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SÁNDOR'S INEQUALITY FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. The Sándor inequality is a highly significant result in both pure and applied mathematics, providing an upper bound for the mean square of a positive convex function. This paper presents an extension of the Sándor inequality to the case of fractional integrals in the sense of Riemann-Liouville, as well as a generalization for any positive power r.

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1. INTRODUCTION

Let I be an interval of real numbers. A function $\mathcal{G}\colon I\to\mathbb{R}$ is said to be convex, if for all $\xi,\varsigma\in I$ and all $t\in[0,1]$, we have

$$\mathcal{G}\left(t\varsigma+(1-t)\xi\right)\leq t\,\mathcal{G}\left(\varsigma\right)+\left(1-t\right)\mathcal{G}(\xi).$$

The renowned Hermite-Hadamard inequality (refer to [3, 4]) is a fundamental result for the class of convex functions, stated as follows:

Theorem 1. *Let $\mathcal{G}\colon\ [\varsigma,\xi]\to\mathbb{R}$ be a convex function. Then, we have*

$$\mathcal{G}\left(\tfrac{\varsigma+\xi}{2}\right)\leq\tfrac{1}{\xi-\varsigma}\int\limits_{\varsigma}^{\xi}\mathcal{G}\left(z\right)\mathrm{d}z\leq\tfrac{\mathcal{G}(\varsigma)+\mathcal{G}(\xi)}{2}. \tag{1.1}$$

Fractional calculus extends classical calculus to non-integer orders, capturing memory and non-local effects. It is widely used to model complex phenomena like anomalous diffusion, viscoelasticity, and heat conduction, offering a more accurate description of systems with hereditary properties. In recent decades, this field has experienced significant development. In particular, in approximation theory, several famous inequalities, already crucial in the classical case, have been extended to the fractional setting (see [1,2,6,9]).

In [7], Sándor proved the following result connected with (1.1).

Theorem 2. *Let* $\mathcal{G}\colon [\varsigma,\xi]\to\mathbb{R}$ *be a non-negative convex function. Then*

$$\frac{1}{\xi-\varsigma}\int\limits_{\varsigma}^{\xi}\mathcal{G}^{2}(z)\,\mathrm{d}z\leq\frac{1}{3}\left(\mathcal{G}^{2}(\varsigma)+\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\mathcal{G}^{2}(\xi)\right).$$

The Sándor inequality is a mathematical inequality that connects different special functions, particularly convex functions, within the framework of inequality analysis. It plays a crucial role in both pure and applied mathematics, especially in optimization and inequality theory. This inequality is used to establish precise bounds and estimates, providing powerful tools for solving various problems in number theory, real and complex analysis, and mathematical physics. Its significance lies in its ability to generalize and unify several classical results. In this note, we generalize Sándor's inequality to the case of Riemann-Liouville fractional integrals.

2. PRELIMINARIES

Definition 1 ([5])**.** Let $\mathcal{G}\in L^{1}[\varsigma,\xi]$. The left- and right-side Riemann-Liouville fractional integrals $\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}$ and $\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}$ of order $\alpha>0$ are defined by

$$\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{\varsigma}^{x}(x-t)^{\alpha-1}\,\mathcal{G}(t)\,\mathrm{d}t,\qquad x>\varsigma,$$

$$\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{\xi}(t-x)^{\alpha-1}\,\mathcal{G}(t)\,\mathrm{d}t,\qquad \xi>x,$$

respectively, where $\Gamma(\alpha)=\int\limits_{0}^{\infty}e^{-t}t^{\alpha-1}\,\mathrm{d}t$ is the gamma function and

$$\mathcal{J}_{\varsigma^{+}}^{0}\mathcal{G}(x)=\mathcal{J}_{\xi^{-}}^{0}\mathcal{G}(x)=\mathcal{G}(x).$$

Definition 2 ([5])**.** The beta function is defined by

$$B(\varsigma,\xi)=\int\limits_{0}^{1}s^{\varsigma-1}(1-s)^{\xi-1}\,\mathrm{d}s=\frac{\Gamma(\varsigma)\Gamma(\xi)}{\Gamma(\varsigma+\xi)},$$

where $\varsigma,\xi>0$ and $\Gamma(.)$ is the Euler gamma function.

Definition 3 ([5]). The incomplete beta function is given by

$$B_{\delta}(\varsigma,\xi)=\int_{0}^{\delta}s^{\varsigma-1}(1-s)^{\xi-1}\,\mathrm{d}s,$$

where $\varsigma,\xi>0$ and $0<\delta<1$.

Definition 4 ([10]).

$$(x+y)^{r}=\sum_{k=0}^{\infty}\frac{(r)_{k}}{k!}x^{r-k}y^{k},$$

$(r)_{k}$ is the Pochhammer symbol.

3. MAIN RESULTS

Theorem 3. *Let* $\mathcal{G}\colon[\varsigma,\xi]\to\mathbb{R}$ *be a non-negative convex mapping. Then, we have*

$$\frac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}^{2}(\xi)+\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}^{2}(\varsigma)\right)\\ \leq\frac{1}{2(\alpha+2)(\alpha+1)}\left[(2+\alpha(\alpha+1))\left(\mathcal{G}^{2}(\varsigma)+\mathcal{G}^{2}(\xi)\right)+4\alpha\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right],$$

where $\alpha>0$.

Proof. Using the convexity of $\mathcal{G}$ on $[\varsigma,\xi]$, we have

$$\mathcal{G}(t)\leq\frac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\frac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi).\tag{3.1}$$

Since $\mathcal{G}$ is non-negative, from (3.1) we deduce

$$\mathcal{G}^{2}(t)\leq\left(\frac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\frac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^{2}\\ =\frac{(\xi-t)^{2}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\varsigma)+2\frac{(\xi-t)(t-\varsigma)}{(\xi-\varsigma)^{2}}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\frac{(t-\varsigma)^{2}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\xi).\tag{3.2}$$

Multiplying both sides of (3.2) by $\frac{1}{\Gamma(\alpha)}(\xi-t)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we get

$$\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1}\mathcal{G}^{2}(t)\,\mathrm{d}t\leq\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\frac{(\xi-t)^{\alpha+1}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\varsigma)\,\mathrm{d}t+\frac{2}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\frac{(\xi-t)^{\alpha}(t-\varsigma)}{(\xi-\varsigma)^{2}}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\,\mathrm{d}t\\ +\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\frac{(t-\varsigma)^{2}(\xi-t)^{\alpha-1}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\xi)\,\mathrm{d}t\tag{3.3}\\ =\frac{(\xi-\varsigma)^{\alpha}}{(\alpha+2)\Gamma(\alpha)}\left(\mathcal{G}^{2}(\varsigma)+\frac{2}{\alpha+1}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\frac{2}{\alpha(\alpha+1)}\mathcal{G}^{2}(\xi)\right).$$

Similarly, multiplying both sides of (3.2) by $\frac{1}{\Gamma(\alpha)}(t-\varsigma)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha-1}\mathcal{G}^{2}(t)\,\mathrm{d}t\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\left(\int_{\varsigma}^{\xi}\frac{(\xi-t)^{2}(t-\varsigma)^{\alpha-1}}{(\xi-\varsigma)^{2}}\mathrm{d}t\right)\mathcal{G}^{2}(\varsigma)+2\left(\int_{\varsigma}^{\xi}\frac{(\xi-t)(t-\varsigma)^{\alpha}}{(\xi-\varsigma)^{2}}\mathrm{d}t\right)\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right.\\
&\qquad\left.+\left(\int_{\varsigma}^{\xi}\frac{(t-\varsigma)^{\alpha+1}}{(\xi-\varsigma)^{2}}\mathrm{d}t\right)\mathcal{G}^{2}(\xi)\right]\\
&=\frac{(\xi-\varsigma)^{\alpha}}{(\alpha+2)\Gamma(\alpha)}\left(\frac{2}{\alpha(\alpha+1)}\mathcal{G}^{2}(\varsigma)+\frac{2}{\alpha+1}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\mathcal{G}^{2}(\xi)\right),
\end{aligned}\tag{3.4}$$

where we have used

$$\int_{\varsigma}^{\xi}(\xi-t)^{\alpha+1}\,\mathrm{d}t=\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha+1}\,\mathrm{d}t=\frac{(\xi-\varsigma)^{\alpha+2}}{\alpha+2},$$

$$\int_{\varsigma}^{\xi}(\xi-t)^{\alpha}(t-\varsigma)\,\mathrm{d}t=\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha}(\xi-t)\,\mathrm{d}t=\frac{1}{(\alpha+1)(\alpha+2)}(\xi-\varsigma)^{\alpha+2}$$

and

$$\int_{\varsigma}^{\xi}(t-\varsigma)^{2}(\xi-t)^{\alpha-1}\,\mathrm{d}t=\int_{\varsigma}^{\xi}(\xi-t)^{2}(t-\varsigma)^{\alpha-1}\,\mathrm{d}t=\frac{2}{\alpha(\alpha+1)(\alpha+2)}(\xi-\varsigma)^{\alpha+2}.$$

Finally, summing (3.3) and (3.4), then multiplying the resulting inequality by $\frac{1}{2(\xi-\varsigma)}$, we get the desired result. ☐

Remark 1. In Theorem 3, if we take $\alpha=1$, we obtain Theorem 2.

Theorem 4. *Let $\mathcal{G}\colon[\varsigma,\xi]\to\mathbb{R}$ be a non-negative convex mapping. Then we have*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}^{r}(\xi)+\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}^{r}(\varsigma)\right)\\
&\leq\frac{\alpha}{2^{2-\frac{r}{[r]}}}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}\\
&\qquad\times\left(B\left(\alpha+r-\frac{kr}{[r]},\frac{kr}{[r]}+1\right)+B\left(\alpha+\frac{kr}{[r]},r-\frac{kr}{[r]}+1\right)\right),
\end{aligned}$$

where $\alpha>0$ and $r\geq 1$.

Proof. Since $\mathcal{G}$ is non-negative and convex, from (3.1) we deduce

$$\begin{aligned}
\mathcal{G}^r(t) &\leq \left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^r \\
&= \left(\left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^{[r]}\right)^{\frac{r}{[r]}} \\
&= \left(\sum_{k=0}^{[r]} C_{[r]}^{k}\left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)\right)^{[r]-k}\left(\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^{k}\right)^{\frac{r}{[r]}} \\
&\leq 2^{\frac{r}{[r]}-1}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\frac{(\xi-t)^{r-\frac{kr}{[r]}}(t-\varsigma)^{\frac{kr}{[r]}}}{(\xi-\varsigma)^{r}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}, \qquad (3.5)
\end{aligned}$$

where we have used the following algebraic inequality $(a+b)^h \leq 2^{h-1}\left(a^h+b^h\right)$ for $a,b>0$ and $h>1$ and $(a+b)^h \leq \left(a^h+b^h\right)$ for $0<h<1$ with $[r]$ denoting the integer part of r.

Multiplying both sides of (3.5) by $\frac{1}{\Gamma(\alpha)}(\xi-t)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1}\mathcal{G}^r(t)\,\mathrm{d}t \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\left(2^{\frac{r}{[r]}-1}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\frac{(\xi-t)^{\alpha-1+r-\frac{kr}{[r]}}(t-\varsigma)^{\frac{kr}{[r]}}}{(\xi-\varsigma)^{r}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\right)\mathrm{d}t \\
&= \frac{2^{\frac{r}{[r]}-1}}{(\xi-\varsigma)^r\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1+r-\frac{kr}{[r]}}(t-\varsigma)^{\frac{kr}{[r]}}\,\mathrm{d}t \\
&= \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\int_{0}^{1}(1-u)^{\alpha-1+r-\frac{kr}{[r]}}u^{\frac{kr}{[r]}}\,\mathrm{d}u \\
&= \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}B\left(\alpha+r-\frac{kr}{[r]},\frac{kr}{[r]}+1\right). \qquad (3.6)
\end{aligned}$$

Similarly, multiplying both sides of (3.5) by $\frac{1}{\Gamma(\alpha)}(t-\varsigma)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we obtain

$$\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha-1}\mathcal{G}^r(t)\,\mathrm{d}t$$

$$
\begin{aligned}
&\leq \tfrac{1}{\Gamma(\alpha)}\int\limits_{\varsigma}^{\xi}\left(2^{\frac{r}{[r]}-1}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\tfrac{(\xi-t)^{r-\frac{kr}{[r]}}(t-\varsigma)^{\alpha-1+\frac{kr}{[r]}}}{(\xi-\varsigma)^{r}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\right)\mathrm{d}t\\
&=\tfrac{2^{\frac{r}{[r]}-1}}{(\xi-\varsigma)^{r}\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\int\limits_{\varsigma}^{\xi}(\xi-t)^{r-\frac{kr}{[r]}}(t-\varsigma)^{\alpha-1+\frac{kr}{[r]}}\mathrm{d}t\\
&=\tfrac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\int\limits_{0}^{1}(1-u)^{r-\frac{kr}{[r]}}u^{\alpha-1+\frac{kr}{[r]}}\mathrm{d}u\\
&=\tfrac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}B\left(\alpha+\tfrac{kr}{[r]},r-\tfrac{kr}{[r]}+1\right), \qquad (3.7)
\end{aligned}
$$

Now, summing (3.6) and (3.7), we get

$$
\begin{aligned}
&\tfrac{1}{\Gamma(\alpha)}\int\limits_{\varsigma}^{\xi}(\xi-t)^{\alpha-1}\mathcal{G}^{r}(t)\,\mathrm{d}t+\tfrac{1}{\Gamma(\alpha)}\int\limits_{\varsigma}^{\xi}(t-\varsigma)^{\alpha-1}\mathcal{G}^{r}(t)\,\mathrm{d}t\\
&\leq\tfrac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\\
&\quad\times\left(B\left(\alpha+r-\tfrac{kr}{[r]},\tfrac{kr}{[r]}+1\right)+B\left(\alpha+\tfrac{kr}{[r]},r-\tfrac{kr}{[r]}+1\right)\right). \qquad (3.8)
\end{aligned}
$$

Multiplying both sides of (3.8) by $\frac{1}{2(\xi-\varsigma)}$, we get the desired result. $\square$

Corollary 1. *For $r=1$, Theorem 4 gives*

$$\tfrac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}(\xi)+\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}(\varsigma)\right)\leq\tfrac{\mathcal{G}(\varsigma)+\mathcal{G}(\xi)}{2},$$

which is the fractional version of Hermite-Hadamard inequality obtained by Sarikaya et al. in [8].

Moreover, if we take $\alpha=1$, then we get the second inequality given in Theorem 1.

Corollary 2. *For $r=2$, Theorem 4 gives*

$$
\begin{aligned}
&\tfrac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}^{2}(\xi)+\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}^{2}(\varsigma)\right)\\
&\leq\tfrac{\alpha}{2}\left(\left(B(1,\alpha+2)+B(3,\alpha)\right)\mathcal{G}^{2}(\varsigma)+4B(2,\alpha+1)\,\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right)\\
&\quad+\left(B(3,\alpha)+B(1,\alpha+2)\right)\mathcal{G}^{2}(\xi)\big)\\
&=\tfrac{1}{2(\alpha+2)(\alpha+1)}\left[(2+\alpha(\alpha+1))\left(\mathcal{G}^{2}(\varsigma)+\mathcal{G}^{2}(\xi)\right)+4\alpha\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right],
\end{aligned}
$$

which is the same result obtained in Theorem 3.

4. CONCLUSION

In conclusion, this paper successfully extends the Sándor inequality to the context of Riemann-Liouville fractional integrals, offering new insights into the application of this important inequality in fractional calculus. By generalizing the inequality for any positive power r, we have expanded its utility in both pure and applied mathematics, particularly in areas such as optimization and inequality theory. The results presented not only build upon classical findings but also provide a broader framework for addressing complex problems in number theory, real and complex analysis, and mathematical physics. This work highlights the continued relevance and versatility of the Sándor inequality in modern mathematical research.

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