \leq y\} = \bigvee\{t \in \uparrow a \mid i(t) \leq y\} = \bigvee\{t \in \uparrow a \mid t \leq i^{-1}(y)\} = i^{-1}(y)$. Hence $k_f = f^*(0) = i^{-1}(0) = a$.

Now we use the above characterizations to exhibit some significant examples for weakly residuated mappings on complete lattices.

Example 1.

(a) By Corollary 1, any automorphism f of L is a weakly regular residuated map, because it is a complete join-endomorphism with $f(L) = [0,1] = L$ and $k_f = 0$. We have $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f} \Leftrightarrow x = y \Leftrightarrow x \vee k_f = y \vee k_f$ for any $x, y \in L$.
(b) Let L be a modular lattice and $s, t \in L$ complements each of other. If we substitute in Proposition 1(ii) $a := t$ and $i(x) := x \wedge s$, then the map $p_{t,s} = i(x \vee a) = (x \vee t) \wedge s$ is called a "*projection*" (see e.g [5]). Since L is modular, the map $i\colon [t,1] \to [0,s]$ is a lattice isomorphism. Thus any projection $p_{t,s}$ is a weakly regular residuated mapping of L, according to Proposition 1.
(c) Let $(V, +, K)$ be a vector space over a field K and $L\colon V \to V$ a linear mapping. Then L induces a weakly regular residuated mapping $f_L\colon \mathrm{Sub}(V) \to \mathrm{Sub}(V)$, $f_L(W) = L(W) := \{L(w) \mid w \in W\}$ of the subspace lattice $(\mathrm{Sub}(V), \subseteq)$, according to [1], Example 4.2 (see also [5]).

We say that a mapping $f\colon L \to L$ is *idempontent* if $f^2 = f$. It is easy to check that on a modular lattice L any projection $p_{t,s} = (x \vee t) \wedge s$ is idempotent.

Remark 2. Let L be a complete modular lattice. In view of [[1], Exercise 4.3] any idempotent weakly regular $f \in \mathrm{Res}(L)$ has the form $f(x) = (x \vee f^*(0)) \wedge f(1)$, $x \in L$, moreover, $f^*(0) = k_f$ and $f(1)$ are complements each of other, according to [[1], Theorem 6.23]. Therefore, f equals to a projection $p_{k_f, f(1)}$. This means that idempotent weakly regular mappings in $\mathrm{Res}(L)$ and the projections of a (complete) modular lattice L coincide.

3. Notions related with weak regularity

For a complemented modular lattice we have another notion of "weak regularity" introduced by G. E. Nicholson and his coauthors, which we will call N-regular. ([6])

A join $\bigvee\{x_i \in L_i | i \in I\}$ in a complete lattice L is called *irredundant*, if $x_j \nleq \bigvee\{x_i | i \in I \setminus \{j\}\}$ holds for each index $j \in I$. We recall that in a complemented modular lattice L of a finite height any element $x \in L \setminus \{0\}$ is equal to an irredundant join of atoms and the *number* of the atoms in this irredundant join is called the *dimension* of the element x, denoted by $d(x)$. We denote the set of atoms of our lattice L by A. It is worth to mention that in the case of a finite dimensional vector space $(V, +, K)$ and of a subspace $X \in \mathrm{Sub}(V)$, $\mathrm{d}(X)$ equals to its usual (linear algebraic) dimension.

Definition 2. A join-endomorphism f of a complemented modular lattice L of finite height is said to be *N-regular* (according to Nicholson) if it has the following properties:
(a) $d(f(x)) \leq d(x)$ for all $x \in L$, and
(b) $d(f(x)) < d(x)$ implies that there exists an atom $a \leq x$ such that $f(a) = 0$.

Proposition 2 ([6, Proposition 3.7])**.** *Let f be a join-endomorphism on a complemented modular lattice of a finite height. Then f is N-regular if and only if the following conditions are satisfied:*

(a) *f maps atoms into atoms or 0.*

(b) *For any nonzero elements $x, y \in L$, $x \wedge y = 0$ and $f(x) \leq f(y)$ imply that there exists an element $z \in L$ such that $0 < z \leq x \vee y$ and $f(z) = 0$.*

We note that all residuated maps presented in Example 1 in the particular case of a complemented modular lattice L of finite height are also N-regular maps. In spite of the fact that the two notions are very closed, they are not equivalent, as our Theorem 1 will show. To prove it, we will need the following lemma:

Lemma 2. *Let L be a complete modular and complemented lattice. Then any weakly regular residuated mapping $f\colon L \to L$ has the form $f = l \circ p_{k_f,s_f}$, where s_f is a complement of k_f, and $l\colon [0, s_f] \to [0, f(1)]$ is a lattice isomorphism.*

Proof. Assume that f is a weakly regular residuated mapping. Let s_f be a complement of k_f and i the isomorphism defined in Proposition 1(ii). Observe that $i\colon [k_f, 1] \to [0, f(1)]$ is the restriction of f to $[k_f, 1]$, according to Remark 2 and consider the projection $p_{k_f,s_f}\colon L \to [0, s_f]$, $p_{k_f,s_f}(x) = (x \vee k_f) \wedge s_f$.

Because L is modular, $\varphi\colon [0, s_f] \to [k_f, 1]$, $\varphi(x) = x \vee k_f$ is a lattice isomorphism (having as inverse $\varphi^{-1}(x) = x \wedge s_f$, $x \in [k_f, 1]$), and the map $l := i \circ \varphi$, $l\colon [0, s_f] \to [0, f(1)]$ is also a lattice isomorphism. It is clear that $\varphi(p_{k_f,s_f}(x)) = \varphi(\varphi^{-1}(x \vee k_f)) = x \vee k_f$, hence $\left(i \circ \varphi \circ p_{k_f,s_f}\right)(x) = i(x \vee k_f) = f(x)$. This implies $l \circ p_{k_f,s_f} = f$. □

Theorem 1. *Let L be a complemented modular lattice of a finite height and $f\colon L \longrightarrow L$ a residuated map. Then the following assertions are equivalent:*

(1) *f is weakly regular,*

(2) *f is N-regular and for any atom $p \leq f(1)$ there exists an atom $a_p \in L$ such that $p = f(a_p)$.*

Proof.

(1)⇒(2): If f is weakly-regular, then $f\colon L \to [0, f(1)]$ is surjective, and hence for any atom $p \leq f(1)$ there exists an $x \in L \setminus \{0\}$ with $p = f(x)$. Clearly, we have $0 \leq f(a) \leq p$, for any atom $a \leq x$. Since $x = \bigvee\{a \in A | a \leq x\}$, the assumption $f(a) = 0$, for all atoms $a \leq x$ would imply $f(x) = 0$, a contradiction. Thus there exists an $a_p \in A$, $a_p \leq x$, with $f(a_p) \neq 0$, i.e. with $f(a_p) = p$. This proves the second part of (2). In order to prove that f is N-regular we will show that conditions (a) and (b) of Proposition 2 are satisfied. Since f is weakly-regular and L is modular and complemented, in view of Lemma 2, f has the form $f = l \circ p_{k_f,s_f}$, where s_f is a complement of k_f denoted as far as follows by $\overline{k_f}$, and $l\colon [0, s_f] \to [0, f(1)]$ is a lattice isomorphism. Now let a be an atom of L. Then the projection p_{k_f,s_f} maps a into an atom or 0. Because an isomorphism $l\colon [0, s_f] \to [0, f(1)]$ maps atoms into atoms and 0 into 0, condition (a) holds. In order to prove (b), take any $x, y \in L$ with $x \wedge y = 0$ and assume that $f(x) \leq f(y)$. Then $l \circ p_{k_f,s_f}(x) \leq l \circ p_{k_f,s_f}(y)$

implies $p_{k_f,s_f}(x) \leq p_{k_f,s_f}(y)$, i.e. $(x \vee k_f) \wedge \overline{k_f} \leq (y \vee k_f) \wedge \overline{k_f}$. Since the mapping $\mu\colon y \mapsto y \wedge \overline{k_f}$ is an order-isomorphism between the intervals $[k_f, 1]$ and $[0, \overline{k_f}]$ (having as inverse the map $\lambda\colon x \mapsto x \vee k_f$), we obtain $x \vee k_f \leq y \vee k_f$, and hence $y \vee k_f = x \vee y \vee k_f$. We claim that now $(x \vee y) \wedge k_f = 0$ is not possible. Indeed, $(x \vee y) \wedge k_f = 0$ would yield also $y \wedge k_f = 0$, and hence we would obtain that $\{0, y, x \vee y, k_f, x \vee y \vee k_f\}$ is a sublattice of L isomorphic to N_5. This is not possible, because L modular. Thus we get $z = (x \vee y) \wedge k_f \neq 0$. Since $z \leq x \vee y$, and because $z \leq k_f$ implies $f(z) = 0$, condition (b) holds, and in view of Proposition 2 we get (2).

(2)⇒(1): Let $y \leq f(1)$. Since $y = \bigvee\{a \in A | a \leq y\}$, and because by (2) for any atom $a \leq y$, there exists an atom $a' \in A$ with $f(a') = a$, we deduce $y = \bigvee\{f(a') | f(a') \leq y\} = f(\bigvee\{a' \in A | f(a') \leq y\})$, and this proves that $f\colon L \to [0, f(1)]$ is surjective. Thus f is range-closed. In order to prove that f is dually range-closed, it is enough to show relation (2.1). By the way of contradiction, suppose that (2.1) fails, i.e. let $x, y \in L$ with $f(x) = f(y)$ and assume $x \vee k_f \neq y \vee k_f$. Then $x \vee k_f < x \vee y \vee k_f$, hence we can choose an atom $a \in L$ such that $a \nleq x \vee k_f$, but $a \leq x \vee y \vee k_f$. Since L is also relatively complemented, there exists a relative complement u of the element k_f in the interval $[0, x \vee k_f]$, i.e. an element $u \leq x \vee k_f$ such that

$u \vee k_f = x \vee k_f$ and $u \wedge k_f = 0$.

Since $a \nleq x \vee k_f$, we have $a \nleq u$, whence $a \wedge u = 0$. On the other hand, we obtain:

$f(u) = f(u) \vee f(k_f) = f(u \vee k_f) = f(x \vee k_f) = f(x) \vee f(k_f) = f(x)$, and
$f(a) \leq f(x \vee y \vee k_f) = f(x) \vee f(y) \vee f(k_f) = f(x)$,

Since, in view of (2), f is N-regular, we can apply condition (b) from Proposition 2, and this implies the existence of a nonzero element $z \leq a \vee u$ with $f(z) = 0$. Now let β an atom of z. Then $\beta \leq a \vee u$ and $f(\beta) = 0$, i.e. $\beta \leq k_f$. Because $u \wedge k_f = 0$, we have also $u \wedge \beta = 0$. Since L is modular, we get $u \prec u \vee \beta \leq u \vee (a \vee u) = u \vee a$. As $a \wedge u = 0$, we have also $u \prec u \vee a$, and hence we obtain $u \vee \beta = u \vee a$. This implies $a \leq u \vee \beta \leq u \vee k_f = x \vee k_f$ - a contradiction to the selection of a. Thus relation (2.1) holds, and hence f is weakly regular.

□

Corollary 2. *Let L be a complemented modular lattice of a finite height and let $f\colon L \to L$ be a residuated map. Then f is weakly regular if and only if f maps atoms into atoms or 0, for any atom $a \leq f(1)$ there is an atom $a' \in L$ with $a = f(a')$, and for any $x, y \in L \setminus \{0\}$ with $x \wedge y = 0$ and $f(x) \leq f(y)$ there exists an atom $p \leq x \vee y$ such that $f(p) = 0$.*

Now, we can provide a simple example for a residuated mapping which is N-regular only, and it is not weakly regular.

Example 2. Let L stand for the direct product of lattices $M_3 = \{0,a,b,c,1\}$ and $M_2 = \{0, p, q, 1\}$ on Figure 2.

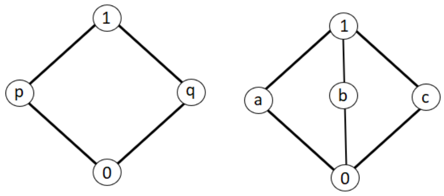

FIGURE 1. Lattices M_2 and M_3

Then clearly $L = M_3 \times M_2$ is a finite modular atomistic lattice with the atoms $(a,0),(b,0),(c,0),(0,p)$ and $(0,q)$. Since L is a geometric lattice, it is complemented, too, and in addition, it has the finite height 4.

Now, define a mapping $f: L \to L$ as follows:

$$f(x,y) = \begin{cases} (0,0), \text{ if } y = 0; \\ (a,0), \text{ if } y = p; \\ (c,0), \text{ if } y = q; \\ (1,0), \text{ if } y = 1. \end{cases}$$

It is clear that $f(x,y) = f(0,y)$, and that $y_1 \leq y_2$ yields $f(0,y_1) \leq f(0,y_2)$, therefore f is order-preserving. We will prove that f it is a join-endomorphism, i.e. $f((x_1,y_1) \vee (x_2,y_2)) = f(x_1,y_1) \vee f(x_2,y_2)$. This is clear, whenever $y_1 \leq y_2$. Assume now that $y_1,y_2 \in M_2$ are incomparable. Then either $y_1 = p$, $y_2 = q$ or vice-versa. In both case
$f((x_1,y_1) \vee (x_2,y_2)) = f(x_1 \vee x_2, p \vee q) = f(x_1 \vee x_2, 1) = (1,0) = (a,0) \vee (c,0) = f(x_1,y_1) \vee f(x_2,y_2)$.

Thus f is a residuated mapping, because L is a finite lattice. Let us observe also that $f(a,0) = f(b,0) = f(c,0) = (0,0)$, and $f(0,p) = (a,0)$, $f(0,q) = (c,0)$, i.e. f maps atoms into atoms or 0, which is the first condition in Proposition 2.

Now, suppose that for some $u = (x_1,y_1) \neq (0,0)$ and $v = (x_2,y_2) \neq (0,0)$ we have $u \wedge v = 0$ and $f(u) \leq f(v)$. Then $x_1 \wedge x_2 = 0$, $y_1 \wedge y_2 = 0$ and $y_1 \leq y_2$. The latter two relations imply $y_1 = 0$, hence we obtain $f(u) = f(x_1,0) = 0$. Since $0 \neq u \leq u \vee v$, this result means that the conditions in Proposition 2 are satisfied, therefore, f is an N-regular residuated mapping.

Observe that f is not weakly regular, because do not satisfies the second statement in Theorem 1(2). Indeed, we have $f(1_L) = f(1,1) = (1,0)$, however the atom $(b,0) \leq f(1_L)$ is not the image of any atom of L.

In [9] two other conditions (called Jordan conditions) imposed on complete join-endomorphisms $f\colon L \to L$, i.e. on residuated mappings on L are discussed:

(J1) For any choice of the elements $x \leq y$ with $f(x) = f(y)$ there is an element $u \in L$ such that $y = x \vee u$ and $f(u) = 0$.

(J2) For any $x \in L$ the mapping $f\colon [0,x] \to [0, f(x)]$ is surjective.

Condition (J2) means that the image under f of every principal ideal $x\downarrow$ is a principal ideal $f(x)\downarrow$; residuated mappings with this property in [1] are called *totally range-closed*. Clearly, any totally range-closed residuated mapping is range-closed. In view of [[1], Thm 6.21], $f \in \mathrm{Res}(L)$ is totally range-closed if and only if

$$f(f^*(x) \wedge y) = x \wedge f(y) \tag{3.1}$$

holds for all $x, y \in L$. As a consequence of this result we obtain

Corollary 3. *Let L be a complete modular lattice. Then any weakly regular residuated mapping $f\colon L \to L$ is totally range-closed.*

Proof. Let $f \in \mathrm{Res}(L)$. In view of Proposition 1, $f(x) = i(x \vee a)$, where $a = k_f$ and $i\colon \uparrow a \to \downarrow b$ is a lattice isomorphism with $b := f(1)$. Then, in view of Remark 2, $f^*(x) = i^{-1}(x)$, for any $x \leq b$. Hence $f(f^*(x) \wedge y) = f(i^{-1}(x) \wedge y) = i((i^{-1}(x) \wedge y) \vee a)$. Since $a \leq i^{-1}(x)$ and L is modular, we get $(i^{-1}(x) \wedge y) \vee a = i^{-1}(x) \wedge (y \vee a)$. Thus we obtain $f(f^*(x) \wedge y) = i(i^{-1}(x) \wedge (y \vee a)) = i(i^{-1}(x)) \wedge i(y \vee a) = x \wedge f(y)$. Thus in virtue of (3.1) f is totally range-closed. □

Condition (J1) seems to be similar to our condition (2.1), see the following

Proposition 3. *Condition (J1) implies condition (2.1) from Lemma 1. If L is a modular lattice, then (J1) and (2.1) are equivalent.*

Proof. Let L be a complete lattice, and assume that $f\colon L \to L$ satisfies (J1). First, take any $x, y \in L$ with $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f}$. Then $(x,y) \in \mathrm{Ker} f$, i.e. $f(x) = f(y)$. Hence $f(x) = f(x) \vee f(y) = f(x \vee y)$. As $x \leq x \vee y$, in view of (J1) there exists an element $u \in L$ with $f(u) = 0$ and such that $x \vee y = x \vee u$. Clearly, $u \leq k_f$. Similarly, $f(y) = f(x \vee y)$ and $y \leq x \vee y$ by (J1) imply $x \vee y = y \vee v$ for some $v \in L$ with $v \leq k_f$. Since $x \vee u = y \vee v$ and $u, v \leq k_f$, we obtain $x \vee k_f = x \vee u \vee k_f = y \vee v \vee k_f = y \vee k_f$. Conversely, for any $x, y \in L$, $x \vee k_f = y \vee k_f$ always yields $f(x) = f(y)$, i.e. $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f}$. Thus we proved that (J1) implies the equivalence $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f} \Leftrightarrow x \vee k_f = y \vee k_f$, for any $x, y \in L$, i.e. it implies condition (2.1).

Suppose now that f satisfies (2.1), and furthermore, that L is modular. Take any $x, y \in L$, $x \leq y$ with $f(x) = f(y)$. Then $[x]_{\mathrm{Ker} f} = [y]_{\mathrm{Ker} f}$ implies $x \vee k_f = y \vee k_f$, according to (2.1). Since L is modular, $x \leq y$ implies $y = (y \vee k_f) \wedge y = (x \vee k_f) \wedge y =$

$x \vee (k_f \wedge y)$. Now for $u := k_f \wedge y$ we obtain $f(u) \leq f(k_f) = 0$, and $y = x \vee u$. Thus (J1) holds. Hence (2.1) is equivalent to (J1), whenever L is modular. □

Since in view of Lemma 1, condition (2.1) is equivalent to the fact that the residuated mapping $f\colon L \to L$ is dually range-closed, using Proposition 3 and Corollary 3 we obtain:

Corollary 4. *Let L be a complete modular lattice and $f\colon L \to L$ a residuated mapping. Then*

(i) *f is dually range-closed if and only it if satisfies condition (J1).*

(ii) *f is weakly regular if and only it satisfies conditions (J1) and (J2)*

Example 3.

(a) Obviously, any automorphism f of L is totally range-closed, and in view of Example 1(a) is also a weakly regular residuated mapping. Hence, it satisfies conditions (J1) and (J2).

(b) In view of [9], any residuated mapping f defined in Example 1(c) by a linear transformation $L\colon V \to V$ on the subspace lattice $\mathrm{Sub}(V)$ of a vector space $(V,+,K)$ satisfies conditions (J1) and (J2).

(c) Projections, being weakly regular maps, satisfy conditions (J1) and (J2) in case of a modular lattice L.

(d) Finally, let L be a complete lattice and p a dual atom and q an atom in L. In view of [5], the mapping

$$e_{p,q} := \begin{cases} 0, & \text{if } x \leq p; \\ q, & \text{if } x \not\leq p. \end{cases}$$

is a weakly regular residuated mapping of L, which is an atom in the lattice $\mathrm{Res}(L)$. Thus by Lemma 1, $e_{p,q}$ satisfies condition (2.1). It is obvious that $e_{p,q}$ also satisfies (J2), i.e. it is totally range-closed. However, it is easy to see that $e_{p,q}$, fails to satisfy (J1), in general. Indeed, let $L = \{0, p, r, q, 1\}$ be an N_5 lattice where $p < r$ and p, q are atoms of L and r, q are dual atoms of L. Then $\ker(e_{p,q}) = \{0, p\}$. If $e_{p,q}$ would satisfy (J1), then $r < 1$ and $e_{p,q}(r) = e_{p,q}(1) = q$, would imply $1 = r \vee u$, for some $u \in \ker(e_{p,q})$. Clearly, this is not possible (because $r \vee 0 = r \vee p = r \neq 1$).

In [9] several properties of residuated mappings sharing properties (J1) and (J2) are proved. For instance, it is shown that for any $f_1, f_2 \in \mathrm{Res}(L)$ satisfying conditions (J1) and (J2), $f_1 \circ f_2 \in \mathrm{Res}(L)$ also satisfies (J1) and (J2). An important result of [9] with applications in module theory was showing the existence of a so-called n-nilpotent Jordan normal base for any n-nilpotent residuated mapping $f\colon L \to L$ defined on a geometric lattice L. We note that a map $f\colon L \to L$ is n-nilpotent, if $f^n = 0$ for some integer $n \geq 1$, but $f^{n-1} \neq 0$. Any linear transformation $L\colon \mathbb{R}^p \to \mathbb{R}^p$, $L(\bar{x}) = A\bar{x}$, with a square matrix A of order $p \geq 1$ is n-nilpotent, if and only if n is the least positive integer with the property that A^n equals the 0 matrix. (i.e. the matrix A is *n-nilpotent*). For example, $L\colon \mathbb{R}^3 \to \mathbb{R}^3$, $L(x_1,x_2,x_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a 3-nilpotent linear transformation.

As a consequence of the first mentioned result of [9] we obtain:

Corollary 5. *Let L be a finite modular lattice L and $f\colon L \to L$ a weakly regular residuated mapping on it. Then $f\colon L \to L$ is either nilpotent or there exists an integer $n \geq 1$ such that the power $f^n\colon L \to L$ is a (nonzero) projection.*

Proof. If our condition hold, then f satisfies conditions (J1) and (J2), according to Corollary 4. In view of [9], for any integer $n \geq 1$ $f^n\colon L \to L$ is also a residuated mapping satisfying conditions (J1) and (J2), i.e. it is a weakly regular residuated mapping. Because L is a finite set, the semigroup formed by the powers $f,\dots,f^n,\dots$ of f is finite, hence in view of [3], it contains an idempotent element g, where $g = f^k$, for some integer $k \geq 1$. If $g = 0$, then f is nilpotent. If $g \neq 0$, then in view of Remark 1 the map $g = f^k$ is a nonzero projection on L. □

Remark 3. We note that the converse of Corollary 5 is not true in general. For instance, the residuated mapping $f\colon L \to L$ defined in Example 2, (where $L = M_3 \times M_2$) is 2-nilpotent, because $f^2(x,y) = (0,0) = 0_L$, for all $(x,y) \in L$, however f is not weakly regular, as it is shown in Example 2. Next, in order to examine the case when a power of residuated map is a projection, consider the lattice $L' := L \times M_2$, i.e. $L' = M_3 \times M_2 \times M_2$, and the mapping $g\colon L' \to L'$, $g(x,y,z) = (f(x,y),z)$. Then L' is a modular atomistic lattice with the atoms $(a,0,0)$, $(b,0,0)$, $(c,0,0)$, $(0,p,0)$, $(0,q,0)$, $(0,0,p)$ and $(0,0,q)$ (see Figure 1). It is easy to check that g is a residuated map. Because $f^2(x,y) = (0,0)$, we get $g^2(x,y,z) = (f^2(x,y),z) = (0,0,z)$, hence g is a projection (corresponding to the pair $s = (1,1,0), t = (0,0,1)$). However, g is not weakly regular, because do not satisfies the second condition in Theorem 1(2). Indeed, $g(1_{L'}) = g(1,1,1) = (1,0,1)$, but the atom $(b,0,0) \leq g(1_{L'})$ is not the image of any atom of L'. (We have $g(a,0,0) = g(b,0,0) = g(c,0,0) = (0,0,0)$, $g(0,p,0) = (a,0,0)$, $g(0,q,0) = (c,0,0)$ and $g(0,0,p) = (0,0,p)$, $g(0,0,q) = (0,0,q)$.)

4. Conclusions

We have shown that the notion of weak regularity for residuated mappings is stronger than the notion of N-regulatiy, even for complemented modular lattices. We also proved that a residuated mapping satisfying the Jordan conditions J1 and J2 is weakly regular and totaly-range closed. Therefore, Jordan condition J1+J2 in general is stronger than the weak regularity condition, but according to our Corollary 4, for complete modular lattices the two conditions are equivalent. The benefit of the residuated maps satisfying the previous Jordan condition lies in the fact that composing them we obtain residuated maps with the same property. As further work, we would like to characterize those residuated maps $f\colon L \to L$ which have the property that f^n $(n \geq 1)$ is weakly regular, and to examine for any atomistic modular lattice L whether there are N-regular but not weakly regular atoms in the lattice $\mathrm{Res}(L)$.

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