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DYNAMICAL BEHAVIOR OF A TWO DIMENSIONAL DIFFERENCE EQUATION SYSTEM WITH QUADRATIC TERM

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Abstract. This article investigates the dynamical behavior of a two-dimensional asymmetric system of fractional difference equations given by

$$\mu_{n+1} = 1 + \hbar\frac{\mu_n}{\nu_{n-1}^2}, \quad \nu_{n+1} = 1 + \hbar\frac{\nu_n}{\mu_{n-1}^2}, \quad n \in \mathbb{N},$$

where $\hbar > 0$ is a parameter and the initial values μ_i, ν_i $(i = -1, 0)$ are positive real numbers. By employing linear stability analysis and eigenvalue localization within the unit disk, we rigorously establish the existence and stability of equilibrium points. For $0 < \hbar \leq \frac{3}{4}$, the system exhibits a unique symmetric equilibrium (ξ, ξ) that is globally asymptotically stable. When $\frac{3}{4} < \hbar < 1$, two distinct asymmetric equilibria emerge, both of which retain local and global asymptotic stability. Furthermore, the boundedness and persistence of the solutions are demonstrated for all $0 < \hbar < 1$ using induction and comparison principles. The convergence rate of solutions toward equilibrium is quantified through error term linearization, revealing the dependence on the spectral radius of the system's Jacobian. Numerical simulations validate the theoretical findings, illustrating bifurcation phenomena and stability transitions as $\hbar$ crosses the critical thresholds. This work extends the existing models by incorporating asymmetric interactions, offering insights into the qualitative behavior of nonlinear discrete dynamical systems with delayed feedback. The results contribute to broader applications in population dynamics, epidemiology, and other fields governed by coupled difference equations.

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1. INTRODUCTION

Difference equations or systems of difference equations have been widely utilized across various disciplines, including physics, economics, ecology, and infectious disease dynamics. These include population growth, HIV/AIDS and tuberculosis transmission, and influenza prevention and control (refer to [1–5,21,22]). Thus, they play a crucial role in applied mathematics. In particular, numerous academics have conducted thorough investigations into the dynamics of difference equations owing to their theoretical and practical relevance. The examination of the dynamic properties of ordinary or partial difference equations and systems of difference equations has emerged as a prominent research focus over the last decade (refer to [6–17]). With the behavior qualitatively examined through various methods and numerically solved using different techniques, the solutions to these equations were determined. Consequently, the study of difference equations has developed rapidly. Subsequently, a detailed discussion of the research background of the equation under investigation in this study was presented.

Gumus [18] focused on the qualitative behavior of discrete difference system

$$x_{n+1}=\alpha+\frac{\sum_{i=1}^{m}x_{n-i}}{y_n},\ \ y_{n+1}=\beta+\frac{\sum_{i=1}^{m}y_{n-i}}{x_n},\ n\in\mathbb{N},$$

where $\alpha,\beta\in(0,\infty)$, m is a positive integer, both x_{-i} and y_{-i} are positive real numbers for $i\in\{0,1,2,\dots,m\}$.

Khan [19] researched the global dynamics of asymmetric difference, which include boundedness and persistence, existence and local dynamics of fixed points, and global dynamics of asymmetric difference systems and convergence rates.

$$x_{n+1}=A+B\frac{y_n}{y_{n-1}^2},\ \ y_{n+1}=C+D\frac{x_n}{x_{n-1}^2},\ n\in\mathbb{N},$$

where $A,B,C,D,$ are positive and $x_i,y_i,(i=-1,0)$ may be positive or negative.

Okumus and Soykan [20] explored asymptotic stability, periodicity, and boundedness of a nonlinear three dimensional system of difference equations

$$x_{n+1}=A+\frac{x_{n-1}}{z_n},y_{n+1}=A+\frac{y_{n-1}}{z_n},z_{n+1}=A+\frac{z_{n-1}}{y_n},\ n\in\mathbb{N},$$

where the parameter $A\in(0,\infty)$ and the initial values $x_i,y_i,z_i\in(0,\infty),(i=-1,0)$.

After examining several equations explored by other researchers, we will now focus on the equation we have studied, aiming to analyze the solutions of a fractional difference system represented by the following equations. At the same time, we will modify Khan's model [19] by incorporating asymmetric terms to obtain the following model:

$$\mu_{n+1}=1+\hbar\frac{\mu_n}{\nu_{n-1}^2},\quad \nu_{n+1}=1+\hbar\frac{\nu_n}{\mu_{n-1}^2},\quad n\in\mathbb{N},\tag{1.1}$$

where the parameter $\hbar>0$ and initial values μ_i, $\nu_i\in(0,\infty)$, $(i=-1,0)$.

2. PRELIMINARIES

In this section, we recall some definitions and theorems that will be used in system (1.1).

Consider the following discrete dynamical system:

$$\begin{aligned} \mu_n &= \rho(\mu_n, \mu_{n-1}, \nu_n, \nu_{n-1}), \quad n \in \mathbb{N}, \\ \nu_n &= \rho(\mu_n, \mu_{n-1}, \nu_n, \nu_{n-1}), \quad n \in \mathbb{N}. \end{aligned} \tag{2.1}$$

Here, ρ and ρ are continuously differentiable functions. The solution (μ_n, ν_n) of system (2.1) is determined by the specified initial values.

Definition 1.

(i) If there exist positive real numbers M and N such that $\operatorname{supp} x_n \in (0, M]$ and $\operatorname{supp} y_n \in (0, N]$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

(ii) If there exist positive real numbers m and n such that $\operatorname{supp} x_n \in [m, \infty)$ and $\operatorname{supp} y_n \in [n, \infty)$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ are persistent.

(iii) If there exist positive real numbers M, N, m, n such that $\operatorname{supp} x_n \in [m, M]$ and $\operatorname{supp} y_n \in [n, N]$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ are both bounded and persistent.

Definition 2 ([13])**.** Assume that ρ and ρ are continuously differentiable at the equilibrium point (ξ, ξ) and that this point is also an equilibrium point of the mapping $\mathcal{F}$, such that the linearized system about the equilibrium point (ξ, ξ) is given by

$$\Phi_{n+1} = \mathcal{F}(\Phi_n) = \Upsilon \Phi_n,$$

where

$$\Phi_n = \begin{pmatrix} \mu_n \\ \mu_{n-1} \\ \nu_n \\ \nu_{n-1} \end{pmatrix}$$

and the Jacobian matrix of system (2.1) about the equilibrium point (ξ, ξ) is denoted as Υ.

Theorem 1 ([13])**.** *Let*

$$\Phi_{n+1} = \mathcal{F}(\Phi_n), \; n \in \mathbb{N}$$

be a difference system. Assume (ξ, ξ) *is a fixed point of* $\mathcal{F}$*. An equilibrium point* (ξ, ξ) *is locally exponentially stable if and only if all eigenvalues of the Jacobian matrix of* $\mathcal{F}$ *evaluated at* (ξ, ξ) *reside within the open unit disk (i.e.,* $|\lambda| < 1$*. Conversely, if at least one eigenvalue exceeds 1 (i.e.,* $|\lambda| > 1$*), the system is unstable.*

This paper presents a solid theoretical foundation for comprehensively analyzing the boundedness, persistence, and stability of the positive solution of system (1.1). These results carry significant implications for comprehending the dynamic characteristics of system (1.1) and addressing relevant practical problems.

3. RESULTS

In this study, we have identified that the point $(\bar{\mu},\bar{\nu}) = (\xi,\xi) = \left(\frac{1+\sqrt{1+4\hbar}}{2}, \frac{1+\sqrt{1+4\hbar}}{2}\right)$ is a fixed point of system (1.1) for $0 < \hbar \le \frac{3}{4}$. For $(\frac{3}{4} < \hbar < 1)$, system (1.1) exhibits additional equilibrium points

$$\left(\frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right) = (\bar{\mu}_1,\bar{\nu}_1),$$
$$\left(\frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right) = (\bar{\nu}_1,\bar{\mu}_1).$$

(1) If $0 < \hbar < 1$, every positive solution of system (1.1) is persistence and boundedness.
(2) If $0 < \hbar \le \frac{3}{4}$, (ξ,ξ) is globally asymptotically stable.
(3) If $\frac{3}{4} < \hbar < 1$,$(\bar{\mu}_1,\bar{\nu}_1)$ and $(\bar{\nu}_1,\bar{\mu}_1)$ is globally asymptotically stable.

3.1. *Boundedness*

Theorem 2. *The following two statements are true:*

(1) *Both $\mu_n > 1$ and $\nu_n > 1$ for all $n \ge 1$, independently of the initial conditions.*
(2) *If $0 < \hbar < 1$, then for all values of $k \ge 3$, we obtain the following inequality:*

$$\begin{cases} \mu_k \le M_1 + \frac{1}{1-\hbar}, \\ \nu_k \le M_2 + \frac{1}{1-\hbar}, \end{cases} \quad k \ge 3, \tag{3.1}$$

where $M_1 = 1 + \frac{\hbar}{\nu_0^2} + \frac{\hbar^2\mu_0}{\nu_0^2\nu_{-1}^2}$, $M_2 = 1 + \frac{\hbar}{\mu_0^2} + \frac{\hbar^2\nu_0}{\mu_0^2\mu_{-1}^2}$.

Proof. Inference (i) is evidently true. We now assess the validity of inference (ii). Utilizing (1.1) and inference (i), we derive the following expressions for $k \ge 3$:

$$\begin{aligned} \mu_k &= 1 + \hbar\frac{\mu_{k-1}}{\nu_{k-2}^2} \le 1 + \hbar\mu_{k-1}, \\ \nu_k &= 1 + \hbar\frac{\nu_{k-1}}{\mu_{k-2}^2} \le 1 + \hbar\nu_{k-1}. \end{aligned} \tag{3.2}$$

Here, let $\{x_k\}$ and $\{w_k\}$ be the solutions of the following system.

$$\begin{cases} x_k = 1 + \hbar x_{k-1}, \\ w_k = 1 + \hbar w_{k-1}, \end{cases} \quad k \ge 3, \tag{3.3}$$

such that

$$\mu_2 = x_2, \quad \nu_2 = w_2. \tag{3.4}$$

We prove by induction that

$$\mu_k \le x_k, \quad \nu_k \le w_k, \quad k \ge 3. \tag{3.5}$$

Assume $\mu_m \leq x_m$ and $\nu_m \leq w_m$ for $k = m \geq 3$. Then from (3.2) we get

$$\begin{aligned} \mu_{m+1} &\leq 1 + \hbar\mu_m \leq 1 + \hbar x_m = x_{m+1}, \\ \nu_{m+1} &\leq 1 + \hbar\nu_m \leq 1 + \hbar w_m = w_{m+1}. \end{aligned} \tag{3.6}$$

Therefore, $\mu_k \leq x_k$ and $\nu_k \leq w_k$ for $k \geq 3$. (3.5) holds.

From (3.3), we have the following inequalities:

$$\begin{cases} x_k \leq \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}x_2 = \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}\mu_2, \\ w_k \leq \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}w_2 = \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}\nu_2. \end{cases} \tag{3.7}$$

From (1.1), we get

$$\mu_2 = 1 + \hbar\frac{\mu_1}{\nu_0^2}, \quad \mu_1 = 1 + \hbar\frac{\mu_0}{\nu_{-1}^2}, \quad \nu_2 = 1 + \hbar\frac{\nu_1}{\mu_0^2}, \quad \nu_1 = 1 + \hbar\frac{\nu_0}{\mu_{-1}^2},$$

where $\mu_{-1}, \mu_0, \nu_{-1}, \nu_0$ are arbitrary positive numbers.

Since $0 < \hbar < 1$, $k \geq 3$ we have $\hbar^{k-2} < 1$, that is

$$\begin{cases} \mu_k \leq M_1 + \frac{1}{1-\hbar}, \\ \nu_k \leq M_2 + \frac{1}{1-\hbar}, \end{cases} \quad k \geq 3, \tag{3.8}$$

where $M_1 = 1 + \frac{\hbar}{\nu_0^2} + \frac{\hbar^2\mu_0}{\nu_0^2\nu_{-1}^2}$, $M_2 = 1 + \frac{\hbar}{\mu_0^2} + \frac{\hbar^2\nu_0}{\mu_0^2\mu_{-1}^2}$. Afterward, by utilizing (3.2), (3.5), and (3.6), (3.1) can be proven.

According to Definition 1, system (1.1) is bounded and persistent. □

3.2. *Linear stability*

This section will explore the stability of system (1.1), first analyzing its local stability and then extending the discussion to global stability.

Theorem 3. *For $0 < \hbar \leq \frac{3}{4}$, the unique positive equilibrium point of system* (1.1) *is $(\xi, \xi) = \left(\frac{1+\sqrt{1+4\hbar}}{2}, \frac{1+\sqrt{1+4\hbar}}{2}\right)$ and exhibits local asymptotic stability. When $\frac{3}{4} < \hbar < 1$, system* (1.1) *possesses two distinct positive equilibrium points $(\bar{\mu}_1, \bar{\nu}_1) = \left(\frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right)$ and $(\bar{\nu}_1, \bar{\mu}_1) = \left(\frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right)$, both of which demonstrate local asymptotic stability.*

Proof of Theorem 3. (i) When $\bar{x} = \bar{y} = \xi$, the steady-state equation for system (3.1) is

$$\xi = 1 + \frac{\hbar}{\xi}.$$

This equation can be solved to yield $\xi = \frac{1\pm\sqrt{1+4\hbar}}{2}$, thus the unique positive equilibrium point of system (3.1) is

$$(\xi, \xi) = \left(\frac{1+\sqrt{1+4\hbar}}{2}, \frac{1+\sqrt{1+4\hbar}}{2}\right).$$

In accordance with Definition 2, the linear equation for system (1.1) about the equilibrium point (ξ,ξ) is

$$\Phi_{n+1} = \Upsilon\Phi_n, \tag{3.9}$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} \frac{\hbar}{\xi^2} & 0 & 0 & -\frac{2\hbar}{\xi^2} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar}{\xi^2} & \frac{\hbar}{\xi^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of (3.9) is

$$\lambda^4 - 2\frac{\hbar}{\xi^2}\lambda^3 + \left(\frac{\hbar}{\xi^2}\right)^2\lambda^2 - 4\left(\frac{\hbar}{\xi^2}\right)^2 = 0,$$

then

$$\lambda^2\left(\lambda - \frac{\hbar}{\xi^2}\right)^2 = 4\left(\frac{\hbar}{\xi^2}\right)^2. \tag{3.10}$$

(a) When $\lambda < \frac{\hbar}{\xi^2}$, (3.10) simplifies to:

$$\frac{\hbar}{\xi^2}\lambda - \lambda^2 = \frac{2\hbar}{\xi^2}.$$

From this, we can calculate:

$$\lambda_{1,2} = \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}}}{2}. \tag{3.11}$$

According to Theorem 1, for the equilibrium point (ξ,ξ) of system (1.1) to be locally asymptotically stable, we need $|\lambda| < 1$, leading to:

$$\left|\frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}}}{2}\right| < 1. \tag{3.12}$$

Since $\frac{\hbar}{\xi^2}, \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}} > 0$, it follows that:

$$|\frac{\hbar}{\xi^2} - \sqrt{(\frac{\hbar}{\xi^2})^2 - 8\frac{\hbar}{\xi^2}}| < |\frac{\hbar}{\xi^2} + \sqrt{(\frac{\hbar}{\xi^2})^2 - 8\frac{\hbar}{\xi^2}}| < 2. \tag{3.13}$$

If $\sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}} > 0$, then $\frac{\hbar}{\xi^2} \in (-\infty,0)\cup(8,+\infty)$. However, given that equation (3.13) holds, it follows that $\frac{\hbar}{\xi^2} < 2$, which precludes any solutions in this case.

(b) When $\lambda > \frac{\hbar}{\xi^2}$, equation (3.10) can be rewritten as

$$\lambda^2 - \frac{\hbar}{\xi^2}\lambda = \frac{2\hbar}{\xi^2}.$$

At this point, we can calculate the eigenvalues:

$$\lambda_{3,4} = \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}}}{2}. \tag{3.14}$$

We set $|\lambda_{3,4}| < 1$:

$$\left| \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}}}{2} \right| < 1. \tag{3.15}$$

Similarly, based on (3.15), we have

$$\left| \frac{\hbar}{\xi^2} - \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}} \right| < \left| \frac{\hbar}{\xi^2} + \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}} \right| < 2. \tag{3.16}$$

From (3.16), we deduce that when $\frac{\hbar}{\xi^2} < 2$:

$$\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2} < \left(2 - \frac{\hbar}{\xi^2}\right)^2. \tag{3.17}$$

Solving (3.17) gives us $\frac{\hbar}{\xi^2} < \frac{1}{3}$. Substituting $\xi = \frac{1 \pm \sqrt{1+4\hbar}}{2}$ yields $0 < \hbar < \frac{3}{4}$. When $\frac{\hbar}{\xi^2} > 2$, (3.16) does not hold. In summary, when $0 < \hbar < \frac{3}{4}$, the unique positive equilibrium point (ξ, ξ) of system (1.1) is locally asymptotically stable.
(ii) When the equilibrium point is $(\bar{\mu}, \bar{\nu})$, system (1.1) can be transformed into

$$\bar{\mu} = 1 + \hbar\bar{\mu}\left(\frac{1}{\hbar} - \frac{1}{\bar{\mu}}\right)^2. \tag{3.18}$$

Solving equation (3.18), we obtain

$$\bar{\mu} = \frac{\hbar \pm \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)}.$$

Similarly, we can find

$$\bar{\nu} = \frac{\hbar \pm \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)}.$$

Since μ_n, ν_n are non-negative real numbers, we have $\frac{3}{4} \leq \hbar < 1$. Notably, when $\hbar = \frac{3}{4}$, we find $\bar{\mu} = \bar{\nu} = 1.5 = \xi$. Because $\bar{\mu} \neq \bar{\nu}$, system (1.1) has two different positive equilibrium points:

$$(\bar{\mu}_1,\bar{\nu}_1)=\left(\frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)},\frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right),$$

$$(\bar{\nu}_1,\bar{\mu}_1)=\left(\frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)},\frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right).$$

In system (1.1), the Jacobian matrix at the equilibrium point $(\bar{\mu}_1,\bar{\nu}_1)$ is given by:

$$\Upsilon=\begin{pmatrix} \frac{\hbar}{\bar{\nu}_1^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}_1}{\bar{\nu}_1^3} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}_1}{\bar{\mu}_1^3} & \frac{\hbar}{\bar{\mu}_1^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let the eigenvalues of matrix Υ be $\lambda_1,\lambda_2,\lambda_3,\lambda_4$. Define Θ as a 4×4 diagonal matrix with elements d_1,d_2,d_3,d_4, where $d_1=d_3=1$ and $d_2=1-2\varepsilon$ and $d_4=1-4\varepsilon$. Furthermore, we have

$$0<\varepsilon<\min\left\{\frac{1}{4}\left(1-(\hbar+\sqrt{4\hbar-3})(1-\hbar)\right),\frac{1}{4}\left(1-\frac{(1-\hbar)^2}{\hbar}\right)\right\}.$$

It is evident that matrix Θ is invertible. To show this, we can compute $\Theta\Upsilon\Theta^{-1}$:

$$\Theta\Upsilon\Theta^{-1}=\begin{pmatrix} \frac{\hbar}{\bar{\nu}_1^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}_1}{\bar{\nu}_1^3}d_1d_4^{-1} \\ d_2d_1^{-1} & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}_1}{\bar{\mu}_1^3}d_3d_2^{-1} & \frac{\hbar}{\bar{\mu}_1^2} & 0 \\ 0 & 0 & d_4d_3^{-1} & 0 \end{pmatrix}.$$

Since $d_1>d_2>0$ and $d_3>d_4>0$, we have

$$d_2d_1^{-1}<1,\quad d_4d_3^{-1}<1.$$

Additionally, we have

$$\begin{aligned}&\frac{\hbar}{\bar{\nu}_1^2}+\frac{2\hbar\bar{\mu}_1}{\bar{\nu}_1^3}d_1d_4^{-1}=\frac{\hbar}{\bar{\nu}_1^2}(1+\frac{2\bar{\mu}_1}{\bar{\nu}_1}d_1d_4^{-1})=\frac{\hbar}{\bar{\nu}_1^2}(1+\frac{2\bar{\mu}_1}{\bar{\nu}_1}\frac{1}{1-4\varepsilon})\\ &<\frac{\hbar}{\bar{\nu}_1^2}(1+\frac{2\bar{\mu}_1}{\bar{\nu}_1})\frac{1}{1-4\varepsilon}=\frac{1}{1-4\varepsilon}\cdot\frac{2(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2}}<1,\end{aligned}\tag{3.19}$$

$$\begin{aligned}&\frac{\hbar}{\bar{\mu}_1^2}+\frac{2\hbar\bar{\nu}_1}{\bar{\mu}_1^3}d_3d_2^{-1}=\frac{\hbar}{\bar{\mu}_1^2}(1+\frac{2\bar{\nu}_1}{\bar{\mu}_1}d_3d_2^{-1})<\frac{\hbar}{\bar{\mu}_1^2}(1+\frac{2\bar{\nu}_1}{\bar{\mu}_1})d_3d_2^{-1}\\ &=\frac{1}{1-4\varepsilon}\cdot\frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}}\cdot\frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}}<1.\end{aligned}\tag{3.20}$$

From (3.19), we have

$$1-4\varepsilon > \frac{2(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2}}.$$

When $\frac{3}{4} \le \hbar < 1$, it follows that $2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2} < \hbar-\sqrt{4\hbar^3-3\hbar^2}$, leading to

$$\begin{aligned} 1-4\varepsilon &> \frac{(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2}} > \frac{(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{\hbar-\sqrt{4\hbar^3-3\hbar^2}} \\ &> \frac{(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{\hbar} > (\hbar+\sqrt{4\hbar-3})(1-\hbar). \end{aligned} \tag{3.21}$$

Thus, we establish that $\varepsilon < \frac{1}{4}(1-(\hbar+\sqrt{4\hbar-3})(1-\hbar))$.

From (3.20), we obtain

$$1-4\varepsilon > \frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}}. \tag{3.22}$$

If $\frac{3}{4} \le \hbar < 1$, then

$$2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2} < 3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}, \;\; 2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2} < 2\hbar,$$

$$\begin{aligned} 1-4\varepsilon &> \frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \\ &> \frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}} \\ &= \frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} > \frac{2(1-\hbar)^2}{2\hbar} = \frac{(1-\hbar)^2}{\hbar}. \end{aligned} \tag{3.23}$$

From (3.23), we conclude that $\varepsilon < \frac{1}{4}(1-\frac{(1-\hbar)^2}{\hbar})$.

Since Υ and $\Theta\Upsilon\Theta^{-1}$ have the same eigenvalues, we obtain

$$\begin{aligned} &\max_{1\le i\le 4} |\lambda_i| \le \|\Theta\Upsilon\Theta^{-1}\|_\infty \\ &= \max\left\{ d_2d_1^{-1}, d_4d_3^{-1}, \frac{\hbar}{\bar{\nu}_1^2}\left(1+\frac{2\bar{\mu}_1}{\bar{\nu}_1}d_1d_4^{-1}\right), \frac{\hbar}{\bar{\mu}_1^2}\left(1+\frac{2\bar{\nu}_1}{x_1}d_3d_2^{-1}\right)\right\} \\ &< 1. \end{aligned}$$

So the equilibrium point $(\bar{\mu}_1,\bar{\nu}_1)$ of system (1.1) is locally asymptotically stable for $\frac{3}{4} \le \hbar < 1$. The local asymptotic stability of system (1.1) at the equilibrium point $(\bar{\nu}_1,\bar{\mu}_1)$ is similar to that of $(\bar{\mu}_1,\bar{\nu}_1)$, so we omit that proof.

Thus, the proof of Theorem 3 is complete. □

Theorem 4. *The equilibrium point* $(\bar{\mu},\bar{\nu})$ *of system* (1.1) *is globally asymptotically stable if* $\hbar \in (0,1)$.

Proof of Theorem 4. If we take into account that (μ_n, ν_n) serves as a positive solution to system (1.1), we can deduce that

$$\lim_{n\to\infty} \mu_n = \bar{\mu}, \qquad \lim_{n\to\infty} \nu_n = \bar{\nu}. \tag{3.24}$$

By applying Theorem 2, we derive this

$$\begin{aligned} \Gamma_1 &= \limsup_{n\to\infty} \mu_n < \infty, & \Gamma_2 &= \limsup_{n\to\infty} \nu_n < \infty, \\ \gamma_1 &= \liminf_{n\to\infty} \mu_n \geq 1, & \gamma_2 &= \liminf_{n\to\infty} \nu_n \geq 1. \end{aligned} \tag{3.25}$$

Then from system (1.1) and (3.25), we get

$$\Gamma_1 \leq 1 + \hbar \frac{\Gamma_1}{\gamma_2^2}, \;\; \Gamma_2 \leq 1 + \hbar \frac{\Gamma_2}{\gamma_1^2}, \;\; \gamma_1 \geq 1 + \hbar \frac{\gamma_1}{\Gamma_2^2}, \;\; \gamma_2 \geq 1 + \hbar \frac{\gamma_2}{\Gamma_1^2}. \tag{3.26}$$

Now we set $\alpha = \frac{\gamma_2}{\Gamma_1}$, $\beta = \frac{\gamma_1}{\Gamma_2}$, then from (3.26) we have

$$\alpha \geq \frac{1 + \hbar \frac{\gamma_2}{\Gamma_1^2}}{1 + \hbar \frac{\Gamma_1}{\gamma_2^2}} = \frac{1 + \hbar \frac{\alpha}{\Gamma_1}}{1 + \hbar \frac{1}{\alpha \gamma_2}} \geq \frac{1}{1 + \frac{\hbar}{\alpha}}. \tag{3.27}$$

Then the following relationship holds:

$$\alpha + \hbar \geq 1. \tag{3.28}$$

If $0 < \hbar < 1$, then it follows that $\alpha \geq 1$, which implies $\gamma_2 \geq \Gamma_1$. Similarly, when $0 < \hbar < 1$, it follows that $\gamma_1 \geq \Gamma_2$. Therefore, when $0 < \hbar < 1$, we have the following relationship:

$$\gamma_2 \geq \Gamma_1 \geq \gamma_1 \geq \Gamma_2. \tag{3.29}$$

Based on (3.25) and (3.29), we obtain the following results:

$$\Gamma_1 = \gamma_1 = \Gamma_2 = \gamma_2. \tag{3.30}$$

Hence from system (1.1) and (3.30), there exist the $\lim \mu_n$, $\lim \nu_n$, as $n \to \infty$,

$$\lim_{n\to\infty} \mu_n = \bar{\mu}, \quad \lim_{n\to\infty} \nu_n = \bar{\nu}.$$

The proof is fully established when the point $(\bar{\mu}, \bar{\nu})$ is identified as a positive equilibrium within system (1.1) if $0 < \hbar < 1$. ☐

4. RATE OF CONVERGENCE

In this section,we will discuss the rate of convergence result of the solution that tends to $(\bar{\mu}, \bar{\nu})$. Consider the system

$$\text{Ł}_{n+1} = (\iota + \kappa(n))\text{Ł}_n. \tag{4.1}$$

Here, Ł_n represents an n-dimensional vector, ι denotes a constant matrix, and κ signifies a constant matrix transformation satisfying the condition:

$$\|\kappa(n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.2}$$

In this context, $\|\cdot\|$ represents any chosen matrix norm associated with a vector norm.

Theorem 5. *Assuming condition* (4.2) *holds,* $0 < \hbar < 1$ *and* $\{Ł_n\}$ *serves as a solution for equation* (4.1)*, then either* $Ł_n = 0$ *for all large n or* $\Lambda = \lim_{n\to\infty} \frac{||Ł_{n+1}||}{||Ł_n||}$ *or* $\Lambda = \lim_{n\to\infty}(||Ł||)^{\frac{1}{n}}$ *exists and* Λ *is equal to the modulus of an eigenvalue of the matrix* ι.

Theorem 6. *Assume that the solution* (μ_n, ν_n) *of system* (1.1) *converges to* $(\bar{\mu}, \bar{\nu})$. *Then, the error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} \mu_n - \bar{\mu} \\ \mu_{n-1} - \bar{\mu} \\ \nu_n - \bar{\nu} \\ \nu_{n-1} - \bar{\nu} \end{pmatrix}$$

for all solutions of system (1.1) *satisfies the following asymptotic relation:*

$$\lim_{n\to\infty} \frac{||Ł_{n+1}||}{||Ł_n||} = \lambda_{1,2,3,4}\Upsilon(\bar{\mu}, \bar{\nu}), \qquad \lim_{n\to\infty}(||Ł||)^{\frac{1}{n}} = \lambda_{1,2,3,4}\Upsilon(\bar{\mu}, \bar{\nu}),$$

where $\lambda_{1,2,3,4}\Upsilon(\bar{\mu}, \bar{\nu})$ *are the characteristic roots of* $\Upsilon(\bar{\mu}, \bar{\nu})$.

Proof of Theorem 6. The error terms can be expressed as follows:

$$\begin{aligned} \mu_{n+1} - \bar{\mu} &= 1 + \hbar\frac{\mu_n}{\nu_{n-1}^2} - (1 + \hbar\frac{\bar{\mu}}{\bar{\nu}^2}), \\ \nu_{n+1} - \bar{\nu} &= 1 + \hbar\frac{\nu_n}{\mu_{n-1}^2} - (1 + \hbar\frac{\bar{\nu}}{\bar{\mu}^2}). \end{aligned} \tag{4.3}$$

(4.3) can be transformed into

$$\begin{aligned} \mu_{n+1} - \bar{\mu} &= \frac{\hbar}{\nu_{n-1}^2}(\mu_n - \bar{\mu}) - \frac{\hbar\bar{\mu}(\nu_{n-1} + \bar{\nu})}{\bar{\nu}^2\nu_{n-1}^2}(\nu_{n-1} - \bar{\nu}), \\ \nu_{n+1} - \bar{\nu} &= \frac{\hbar}{\mu_{n-1}^2}(\nu_n - \bar{\nu}) - \frac{\hbar\bar{\nu}(\mu_{n-1} + \bar{\mu})}{\bar{\mu}^2\mu_{n-1}^2}(\mu_{n-1} - \bar{\mu}). \end{aligned} \tag{4.4}$$

Let $e_n^1 = \mu_n - \bar{\mu}$, $e_{n-1}^1 = \mu_{n-1} - \bar{\mu}$, $e_n^2 = \nu_n - \bar{\nu}$, $e_{n-1}^2 = \nu_{n-1} - \bar{\nu}$, system (4.4) can be expressed as

$$e_{n+1}^1 = a_n e_n^1 + b_n e_{n-1}^2, \quad e_{n+1}^2 = c_n e_n^2 + d_n e_{n-1}^1, \tag{4.5}$$

where

$$a_n = \frac{\hbar}{\nu_{n-1}^2}, \quad b_n = -\frac{\hbar\bar{\mu}(\nu_{n-1} + \bar{\nu})}{\bar{\nu}^2\nu_{n-1}^2}, \quad c_n = \frac{\hbar}{\mu_{n-1}^2}, \quad d_n = -\frac{\hbar\bar{\nu}(\mu_{n-1} + \bar{\mu})}{\bar{\mu}^2\mu_{n-1}^2}. \tag{4.6}$$

By taking the limits of (4.6) to obtain

$$\lim_{n\to\infty} a_n = \frac{\hbar}{\bar{\nu}^2}, \quad \lim_{n\to\infty} b_n = -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3}, \quad \lim_{n\to\infty} c_n = \frac{\hbar}{\bar{\mu}^2}, \quad \lim_{n\to\infty} d_n = -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3}. \tag{4.7}$$

That is

$$a_n = \frac{\hbar}{\bar{\nu}^2} + \kappa_a, \quad b_n = -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3} + \kappa_b, \quad c_n = \frac{\hbar}{\bar{\mu}^2} + \kappa_c, \quad d_n = -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3} + \kappa_d, \tag{4.8}$$

where $\kappa_a, \kappa_b, \kappa_c, \kappa_d \to 0$ as $n \to \infty$. Thus,we get the system of the form (4.1)

$$\text{Ł}_{n+1} = (\iota + \kappa(n))\text{Ł}_n, \tag{4.9}$$

where

$$\iota = \begin{pmatrix} \frac{\hbar}{\bar{\nu}^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3} & \frac{\hbar}{\bar{\mu}^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \kappa(n) = \begin{pmatrix} \kappa_a & 0 & 0 & \kappa_b \\ 1 & 0 & 0 & 0 \\ 0 & \kappa_d & \kappa_c & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.10}$$

and when $n \to \infty$, $||\kappa(n)|| \to 0$. Therefore, the limiting system of error terms can be formulated as follows:

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ e^2_{n+1} \\ e^2_n \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{\bar{\nu}^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3} & \frac{\hbar}{\bar{\mu}^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}. \tag{4.11}$$

$\square$

5. FIGURES

To validate the theoretical findings, we present numerical simulations of system (1.1) under distinct parameter regimes. The initial conditions are chosen as $\mu_{-1} = 0.5$, $\mu_0 = 1.0$, $\nu_{-1} = 0.6$, and $\nu_0 = 1.2$ for all cases unless specified otherwise.

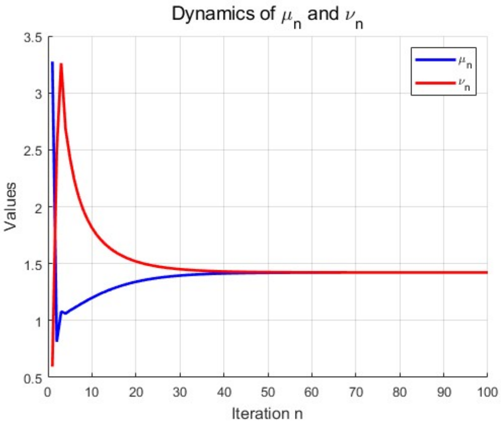

FIGURE 1. Behavior of system (1.1) at $\hbar = 0.6$

Case 1: $\hbar = 0.6$ $(0 < \hbar \leq 3/4)$. Figure 1 illustrates the trajectories of μ_n and ν_n. Both variables converge monotonically to the symmetric equilibrium $(\xi, \xi) = (1.366, 1.366)$, consistent with Theorem 3. This confirms the global asymptotic stability of the unique equilibrium in this parameter range.

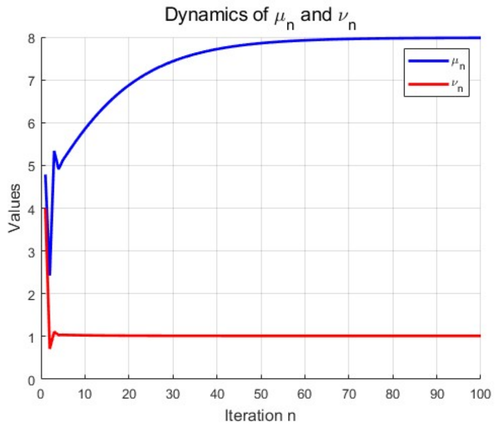

FIGURE 2. Behavior of system (1.1) at $\hbar = 0.9$

Case 2: $\hbar = 0.9$ $(3/4 < \hbar < 1)$. As shown in Figure 2, the system exhibits bistability. Depending on initial perturbations, solutions converge to either $(\bar{\mu}_1, \bar{\nu}_1) = (2.12, 0.48)$ or $(\bar{\nu}_1, \bar{\mu}_1) = (0.48, 2.12)$. This bifurcation aligns with the emergence of asymmetric equilibria predicted in Section 3.

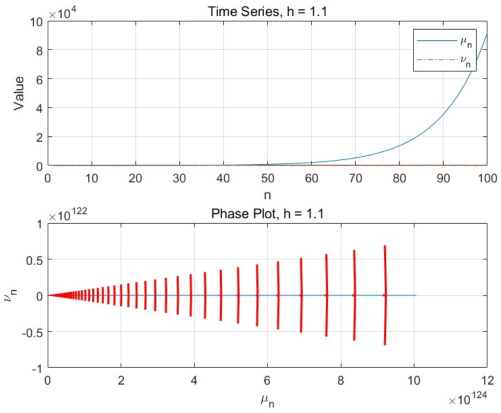

FIGURE 3. Behavior of system (1.1) at $\hbar = 1.1$

Case 3: $\hbar = 1.1$ ($\hbar > 1$). While our theoretical analysis focuses on $0 < \hbar < 1$, Figure 3 demonstrates oscillatory divergence when $\hbar$ exceeds 1. This highlights the critical role of $\hbar$ in maintaining system stability.

6. CONCLUSION

This paper investigates a two-dimensional asymmetric fractional difference equation system that reveals rich dynamic properties. A unique symmetric equilibrium (ξ, ξ) is globally asymptotically stable for $0 < \hbar \leq \frac{3}{4}$, whereas two asymmetric equilibria emerge, and are stable for $\frac{3}{4} < \hbar < 1$. Solution boundedness and persistence were proven using the induction and comparison principles. Numerical simulations confirmed monotonic convergence for $\hbar = 0.6$, bistability for $\hbar = 0.9$, and oscillatory divergence for $\hbar > 1$. This study highlights the threshold effect of $\hbar$ on stability, offering insights for discrete dynamical modeling in fields such as population dynamics and epidemiology. Future studies may explore higher-dimensional systems or specific applications.

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