



Miskolc Mathematical Notes
Vol. 13 (2012), No 2, pp. 543-555

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2012.513

Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations

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STABILITY OF JUNGCK-TYPE ITERATIVE PROCEDURE FOR SOME CONTRACTIVE TYPE MAPPINGS VIA IMPLICIT RELATIONS

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Received 19 March, 2012

Abstract. We obtain some stability results of the Jungck-type iterative procedure for common fixed points and weakly compatible mappings defined by a suitable implicit relation satisfying (E.A) property in metric spaces.

2000 Mathematics Subject Classification: 54H25; 47H10

Keywords: fixed point, stability, implicit relation, weakly compatible mappings, property (E.A)

1. INTRODUCTION

The stability of a fixed point iterative procedure was first studied by Ostrowski [35] in the case of Banach contraction mappings and this subject was later developed for certain contractive definitions by several authors (see Harder and Hicks [12], Rhoades [42], [44], Osilike [33] and [32], Berinde [6], [5], Jachymski [18], Olatinwo, Owojori and Imoru [28], [30] etc.). Osilike and Udomene [34] introduced a shorter method in order to prove stability results and this has also been applied by Imoru and Olatinwo [17], Imoru, Olatinwo and Owojori [16], [29] and some others. Moreover, Olatinwo [26] made generalizations and obtained first stability results using the concepts of pointwise convergence of sequences of operators and the fixed point iteration procedure was investigated for the case of two metrics.

Jungck [20] generalized the Banach's contraction principle, by replacing the identity map with a continuous map, thus obtaining a common fixed point theorem. Following the Jungck's contraction principle, many authors proved general common fixed points theorems and coincidence theorems (see Imdad and Ali [15], Aamri and Moutawakil [1]). Stability results of common fixed point iterative procedures and coincidence points were obtained by some authors. Singh, Bhatnagar and Mishra [46] established some stability results for Jungck and Jungck-Mann iteration procedures by employing two contractive definitions which generalized those of Osilike [32] but independent of that of Imoru and Olatinwo [17]. Moreover, Olatinwo [25], [27] obtained some stability results for nonself mappings in normed linear spaces which

are generalizations and extensions of [5], [17], [16]. Olatinwo and Postolache [31] also studied the stability in convex metric spaces for nonself mappings satisfying certain general contractive definitions in the case of Jungck-Mann and Jungck-Ishikawa iteration procedures. Singh and Prasad [47] studied the problem of stability for coincidence points on b-metric spaces. Timiș and Berinde [49] studied the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings and gave an illustrative example of weakly stable but not stable iterative fixed point procedure.

Using Popa's approach of implicit functions (see [40], [37], [38]) and the advantage of their versatility, several classical fixed point theorems and common fixed point theorems have been unified since they cover several contractive conditions rather than one contractive condition. Moreover, Berinde [9], [8] established stability results for fixed point iteration procedures associated to contractive mappings defined by an implicit relation.

Since a metrical common fixed point theorem generally involves conditions of commutativity, a lot of researches in this domain are aimed at weakening these conditions. The evolution of weak commutativity of Sessa [45] and compatibility of Jungck [22] developed weak conditions in order to improve common fixed points theorems.

This paper gives a general stability result for the common fixed point iteration procedure of Jungck-type in the class of weakly compatible mappings defined by means of an implicit contraction condition with five parameters.

2. PRELIMINARIES

All common fixed point theorems are based on a commuting property. We start by recalling the main important concepts of commutativity and weak commutativity.

Let (X, d) be a metric space and $S, T : X \rightarrow X$ be two mappings. We say that S and T are *commuting* if

$$STx = TSx, \quad \forall x \in X.$$

As a generalization of this notion, Sessa [45] defined S and T to be *weakly commuting* if

$$d(STx, TSx) \leq d(Sx, Tx), \quad \forall x \in X.$$

Jungck [22] defined S and T to be *compatible*, as a generalization of weakly commuting, if

$$\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t, \quad t \in X.$$

Jungck [22] also showed that commuting implies weakly commuting which, in turn, implies compatibility property but the converse property is not true in general, as show the following illustrative example.

Example 1 ([22]). Let the functions $f(x) = x^3$ and $g(x) = 2x^3$, with $X = \mathbb{R}$. They are compatible, since

$$|f(x) - g(x)| = |x^3| \rightarrow 0 \Leftrightarrow |fg(x) - gf(x)| = 6|x^9| \rightarrow 0,$$

but the pair (f, g) is not weakly commuting.

Moreover, Jungck [21] defined S and T to be *weakly compatible* if they commute at their coincidence points, i.e., if

$$Sz = Tz, z \in X, \text{ then } STz = TSz.$$

Jungck [22] established the inclusions between these notions, respectively that the *commuting property* implies *weakly commuting* which, in turn, implies *compatible* that implies *weakly compatible* but the reverse is not generally true.

Secondly, Aamri and Moutawakil [1] introduced a notion which is independent of the notion of weakly compatibility.

Definition 1 ([1]). S and T mappings satisfy (E.A) property if there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t, \text{ for some } t \in X.$$

The following example shows that a pair of mappings can satisfy the (E.A) property without being weakly compatible.

Example 2 ([3]). Let $(\mathbb{R}_+, |\cdot|)$ and define S and T by $Sx = x^2$ and $Tx = x + 2$. We have that $Sx = Tx \Leftrightarrow x = 2$. Let $\{x_n\}$ be a sequence in X , given by $x_n = 2 + \frac{1}{n}, n \geq 1$. Then, $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = 4$, so, S and T satisfy property (E.A).

As $ST(2) = S(4) = 16$, and $TS(2) = T(4) = 6$, (S, T) is not weakly compatible.

In general, a pair satisfying (E.A) property need not follow the pattern of containment of range of one map into the range of other as it is utilized in common fixed point considerations but still it relaxes such requirements.

Example 3 ([15]). Consider $X = [-1, 1]$ with the usual metric. Define $S, T : X \rightarrow X$, as follows:

$$T(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ \frac{x}{4}, & x \in (-1, 1), \\ \frac{3}{5}, & x = 1, \end{cases}$$

and

$$S(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ \frac{x}{2}, & x \in (-1, 1), \\ -\frac{1}{2}, & x = 1. \end{cases}$$

Let the sequence $\{x_n\}$ be given by $x_n = \frac{1}{n}$. Then,

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = 0,$$

so the pair (S, T) satisfies (E.A) property.

The mappings T and S are also weakly compatible because $T(0) = S(0) = 0$ and $ST(0) = TS(0) = 0$.

On the other hand, $T(X) = \{\frac{1}{2}, \frac{3}{5}\} \cup (\frac{-1}{4}, \frac{1}{4})$ and $S(X) = [\frac{-1}{2}, \frac{1}{2}]$. Hence, neither $T(X)$ is contained in $S(X)$ nor $S(X)$ is contained in $T(X)$.

3. STABILITY OF COMMON FIXED POINT ITERATIVE PROCEDURES

For the following stability study, we shall use weakly compatible mappings satisfying the above (E.A) property.

Let the mappings $S, T : X \rightarrow X$ and $T(X) \subseteq S(X)$. For any $x_0 \in X$, consider $Sx_{n+1} = Tx_n$, $n = 0, 1, \dots$ which is the iterative procedure introduced by Jungck [20] which becomes the Picard iterative procedure when $S = id$, the identity map on X .

Jungck showed in [20] that the mappings S and T satisfying

$$d(Tx, Ty) \leq kd(Sx, Sy), \quad 0 \leq k < 1, \quad \forall x, y \in X, \quad (3.1)$$

have a common fixed point in X , provided that S and T are commuting, $T(X) \subseteq S(X)$ and S is continuous.

The following significant improved version of this result is generally called the Jungck common fixed point principle.

Theorem 1 ([47]). *Let $S, T : X \rightarrow X$ satisfying (3.1). If $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X , then S and T have a coincidence. Indeed, for any $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that*

- $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, \dots$
- $\{Sx_n\}$ converges to Sz for some z in X and $Sz = Tz = t$, that is, S and T have a coincidence at z .

Further, if S and T commute just at z , then they have an unique common fixed point.

Definition 2 ([47]). Let (X, d) be a metric space and $S, T : X \rightarrow X$. Let z to be a coincidence point of T and S , that is, $Sz = Tz = t$. For any $x_0 \in X$, the sequence

$\{Sx_n\}$ generated by the general iterative procedure

$$Sx_{n+1} = Tx_n, \quad n = 1, 2, \dots, \tag{3.2}$$

converges to $t \in X$. Let $\{Sy_n\} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0, 1, 2, \dots$$

Then the iterative procedure 3.2 is said to be (S, T) -stable or stable with respect to (S, T) if and only if

$$\lim_{n \rightarrow \infty} \epsilon_n = 0 \quad \text{implies that} \quad \lim_{n \rightarrow \infty} Sy_n = t.$$

This definition reduces to that of the stability of a fixed point iterative procedure, due to Harder and Hicks [13], [12], when $S = id$.

For several examples discussing the practical aspect and theoretical importance of the stability when S is the identity mapping on X in the above definition, see Berinde [6].

In order to prove our main stability result in this paper, we shall need the next lemma.

Lemma 1 ([6]). *Let $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$ be sequences of nonnegative numbers and a constant $h, 0 \leq h < 1$, so that*

$$a_{n+1} \leq ha_n + b_n, \quad n \geq 0.$$

- *If $\lim_{n \rightarrow \infty} b_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.*
- *If $\sum_{n=0}^\infty b_n < \infty$, then $\sum_{n=0}^\infty a_n < \infty$.*

4. IMPLICIT CONTRACTIVE CONDITIONS

From the class of implicit functions due to Popa [40], [37], [38], let \mathbb{F} to be the set of all continuous functions $F : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (1) F is continuous in each coordinate variable;
- (2) If for some $u, v, w \geq 0$, we have
 - (a) $F(u, v, u, v, w) \leq 0$ or
 - (b) $F(u, v, v, u, w) \leq 0$,
 then there exists $h \in [0, 1)$, such that $u \leq h \max\{v, w\}$;
- (3) $F(u, u, u, u, 0) > 0$, for all $u > 0$.

In the sequel, we present some examples of contractive type mappings defined by an implicit function depending of five parameters.

Example 4 ([40]). The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - at_2,$$

where $a \in [0, 1)$, satisfies (1), (2a), (2b) and (3), with $h = a$.

Example 5. The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by one of the following:

- (1) $F(t_1, \dots, t_5) = t_1 - at_2$,
- (2) $F(t_1, \dots, t_5) = t_1 - bt_5$,
- (3) $F(t_1, \dots, t_5) = t_1 - c(t_3 + t_4)$,

where $a, b \in [0, 1)$, $c \in [0, \frac{1}{2})$, satisfies (1), (2a), (2b) and (3), with $h = a$, b , respectively $\frac{b}{1-b} < 1$.

Example 6. The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - kt_5,$$

where $k \in (0, 1)$, satisfies (1), (2a), (2b) and (3), with $h = k$.

Example 7. The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - at_2 - bt_5,$$

where $a, b \in (0, 1)$, with $a + 2b < 1$, satisfies (1), (2a), (2b) and (3), with $h = a$, if $\max\{v, w\} = v$ and $h = b$, if $\max\{v, w\} = w$.

Example 8 ([40]). The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - a(t_3 + t_4),$$

where $a \in (0, \frac{1}{2})$, satisfies (1), (2a), (2b) and (3), with $h = \frac{a}{1-a} \in (0, 1)$.

Example 9. The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - h \max\{t_3, t_4\},$$

where $h \in [0, 1)$, satisfies (1), (2a), (2b) and (3).

Example 10. The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - at_2 - bt_3 - ct_4,$$

where $a, b, c \in [0, 1)$, with $a + b + c < 1$, satisfies (1), (2a) with $h = \frac{a+c}{1-b} \in [0, 1)$, (2b) with $h = \frac{a+b}{1-c} \in [0, 1)$, and (3).

Example 11. The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5,$$

where $a, b, c, d \in [0, 1)$, with $a + b + c + 2d < 1$, satisfies (1), (2a) with $h = \frac{a+c}{1-b} \in [0, 1)$, (2b) with $h = \frac{a+b}{1-c} \in [0, 1)$, and (3), where $h = \frac{a+c}{1-b} \in [0, 1)$, if $\max\{v, w\} = v$ and $h = \frac{a+b}{1-c} \in [0, 1)$, if $\max\{v, w\} = w$.

Example 12 ([36]). The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - a \max \left\{ t_2, \frac{t_3 + t_4}{2}, t_5 \right\},$$

where $a \in [0, 1)$, satisfies (1), (2a), (2b) and (3), respectively when $\max = t_2$ or $\max = t_5$, then $h = a$, when $\max = \frac{t_3 + t_4}{2}$, then $h = \frac{a}{1 - \frac{a}{2}}$.

Example 13 ([40]). The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1 - c \max \{t_2, t_3, t_4, t_5\},$$

where $h = c \in [0, 1)$, satisfies (1), (3), when $\max = t_2, \max = t_4$ or $\max = t_5$ is satisfied (2a) and when $\max = t_3$ is satisfied (2b).

Example 14 ([40]). The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ given by

$$F(t_1, \dots, t_5) = t_1^2 - c \max \{t_2 t_3, t_2 t_4, t_3 t_4, t_5^2\},$$

where $c \in [0, 1)$, satisfies (1), (2a) and (3), with $h = c$.

5. MAIN RESULTS

Using the common fixed point theorem of Imdad and Ali [15], we give the following general stability result for the common fixed point iteration procedure of Jungck-type using weakly compatible mappings defined by an implicit contraction condition satisfying (E.A) property.

Theorem 2. *Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X .*

There exists $F \in \mathbb{F}$ such that

$$F \left(d(Tx, Ty), d(Sx, Sy), d(Sx, Ty), d(Sy, Tx), \frac{d(Sx, Tx) + d(Sy, Ty)}{2} \right) \leq 0, \tag{5.1}$$

for all $x, y \in X$. Then

- (1) *if F satisfies (2b), then the pair (T, S) has a point of coincidence;*
- (2) *if F satisfies (3), the pair (T, S) has a common fixed point provided it is weakly compatible;*
- (3) *if F satisfies (3), then the associated iterative procedure is (S, T) -stable.*

Proof. Since T and S satisfy (E.A) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t, \quad t \in X.$$

Since $S(X)$ is a complete subspace of X , every convergent sequence of points of $S(X)$ has a limit in $S(X)$. Therefore,

$$\lim_{n \rightarrow \infty} Sx_n = t = Sz = \lim_{n \rightarrow \infty} Tx_n = t, \quad z \in X$$

which in turn yields that $t = Sz \in S(X)$.

Assert that $Sz = Tz$. If not, then $d(Tz, Sz) > 0$ and using (5.1), we have

$$F\left(d(Tz, Tx_n), d(Sz, Sx_n), d(Sz, Tx_n), d(Sx_n, Tz), \frac{d(Sz, Tz) + d(Sx_n, Tx_n)}{2}\right) \leq 0$$

which by letting $n \rightarrow \infty$ reduces to

$$F\left(d(Tz, t), d(Sz, t), d(Sz, t), d(t, Tz), \frac{d(Sz, Tz) + d(t, t)}{2}\right) \leq 0$$

or

$$F\left(d(Tz, Sz), 0, 0, d(Sz, Tz), \frac{d(Sz, Tz) + 0}{2}\right) \leq 0$$

and according to (2b), there exists $h \in [0, 1)$ such that

$$d(Tz, Sz) \leq h \max\left\{0, \frac{d(Sz, Tz)}{2}\right\} = h \frac{d(Sz, Tz)}{2} < d(Sz, Tz),$$

a contradiction.

Hence $Tz = Sz$, so z is a coincidence point of T and S .

Since S and T are weakly compatible, then

$$St = STz = TSz = Tt.$$

Now, assert that $Tt = t$. If not, then $d(Tt, t) > 0$. Again, using (5.1),

$$F\left(d(Tt, Tz), d(Sz, St), d(St, Tz), d(Sz, Tt), \frac{d(St, Tt) + d(Sz, Tz)}{2}\right) \leq 0$$

or

$$F(d(Tt, t), d(Tt, t), d(Tt, t), d(Tt, t), 0) \leq 0$$

which contradicts property (3). Hence, $Tt = t$ which shows that t is a common fixed point of T and S and the uniqueness of the common fixed point is an easy consequence of implicit relation (5.1).

In order to prove the (S, T) -stability, we take the sequence $\{Sx_n\}$ generated by the general iterative procedure $Sx_{n+1} = Tx_n$, $n = 1, 2, \dots$, for any $x_0 \in X$, which converges to t , the common fixed point of the iterative procedure.

Let $\{Sy_n\} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0, 1, 2, \dots$$

By definition, the iterative procedure is (S, T) -stable if and only if

$$\lim_{n \rightarrow \infty} \epsilon_n = 0 \quad \text{implies that} \quad \lim_{n \rightarrow \infty} Sy_n = t.$$

Assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Then

$$d(Sy_{n+1}, t) \leq d(Sy_{n+1}, Ty_n) + d(Ty_n, t) = \epsilon_n + d(Ty_n, t). \tag{5.2}$$

If we take $x := t$ and $y := y_n$ in (5.1), then we obtain $F(u, v, u, v, w) \leq 0$, where $u := d(Ty_n, t)$, $v := d(Sy_n, t)$, $w := \frac{1}{2}d(Sy_n, Ty_n)$. Now, since F satisfies (2a), there exists $h \in [0, 1)$ such that $u \leq h \max\{v, w\}$, respectively $d(Ty_n, t) \leq h \max\left\{d(Sy_n, t), \frac{d(Sy_n, Ty_n)}{2}\right\}$. We discuss two cases.

In the first case, when $\max\left\{d(Sy_n, t), \frac{d(Sy_n, Ty_n)}{2}\right\} = d(Sy_n, t)$, it yields that $d(Ty_n, t) \leq h d(Sy_n, t)$, and then

$$d(Sy_{n+1}, t) \leq h d(Sy_n, t) + \epsilon_n$$

and applying Lemma 1 we get the conclusion.

For the second case, if $\max\left\{d(Sy_n, t), \frac{d(Sy_n, Ty_n)}{2}\right\} = \frac{d(Sy_n, Ty_n)}{2}$, we have

$$d(Ty_n, t) \leq \frac{h}{2}d(Ty_n, Sy_n) \leq \frac{h}{2}d(Ty_n, t) + \frac{h}{2}d(t, Sy_n).$$

Then,

$$\left(1 - \frac{h}{2}\right)d(Ty_n, t) \leq \frac{h}{2}d(t, Sy_n), \text{ so,}$$

$$d(Ty_n, t) \leq \frac{\frac{h}{2}}{1 - \frac{h}{2}}d(Sy_n, p).$$

We denote $q := \frac{\frac{h}{2}}{1 - \frac{h}{2}} \in [0, 1)$, because $h \in [0, 1)$ and then we get

$$d(Ty_n, t) \leq qd(Sy_n, t), \text{ so,}$$

$$d(Sy_{n+1}, t) \leq qd(Sy_n, t) + \epsilon_n.$$

Consequently, the conclusion follow by applying Lemma 1. □

Remark 1. Theorem 2 completes Theorem 3.1 in Imdad and Ali [15] with the information about the stability of the Jungck-type iterative procedure with respect to the mappings S and T , provided that the function F satisfies a certain condition.

Corollary 1. *Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X .*

Suppose there exists $F \in \mathbb{F}$ such that F satisfies (5.1), for all $x, y \in X$.

Then, the Jungck-type iterative procedure is (S, T) -stable.

Proof. We apply Theorem 2, with F given by Example 4 and then we obtain a stability result for Jungck’s contraction principle, see [20]. □

Corollary 2. *Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X .*

Suppose there exists $F \in \mathbb{F}$ such that F satisfies (5.1), for all $x, y \in X$.

Then, in the case of Zamfirescu's contraction conditions, the associated common fixed point iterative procedure is (S, T) -stable.

Proof. We apply Theorem 2, with F given by Example 5 and then we obtain a stability result for the Zamfirescu's fixed point theorem, see [50], corresponding to a pair of mappings with a common fixed point. \square

Remark 2. Particular cases of Theorem 2.

- (1) If F is given by Example 6, then we obtain a stability result for the Kannan's fixed point theorem, see [23], corresponding to a pair of mappings with a common fixed point;
- (2) If F is given by Example 7, then we obtain a stability result for a fixed point theorem obtained by Reich (1971) and Rus (1971), see [48], corresponding to a pair of mappings with a common fixed point;
- (3) If F is given by Example 8, then we obtain a stability result for the Chatterjea's fixed point theorem, see [11], corresponding to a pair of mappings with a common fixed point;
- (4) If F is given by Example 11, then we obtain a stability result for the Hardy and Rogers's fixed point theorem, see [14], corresponding to a pair of mappings with a common fixed point;
- (5) If F is given by Example 12, then we obtain a stability result for the Pathak and Verma's fixed point theorem, see [36], corresponding to a pair of mappings with a common fixed point in symmetric spaces.
- (6) If F is given by Examples 13 and 14, then we obtain stability results for the Popa's fixed point theorem, see [40], corresponding to two pairs of mappings on two metric spaces.

6. CONCLUDING REMARKS

The results obtained in this paper generalize classical fixed point theorems existing in the literature: Jungck [20], Zamfirescu [50], Kannan [23], Reich and Rus [48], Chatterjea [11], Hardy and Rogers [14] and most of their references.

The contractive conditions obtained from (5.1) with F as in Examples 1-11, imply contractive conditions used by Rhoades in [41], [42], [44], [43].

Because of the inclusions between the commutativity definitions, the weakly compatible pair of mappings is the most general type from the mentioned notions and it includes the others. The above theorem use this kind of weakly compatible mappings and it follows that it holds also for compatible, commuting and weakly commuting pair of mappings.

Olaleru [24] showed that the Jungck iteration procedure can be used to approximate the common fixed points of some weakly compatible pair of mappings defined on metric spaces. These are generalizations and extensions of the results of Berinde [7].

Recently, many authors used implicit relations in order to prove common fixed points theorems for weakly compatible mappings satisfying different contractive conditions and various properties.

Bouhadjera and Djoudi [10] proved a common fixed point theorem for four weakly compatible mappings satisfying an implicit relation without need of continuity. This theorem generalizes some results on compatible continuous mappings of Popa [38].

Aliouche [3] proved common fixed point theorems for weakly compatible mappings in metric spaces satisfying an implicit relation using (E.A) property and a common (E.A) property, which generalizes the results of Aamri and Moutawakil [1].

Aliouche [4] also proved common fixed point theorems for weakly compatible mappings satisfying implicit relations without the condition that the map to be decreasing in any variable. These theorems improve results of Ali and Imdad [2], Jeong and Rhoades [19] and Popa [39].

It is possible to obtain corresponding stability results for the above mentioned common fixed point theorems in the presence of implicit contractive conditions, a task which will be completed in forthcoming papers.

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