Miskolc Mathematical Notes
Vol. 27 (2026), No. 1, pp. 77–87

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2026.5123

DEMI VERSION OF QUASI LEVI AND LEVI OPERATORS ON BANACH LATTICES

ÖYKÜ BAL, NAZIFE ERKURŞUN-ÖZCAN, AND AHSEN SENA YURTOĞLU

Received 17 September, 2024

Abstract. Demi versions of different classes of operators have been recently investigated by many researchers. In this paper we study the demi version of quasi Levi and Levi operators. We investigate their properties. Moreover, their relations with other classes of operators are also discussed.

2010 *Mathematics Subject Classification:* 46B42; 46A40; 47B60; 47L05

Keywords: Banach lattice, demi Levi, demi quasi Levi, demi KB, demi quasi KB

1. INTRODUCTION

Many papers have been devoted to studying the properties of different classes of operators in Banach lattice theory. Some of them introduced some operators involving demi criteria, for instance in [10, 11, 15, 18], the notion of demi KB operators using the definition of KB operators from [9] is defined. They also showed that demi KB and b-weakly demicompact operators are the same.

In the present paper we consider the classes of (quasi)-KB and (quasi)-Levi operators as they were introduced in [4, 16]. For alternative versions of the definitions we refer to [6, 9, 11, 20]. The main motivation of this paper is to explore the demi version of operators using the study from [12–14]. We then expand our approach by defining the concepts of demi quasi Levi operators and demi Levi operators. The goal of this paper is to examine these newly defined operators.

The paper is organized as follows. In Section 2, we will address demi quasi Levi operators and study their basic properties depending on order convergence. We characterize Banach lattices on which all operators are demi quasi Levi operators. Section 3 is devoted to demi Levi operators. In the last section, the demi version of KB and quasi KB operators are studied depending on the definition given by [13].

Throughout this paper, E denotes real Banach lattices, B_E is the closed unit ball of E, I is the identity operator on E. The set of all bounded linear operators on E

---

is denoted by $L(E)$. The positive cone of E is denoted by $E_+ := \{x \in E \ \ 0 \leq x\}$. A Banach lattice E is called a KB space if every increasing, norm bounded sequence in E_+ converges in norm. A Banach lattice is called an abstract L_1-space (AL-space), whenever its norm is additive in the sense that $\|x+y\| = \|x\| + \|y\|$ holds for all $x, y \in E_+$ with $x \wedge y = 0$. A normed Riesz space is called a Levi lattice if each increasing norm bounded sequence in E_+ has a supremum in E. For more information, see [1,2,17].

According to [11], an operator $T : E \to E$ is a demi KB operator if for every positive increasing sequence $x_n \in B_E$ such that $(x_n - Tx_n)$ is convergent to some $x \in E$, there is a convergent subsequence of (x_n). Moreover, $T : E \to E$ is called a weak demi KB operator if for every positive increasing sequence $x_n \in B_E$ such that $(x_n - Tx_n)$ is weakly convergent to $x \in E$, there is a weakly convergent subsequence of (x_n). Recall that a subset A of a Banach lattice E is said b-order bounded if there exists some $0 \leq x^{''} \in E^{''}$ such that $|x| \leq x^{''}$ for all $x \in A$. An operator T from a Banach lattice E to a Banach space X is said to be b-weakly compact if the image of every b-order bounded subset of E under T is relatively weakly compact. An operator $T : E \to E$ is said to be b-weakly demicompact if for every b-order bounded sequence in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$, we have $\|x_n\| \to 0$. The definiton of demicompact operators was firstly given by [18]. $T : D(T) \subseteq E \to E$, where $D(T)$ is a subspace of E, is said to be demicompact if, for every bounded sequence (x_n) in the domain $D(T)$ such that $(x_n - Tx_n)$ converges to $x \in E$, there is a convergent subsequence of (x_n). For the other necessary definitions, see [3,5–8].

We provide four fundamental definitions given in [4,13,14], which form the basis of this paper.

Let E be a normed Riesz space. An operator $T : E \to E$ is called a quasi Levi (σ-quasi Levi) operator if T takes increasing norm bounded net (sequence) in E_+ to an order Cauchy net (sequence). The collection of the quasi Levi (σ-quasi Levi) operators is denoted by $L_{ql}(E)$ ($L^{\sigma}_{ql}(E)$). An operator $T : E \to E$ is called Levi (σ-Levi) operator if for every norm bounded increasing net (sequence) in E_+ such that $Tx_\alpha \xrightarrow{o} Tx$, $x \in E$. The collection of the Levi (σ-Levi) operators is denoted by $L_l(E)$ ($L^{\sigma}_l(E)$).

Let T be an operator from a Banach lattice E to Banach space Y. T is a quasi KB (σ-quasi KB) operator if (Tx_α) converges in the norm for every increasing norm bounded net (sequence) in E_+. The collection of quasi KB (σ-quasi KB) operators is denoted by $L_{qKB}(E,Y)$ ($L^{\sigma}_{qKB}(E,Y)$). T is a KB (σ-KB) operator if for every increasing norm bounded net (sequence) in E_+, $\|Tx_\alpha - Tx\| \to 0$ for some $x \in E$. The collection of the KB (σ-KB) operators is denoted by $L_{KB}(E,Y)$ ($L^{\sigma}_{KB}(E,Y)$). In [4], Proposition 1.2 shows that $L^{\sigma}_{qKB}(E,Y) = L_{qKB}(E,Y)$. In the following we study the demi version of such operators.

2. DEMI QUASI LEVI OPERATORS

In this section, demi quasi Levi operators are introduced and their properties are studied. Since the ideas of the general proofs in most cases are similar to the σ-case, we only restrict to sequences.

Proposition 1. *Let E be a normed Riesz space. $L_{ql}(E)$ is a vector space. Moreover every quasi Levi operator $T: E \to E$ is σ-order bounded.*

Proof. The proof is directly from [20, Theorem 2.2]. ☐

Definition 1. Let E be a normed Riesz space. An operator $T: E \to E$ is called a demi quasi Levi (σ-demi quasi Levi) operator if for every increasing and norm bounded net (sequence) in E_+ such that $(x_\alpha - Tx_\alpha)$ is order Cauchy, there exists a subnet (subsequence) of (x_α) that is order convergent. The collection of the demi quasi Levi (σ-demi quasi Levi) operators is denoted by $L_{Dql}(E)$ $(L^{\sigma}_{Dql}(E))$.

Lemma 1. *Let E be a Dedekind complete normed Riesz space. Then λI is a demi quasi Levi operator for every $\lambda \neq 1$.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and $(x_n - \lambda I(x_n))$ be order Cauchy. We have

$$\big(x_n - \lambda I(x_n) - (x_m - \lambda I(x_m))\big) \xrightarrow{o} 0.$$

Then there exists $y_k \downarrow 0$ such that $|x_n - \lambda I(x_n) - x_m + \lambda I(x_m)| \leq y_k$. Therefore we get

$$|x_n - x_m| \leq \frac{y_k}{|1-\lambda|},\ \lambda \neq 1,$$

and so (x_n) is an order Cauchy sequence. Hence (x_n) is order convergent. ☐

Proposition 2. *Let E be a Dedekind complete normed Riesz space. Every quasi Levi operator $T: E \to E$ is a demi quasi Levi operator.*

Proof. Let T be a quasi Levi operator and $0 \leq x_n \uparrow$, $x_n \in B_E$ such that $(x_n - Tx_n)$ is an order Cauchy sequence. Therefore $(I - T)$ is a quasi Levi operator. Since $L_{ql}(E)$ is a vector space, we have $I = (I - T) + T$ that the identity operator is a quasi Levi operator. It follows that (x_n) is order convergent. ☐

Demi quasi Levi operator need not to be quasi Levi as the following example shows.

Example 1. Let $E = c_{00}$ and consider the operator $2I: E \to E$. $2I$ is not a quasi Levi operator but from Lemma 1, it is a demi quasi Levi operator.

Proposition 3. *Let E be a normed Riesz space and $T, S: E \to E$ be two operators. If T is a demi quasi Levi operator and S is a quasi Levi operator, then $T + S$ is a demi quasi Levi operator.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ be such that $(x_n - (T+S)x_n)$ is an order Cauchy sequence. Since S is a quasi Levi operator, (Sx_n) is an order Cauchy sequence. We can write

$$\begin{aligned}(x_n - Tx_n) - (x_m - Tx_m) &= (x_n - Tx_n \pm Sx_n) - (x_m - Tx_m \pm Sx_m)\\ &= (x_n - (T+S)x_n) - (x_m - (T+S)x_m) + (Sx_n - Sx_m).\end{aligned}$$

Since $(x_n - (T+S)x_n)$ and (Sx_n) are order Cauchy sequences, then $(x_n - Tx_n)$ is order Cauchy. Hence there exists an order convergent subsequence (x_{n_k}) as T is demi quasi Levi. Therefore $T+S$ is a demi quasi Levi operator. ☐

Theorem 1. *Let E be a Banach lattice with σ-order continuous norm. Then the following statements are equivalent.*

- (i) *E is a KB space.*
- (ii) *Every σ-order continuous operator $T: E \to E$ is a σ-quasi Levi operator.*
- (iii) *Every σ-order continuous operator $T: E \to E$ is a σ-demi quasi Levi operator.*
- (iv) *The identity operator of E is a σ-demi quasi Levi operator.*

Proof. $(i) \Rightarrow (ii)$ Let E be a KB space and $0 \leq x_n \uparrow$, $x_n \in B_E$. Since E is a KB space, (x_n) is norm convergent. There exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{o} x$. If $x_{n_k} \uparrow$ and $x_{n_k} \xrightarrow{o} x$, then $x_n \xrightarrow{o} x$. Therefore $Tx_n \xrightarrow{o} Tx$ as T is a σ-order continuous operator.
$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ From Proposition 2.
$(iv) \Rightarrow (i)$ Let I be a σ-demi quasi Levi operator and $0 \leq x_n \uparrow$, $x_n \in B_E$. $(x_n - I(x_n))$ is an order Cauchy sequence. Since I is a σ-demi quasi Levi operator then there exists order convergent subsequence (x_{n_k}). Since E has a σ-order continuous norm, (x_{n_k}) is norm convergent and (x_n) is increasing, then (x_n) is a norm convergent sequence which means that E is a KB space. ☐

In Theorem 1, it is important that the space E must be a KB space. For example, let E not be a KB space. The operators $T = 2I$ and $S = -I$ are demi quasi Levi operators by Lemma 1. However the operator $T + S = I$ is not a demi quasi Levi operator by Theorem 1. Additionally, the collection of operators $L_{Dql}(E)$ is not closed under multiplication and does not form a vector space structure. For instance, let $E = c_{00}$ and consider the operator $(-I) \in L_{Dql}(E)$. Since $E = c_{00}$ is not a KB space, by Theorem 1, the operator $(-1)(-I) = I$ is not a demi quasi Levi operator.

Moreover, on AL space, more generally on KB space, every operator $T: E \to E$ is demi quasi Levi.

The following theorem demonstrates that the domination property holds for demi quasi Levi operators. In the following proofs σ version is only considered. However net version is the same.

Theorem 2. *Let E be a normed Riesz space and T be a positive demi quasi Levi operator. Then every operator S satisfying $0 \leq S \leq T \leq I$ is a demi quasi Levi operator.*

Proof. Let (x_n) be a sequence such that $(x_n - Sx_n)$ is a Cauchy sequence, where $0 \leq x_n \uparrow$ and $x_n \in B_E$. Since S is a central operator, we can write $|x_n - Sx_n| = |(I-S)x_n| = |I-S||x_n|$. Hence we have,

$$|(I-S)(x_n - x_m)| = |I-S||x_n - x_m| = (I-S)|x_n - x_m| \xrightarrow{o} 0.$$

On the other hand from the inequality $0 \leq S \leq T \leq I$, we get $(I-T)|x_n| \leq (I-S)|x_n|$ and $(I-T)|x_n - x_m| \xrightarrow{o} 0$. By the assumption that $T\colon E \to E$ is a demi quasi Levi operator, there exists a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{o} y$. Consequently $S\colon E \to E$ is also a demi quasi Levi operator. □

The collection of $L_{Dql}(E)$ is closed under order convergence.

Proposition 4. *Let E be a Banach lattice and $T_\gamma \in L_{Dql}(E)$. If $T_\gamma \xrightarrow{o} T$, then $T \in L_{Dql}(E)$.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ be such that $(x_n - Tx_n)$ is an order Cauchy sequence. From assumption when $T_\gamma \xrightarrow{o} T$, we have $T_\gamma - T \xrightarrow{o} 0$. This implies that the sequence $(T_\gamma - T)$ is order Cauchy. For each (x_n) in B_E, the sequence $(T_\gamma x_n - Tx_n)$ is also order Cauchy. We can write $x_n - T_\gamma x_n = x_n - Tx_n + Tx_n - T_\gamma x_n$. Since both $(x_n - Tx_n)$ and $(T_\gamma x_n - Tx_n)$ are order Cauchy, it follows that $(x_n - T_\gamma x_n)$ is an order Cauchy sequence. Given that $T_\gamma \in L_{Dql}(E)$, there exists a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{o} y$. Therefore $T\colon E \to E$ is a demi quasi Levi operator. □

The other important question is whether demi quasi Levi operators satisfy the modul property. Even E is a Dedekind complete Banach lattice and T is order continuous, $|T|$ is not necessarily to be demi quasi Levi. For instance $-I\colon c_0 \to c_0$ is demi quasi Levi but $|-I| = I$ is not demi quasi Levi by Theorem 1.

3. DEMI LEVI OPERATORS

In this section the new operator class called demi Levi operators are introduced. Since the ideas of the general proofs in most cases are similar to the σ-case, we only restrict to sequences.

Every Levi operator is quasi Levi. For the converse, necessary conditions are given in [14, Theorem 3.3] as the following.

Theorem 3 ([14])**.** *Let E be a Banach lattice with order continuous norm. The followings are equivalent.*

- (i) *E is a KB space.*
- (ii) $L_+(E) = L_{l+}(E)$.
- (iii) $L_{ql+}(E) = L_{l+}(E)$.

Definition 2. Let E be a normed Riesz space. An operator $T\colon E \to E$ is called a demi Levi operator if for every increasing and norm bounded net (sequence) in E_+ such that $x_\alpha - Tx_\alpha \xrightarrow{o} Tx$ and $x \in E$, there exists an order convergent subnet (subsequence) of (x_α). The collection of the demi Levi operators is denoted by $L_{Dl}(E)$ ($L^\sigma_{Dl}(E)$).

Lemma 2. *Let E be a normed Riesz space, λI is a demi Levi operator for every $\lambda \neq 1$.*

Proof. Let $0 \le x_n \uparrow, x_n \in B_E$ and $x_n - \lambda I(x_n) \xrightarrow{o} \lambda I(x), x \in E$. Since $(1-\lambda)x_n \xrightarrow{o} \lambda x$, $x \in E$, so there exists order convergent subsequence x_{n_k} of x_n. ☐

In case $\lambda = 1$, the identify operator is not necessarily demi Levi. Consider the operator $I\colon c_0 \to c_0$ and the sequence

$$x_n = (x_m)_n = \begin{cases} 1, & m \le n, \\ 0, & m > n. \end{cases}$$

It is clear that $0 \le x_n \uparrow$ ve $\|x\|_\infty = 1$. Hence $(x_n - I(x_n))$ is an order convergent but (x_n) is not.

The natural question is the following. Is $L_{Dl}(E)$ a vector space? The answer is negative. Consider, $T = -I$ and $S = 2I$ on c_0. From Lemma 2, T and S are demi Levi operators. But $T + S$ is not a demi Levi operator. Therefore $L_{Dl}(E)$ is not closed with respect to addition. Also, consider $(-I) \in L_{Dl}(E)$. $(-1)(-I) = I \notin L_{Dl}(E)$. Hence $L_{Dl}(E)$ is not a vector space.

Proposition 5. *Let E be a normed Riesz space. Every Levi operators $T\colon E \to E$ is a demi Levi operator.*

Proof. Let $0 \le x_n \uparrow$, $x_n \in B_E$ and consider $x_n - Tx_n \xrightarrow{o} Tx_1,\ x_1 \in E$. Since T is a Levi operator, there exists $x_2 \in E$ such that $Tx_n \xrightarrow{o} Tx_2$. We can write

$$x_{n_k} = Tx_{n_k} - Tx_{n_k} + x_{n_k} \xrightarrow{o} Tx_1 + Tx_2,$$

and so operator T is a demi Levi operator. ☐

Lemma 3. *Let E be a normed Riesz space. If $T\colon E \to E$ is a demi quasi Levi operator, then T is a demi Levi operator.*

Proof. Let $0 \le x_n \uparrow$, $x_n \in B_E$ such that $x_n - Tx_n \xrightarrow{o} Tx$, $x \in E$. Since an order convergent sequence is order Cauchy, $(x_n - Tx_n)$ is order Cauchy. So there exists order convergent subsequence (x_{n_k}) of (x_n). ☐

[14, Example 3.4] is the example for the strict inclusion of $L_l(E) \subset L_{Dl}(E)$. This example is quasi Levi but it is not Levi. So it is demi quasi Levi by Proposition 2. Therefore by Lemma 3, it is demi Levi.

Before the following theorem, the definition of a Levi lattice should be given. A normed Riesz space E is called a Levi lattice if every increasing norm bounded positive sequence (x_n) in E has a supremum. Every Levi lattice is Dedekind complete.

Theorem 4. *Let E be a normed Riesz space. The followings are equivalent:*

- (i) *E is a Levi lattice.*
- (ii) *Every order continuous operator $T\colon E \to E$ is a Levi operator.*
- (iii) *Every order continuous operator $T\colon E \to E$ is a demi Levi operator.*
- (iv) *The identity operator of E is a demi Levi operator.*

Proof. $(i) \Rightarrow (ii)$ Let $0 \leq x_n \uparrow$, $x_n \in B_E$. Since E is Levi lattice, $\sup x_n = x$. (x_n) is order convergent to $x \in E$ as $x_n \uparrow$ and $\sup x_n = x$. It is clear that $Tx_n \xrightarrow{o} Tx$, $x \in E$ as T is order continuous. Hence T is Levi.
$(ii) \Rightarrow (iii)$ From Proposition 5.
$(iii) \Rightarrow (iv)$ Obvious.
$(iv) \Rightarrow (i)$ Let $0 \leq x_n \uparrow$, $x_n \in B_E$. We have to show that (x_n) has a supremum in E. It is obvious that $(x_n - I(x_n))$ order convergent. Since I is a demi Levi operator, there exists an order convergent subsequence of (x_n) such that $x_{n_k} \xrightarrow{o} y$, $y \in E$. $x_n \uparrow$ and $x_{n_k} \xrightarrow{o} y$ implies $x_n \xrightarrow{o} y$, hence $\sup x_n = y$, $y \in E$ is satisfied and so E is a Levi lattice. ☐

Since every KB space is a Levi lattice with an order continuous complete norm, it follows from Theorem 4 that every order continuous operator is a demi quasi Levi operator if and only if it is demi Levi.

It is an important topic to examine whether the same properties hold for the modul of operators. The modulus of a demi Levi operator is not necessarily a demi Levi operator. For example, the operator $-I\colon c_0 \to c_0$ is a demi Levi operator, but the operator $|-I| = I$ is not a demi Levi operator, as c_0 is not a Levi lattice by Theorem 4.

4. DEMI KB AND DEMI QUASI KB OPERATORS

In this section, demi KB and demi quasi KB operators are introduced and their relations are discussed. We restrict ourselves to the sequence case.

Definition 3. Let T be an operator from a Banach lattice E to E. We say that:

- (i) T is a demi quasi KB (σ-demi quasi KB) operator if for every increasing norm bounded net (sequence) in E_+ such that $(x_\alpha - Tx_\alpha)$ is norm Cauchy net (sequence), there is a norm convergent subnet (subsequence) of (x_α). The collection of demi quasi KB (σ-demi quasi KB) operators is denoted by $L_{qKB}(E)$ ($L^\sigma_{qKB}(E)$).
- (ii) T is a demi KB (σ-demi KB)operator if for every increasing norm bounded net (sequence) in E_+ such that $x_\alpha - Tx_\alpha \xrightarrow{\|\cdot\|} Tx$ and $x \in E$, there exists a

norm convergent subnet (subsequence) of (x_α). The collection of the demi KB (σ-demi KB)operators is denoted by $L_{DKB}(E)$ $(L^{\sigma}_{DKB}(E))$.

Lemma 4. *Let E be a Banach lattice and $T: E \to E$ be an operator. Every KB operator is demi KB, and every quasi KB operator is demi quasi KB.*

Lemma 5. *Let E be a Banach lattice. If $T: E \to E$ is a demi quasi KB operator, then T is a demi KB operator.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and for $x \in E$, $x_n - Tx_n \xrightarrow{\|\cdot\|} Tx$. Since the norm convergent sequence is norm Cauchy, $(x_n - Tx_n)$ is norm Cauchy. So there exists norm convergent subsequence (x_{n_k}) of (x_n). ☐

Example 1 in [13] is the example of an operator that is quasi KB but not KB. By Lemma 4, it is demi quasi KB. This operator is also demi KB by Lemma 5. Therefore it is considered for demi KB operator not necessarly to be KB.

Lemma 6. *Let E be a Banach lattice. λI is a demi KB operator for every $\lambda \neq 1$. Hence $\lambda I \in L_{DqKB}(E)$ for every $\lambda \neq 1$.*

Proof. Let $0 \leq x_n \uparrow, x_n \in B_E$ and $x_n - \lambda I(x_n) \xrightarrow{\|\cdot\|} I(x)$. Since $(1-\lambda)x_n \xrightarrow{\|\cdot\|} x$, there exists a norm convergent subsequence of (x_n). ☐

The following corollary gives an important fact on Banach lattices that b-weakly demicompact operators, demi KB operators and weak demi KB operators of [11] and the demi quasi KB operators coincide. For the proof, see [19].

Corollary 1. *Let E be a Banach lattice and $T: E \to E$ be an operator. The following statements are equivalent.*

- (i) *For every b-order bounded sequence (x_n) in E_+ such that $x_n \to 0$ in $\sigma(E,E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$.*
- (ii) *For every positive, increasing sequence (x_n) in B_E such that $(x_n - Tx_n)$ is convergent to some $x \in E$, there is a norm convergent subsequence of (x_n).*
- (iii) *For every positive, increasing sequence (x_n) in B_E such that $(x_n - Tx_n)$ is a norm Cauchy sequence, there is a norm convergent subsequence of (x_n).*
- (iv) *For every positive, increasing sequence (x_n) in B_E such that $(x_n - Tx_n)$ is weakly convergent to some $x \in E$, there is a weakly convergent subsequence of (x_n).*

Therefore, if a b-weakly demi-compact operator fulfills certain properties, then any demi quasi-KB operator inherently satisfy those same properties. Consequently, we can restate the following theorem.

Theorem 5. *Let E be a Banach lattice. Then, the following statements are equivalent.*

- (i) *E is a KB space.*

(ii) *Every operator* $T: E \to E$ *is a quasi KB operator.*
(iii) *The identity operator of* E *is a demi quasi KB operator.*

Proof. $(i) \Rightarrow (ii)$ from Proposition 2.13 [9]. By Lemma 4, (ii) implies (iii). (iii) implies (i) is obtained directly by KB space definition. ☐

The following two propositions examine the relationships between different operators.

Proposition 6. *Let* E *be a Dedekind complete Banach lattice.*
(i) *Every quasi KB operator* $T: E \to E$ *is demi quasi Levi.*
(ii) *Every quasi KB operator* $T: E \to E$ *is demi Levi.*

Proof. (i) Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and $(x_n - Tx_n)$ be order Cauchy sequence. Since E is Dedekind complete, then $x_n - Tx_n \xrightarrow{o} x,\ x \in E$. Since $T \in L_{qKB}(E)$, (Tx_n) is norm Cauchy. Hence, $Tx_n \xrightarrow{\|\cdot\|} y,\ y \in E$. Then there exists subsequence (x_{n_k}) of (x_n) such that $Tx_{n_k} \xrightarrow{o} y$. We can write

$$x_{n_k} = Tx_{n_k} - Tx_{n_k} + x_{n_k} \xrightarrow{o} x + y,\ x + y \in E,$$

and so T is demi quasi Levi.
(ii) Since every demi quasi Levi operator is demi Levi by Lemma 3, the proof is completed. ☐

Proposition 7. *Let* E *be a Banach lattice with order continuous norm. Then the followings are valid.*
(i) *Every demi quasi KB operator* $T: E \to E$ *is demi quasi Levi.*
(ii) $T: E \to E$ *is a demi KB operator iff* $T: E \to E$ *is demi Levi.*
(iii) *Every quasi Levi operator* $T: E \to E$ *is quasi KB.*
(iv) *Every Levi operator* $T: E \to E$ *is quasi KB.*

Proof. (i) Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and $(x_n - Tx_n)$ be order Cauchy. We aim to show that $x_{n_k} \xrightarrow{o} y,\ y \in E$. Since E has an order continuous norm, $(x_n - Tx_n)$ is norm Cauchy. Since T is a demi quasi KB operator, there exists subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is norm convergent. Then there exists an order convergent subsequence $(x_{n_{k_l}})$. As $(x_{n_{k_l}})$ is increasing and order convergent, we can write $x_{n_k} \xrightarrow{o} y,\ y \in E$. Therefore, T is demi quasi Levi.
(ii) $(\Rightarrow)$ Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and $x_n - Tx_n \xrightarrow{o} Tx,\ x \in E$. Since E has an order continuous norm, $x_n - Tx_n \xrightarrow{\|\cdot\|} Tx,\ x \in E$. From assumption T is a demi KB operator, we obtain $x_{n_k} \xrightarrow{\|\cdot\|} y$, so $x_n \xrightarrow{\|\cdot\|} y$. Therefore $x_{n_k} \xrightarrow{o} y$. It implies that T is a demi Levi operator.
(ii) $(\Leftarrow)$ Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and $x_n - Tx_n \xrightarrow{\|\cdot\|} Tx,\ x \in E$. We know that, $x_{n_k} - Tx_{n_k} \xrightarrow{o} Tx$. Since T is a demi Levi operator, a subsequence $(x_{n_{k_t}})$ exists and

it converges to $y \in E$ in order. As (x_{n_k}) is increasing and $x_{n_{k_t}} \xrightarrow{o} y$, then we obtaing $x_{n_k} \xrightarrow{o} y$. On the other hand, since E has an order continuous norm, then $x_{n_k} \xrightarrow{\|\cdot\|} y$. Therefore, T is a demi KB operator.
(iii) Let $0 \leq x_n \uparrow$, $x_n \in B_E$. From the assumption (Tx_n) is order Cauchy. Since E is Dedekind complete, $Tx_n \xrightarrow{o} y$ and E has order continuous norm, we obtain $Tx_n \xrightarrow{\|\cdot\|} y$. So (Tx_n) is norm Cauchy. Therefore, T is a quasi KB operator.
(iv) Since every Levi operator is quasi Levi, the proof is completed. □

Items (iii) and (iv) of the above proposition are also mentioned in [4, 16].

REFERENCES

[1] C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, ser. Pure and Applied Mathematics. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, vol. Vol. 76, doi: 10.1016/S0079-8169(08)61391-4.

[2] C. D. Aliprantis and O. Burkinshaw, *Positive operators*. Springer, Dordrecht, 2006, reprint of the 1985 original, doi: 10.1007/978-1-4020-5008-4.

[3] S. Alpay, B. Altin, and C. Tonyali, "On property (b) of vector lattices," *Positivity*, vol. 7, no. 1-2, pp. 135–139, 2003, positivity and its applications (Nijmegen, 2001), doi: 10.1023/A:1025840528211.

[4] S. Alpay, E. Emelyanov, and S. Gorokhova, "$o\tau$-continuous, Lebesgue, KB, and Levi operators between vector lattices and topological vector spaces," *Results Math.*, vol. 77, no. 3, pp. Paper No. 117, 25, 2022, doi: 10.1007/s00025-022-01650-3.

[5] B. Altin, "On b-weakly compact operators on Banach lattices," *Taiwanese J. Math.*, vol. 11, no. 1, pp. 143–150, 2007, doi: 10.11650/twjm/1500404641.

[6] B. Altin and N. Machrafi, "Some characterizations of KB-operators on Banach lattices and ordered Banach spaces," *Turkish J. Math.*, vol. 44, no. 5, pp. 1736–1743, 2020, doi: 10.3906/mat-2004-106.

[7] B. Aqzzouz and A. Elbour, "Some properties of the class of b-weakly compact operators," *Complex Anal. Oper. Theory*, vol. 6, no. 6, pp. 1139–1145, 2012, doi: 10.1007/s11785-010-0108-z.

[8] B. Aqzzouz, A. Elbour, M. Moussa, and J. Hmichane, "Some characterizations of b-weakly compact operators," *Math. Rep. (Bucur.)*, vol. 12(62), no. 4, pp. 315–324, 2010.

[9] A. Bahramnezhad and K. Haghnejad Azar, "*KB*-operators on Banach lattices and their relationships with Dunford-Pettis and order weakly compact operators," *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, vol. 80, no. 2, pp. 91–98, 2018.

[10] H. Benkhaled, M. Hajji, and A. Jeribi, "On the class of demi Dunford-Pettis operators," *Rend. Circ. Mat. Palermo (2)*, vol. 72, no. 2, pp. 901–911, 2023, doi: 10.1007/s12215-021-00702-x.

[11] H. Benkhaled and A. Jeribi, "The class of demi KB-operators on Banach lattices," *Turkish J. Math.*, vol. 47, no. 1, pp. 387–396, 2023, doi: 10.55730/1300-0098.3366.

[12] E. Emelyanov, "On collectively σ-Levi sets of operators," *Vladikavkaz Mathematical Journal*, vol. 27, no. 1, pp. 15–25, 2025, doi: 10.48550/arXiv.2408.03686.

[13] E. Emelyanov, "On compact KB-operators in Banach lattices," *Positivity*, vol. 29, no. 1, p. 15, 2025, doi: 10.1007/s11117-024-01109-5.

[14] E. Emelyanov, "On Levi operators between normed and vector lattices," *Sib. Math. J.*, vol. 66, no. ., pp. 1270–1275, 2025, doi: 10.1134/S0037446625050167.

[15] N. Erkurşun Özcan and E. H. Eryüksel, "Investigation of Demi-ab continuous operators," *J. Math. Sci. (N.Y.)*, vol. 270, no. 1, pp. 59–66, 2024, doi: 10.1007/s10958-024-07091-3.

[16] S. Gorokhova and E. Emelyanov, "On operators dominated by the Kantorovich-Banach and Levi operators in locally solid lattices," *Sib. Math. J.*, vol. 64, no. 3, pp. 720–724, 2023, doi: 10.1134/S0037446623030199.

[17] P. Meyer-Nieberg, *Banach Lattices*, ser. Universitext. Springer-Verlag, Berlin, 1991. doi: 10.1007/978-3-642-76724-1.

[18] W. V. Petryshyn, "Construction of fixed points of demicompact mappings in Hilbert space," *J. Math. Anal. Appl.*, vol. 14, pp. 276–284, 1966, doi: 10.1016/0022-247X(66)90027-8.

[19] B. Turan and B. Altin, "The relation between b-weakly compact operator and KB-operator," *Turkish J. Math.*, vol. 43, no. 6, pp. 2818–2820, 2019, doi: 10.3906/mat-1908-11.

[20] F. Zhang and Z. Chen, "Some results of σ-Levi operators in Banach lattices," *Positivity*, vol. 26, no. 3, p. 49, 2022, doi: 10.1007/s11117-022-00903-3.

Authors' addresses

Öykü Bal
Marmara University, Department of Mathematics, 34722 İstanbul, Türkiye
E-mail address: `oyku.bal@marmara.edu.tr`

Nazife Erkurşun-Özcan
Hacettepe University, Department of Mathematics, 06800 Ankara, Türkiye
E-mail address: `erkursun.ozcan@hacettepe.edu.tr`

Ahsen Sena Yurtoğlu
(**Corresponding author**) Bursa Technical University, Department of Mathematics, 16310 Bursa, Türkiye
E-mail address: `ahsen.yurtoglu@btu.edu.tr`