



## INTEGRAL CHARACTERIZATIONS OF UNIFORM H-DICHOTOMY IN MEAN FOR DISCRETE-TIME STOCHASTIC SKEW-EVOLUTION SEMIFLOWS

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*Abstract.* The paper provides integral characterizations for the concept of uniform dichotomy in the mean with growth rates for discrete-time stochastic skew-evolution semiflows in Banach spaces. More precisely, necessary and sufficient conditions are given using both invariant projection families and strongly invariant projection families to the discrete-time stochastic skew-evolution semiflows. As a consequence, we obtain integral characterizations for uniform exponential dichotomy in mean.

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### 1. INTRODUCTION

Occurrences in the real world, within domains such as biology, economics, and environmental sciences, happen at specific points in time rather than continuously. Consequently, the discrete-time approach has become essential. By utilizing stochastic skew-evolution semiflows, we aim to develop a framework that enhances the analysis of discrete dynamical systems, providing a deeper understanding of their behavior and properties.

The concept of dichotomy is a key focus in the study of asymptotic behavior for evolution equations. O. Perron [21] introduced the concept of exponential dichotomy for linear differential equations. This concept was further explored in the monograph by J. L. Daleckiĭ and M. G. Krein [10], as well as in a more recent work by Dragičević, Sasu, and Sasu [12]. In their study, the authors introduce new admissibility conditions for uniform exponential dichotomy and provide novel characterizations of polynomial dichotomy through double admissibilities.

An alternative perspective on studying dichotomic behavior focuses on cases where the asymptotic behaviours are of polynomial type. In this context, we address the

concepts of nonuniform polynomial dichotomy, initially introduced by L. Barreira and C. Valls in [3] for the continuous case of evolution operators, and subsequently extended by A.J.G. Bento and C. Silva in [4] for discrete-time systems. Additional results related to polynomial behavior are discussed in [5],[14],[15].

Several significant papers have addressed the problem of the existence of stochastic semiflows for stochastic evolution equations. Notable examples of stochastic evolution semiflows arise from these equations, and readers can refer to the monographs by Arnold [1] and Prato and Zabczyk [9] for more information. The exponential dichotomy in a stochastic setting was discussed by many authors, such as A. M. Atewi in [2], T. Caraballo et al. in [8] or D. Stoica and M. Megan in [23].

The concept of skew-evolution semiflow was introduced by Megan and Stoica for the continuous case in [18] and for the discrete case in [17]. This research was further developed by M. Megan and C. Stoica in [19], as well as by C. Stoica in [22].

This investigation aims to outline several characterisations of uniform  $h$ -dichotomy in mean of discrete-time stochastic skew-evolution semiflows in Banach spaces, where  $h : \mathbb{N} \rightarrow [1, \infty)$  acts as a growth rate function. Specifically,  $h$  is a non-decreasing and bijective function with the property that  $\lim_{m \rightarrow \infty} h(m) = \infty$ . For recent contributions, we refer to the works [6, 7], [13], [16], [20], and [24].

This paper builds on the foundational work of Datko [11] who provided an integral characterization of uniform exponential stability for evolution operators. Expanding on Datko's results, and the aforementioned studies, we present integral characterizations for the concept of uniform dichotomy in mean with growth rates for discrete-time stochastic skew-evolution semiflows considering invariant projections families and, respectively, strongly invariant projection families.

## 2. DEFINITIONS AND NOTATIONS

Let  $(\Omega, \mathcal{B}, \mu)$  be a probability space. Let  $\tilde{\Delta}$  be the set defined by  $\tilde{\Delta} = \{(m, n) \in \mathbb{N}^2 : m \geq n \geq 0\}$  and let  $\tilde{T}$  be the set defined by  $\tilde{T} = \{(m, n, p) \in \mathbb{N}^3 : m \geq n \geq p\}$ . For a real or complex Banach space  $X$  we denote by  $\mathcal{B}(X)$  the Banach algebra of all bounded linear operators on  $X$ .

**Definition 1.** A measurable random field  $\varphi : \tilde{\Delta} \times \Omega \rightarrow \Omega$  is said to be a *discrete-time stochastic evolution semiflow* on  $\Omega$  if the following properties hold:

- (es<sub>1</sub>)  $\varphi(m, m, \omega) = \omega$ , for all  $(m, \omega) \in \mathbb{N} \times \Omega$ ,
- (es<sub>2</sub>)  $\varphi(m, n, \varphi(n, p, \omega)) = \varphi(m, p, \omega)$ , for all  $m \geq n \geq p \geq 0$  and all  $\omega \in \Omega$ .

**Definition 2.** Let  $\Phi : \tilde{\Delta} \times \Omega \rightarrow \mathcal{B}(X)$  be a measurable map. We say that  $\Phi$  is a *discrete-time stochastic evolution cocycle* associated to the stochastic evolution semiflow  $\varphi : \tilde{\Delta} \times \Omega \rightarrow \Omega$  if the following conditions hold:

- (ec<sub>1</sub>)  $\Phi(m, m, \omega) = I$  (the identity operator on  $X$ ), for all  $(m, \omega) \in \mathbb{N} \times \Omega$ ,
- (ec<sub>2</sub>)  $\Phi(m, n, \varphi(n, p, \omega))\Phi(n, p, \omega) = \Phi(m, p, \omega)$ , for all  $m \geq n \geq p \geq 0$  and all  $\omega \in \Omega$ .

If  $\tilde{\Phi}$  represents a discrete-time stochastic evolution cocycle over a discrete-time stochastic evolution semiflow  $\varphi$ , then the pair  $\tilde{C} = (\tilde{\Phi}, \varphi)$  is referred to as a *discrete-time stochastic skew-evolution semiflow*.

**Definition 3.** A map  $\tilde{P} : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  with the property  $\tilde{P}^2(n, \omega) = \tilde{P}(n, \omega)$  for all  $(n, \omega) \in \mathbb{N} \times \Omega$  is called *projections family* on  $X$ .

*Remark 1.* If  $\tilde{P} : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  is a projections family, then the map  $\tilde{Q} : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  defined as  $\tilde{Q}(n, \omega) = I - \tilde{P}(n, \omega)$  is called a projection family. This is referred to as the *complementary projections family* of  $\tilde{P}$ .

**Definition 4.** A projections family  $\tilde{P} : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  is said to be *invariant* to the discrete-time stochastic skew-evolution semiflow  $\tilde{C} = (\tilde{\Phi}, \varphi)$  if

$$\tilde{\Phi}(m, n, \omega)\tilde{P}(n, \omega) = \tilde{P}(m, \varphi(m, n, \omega))\tilde{\Phi}(m, n, \omega),$$

for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ .

If  $\tilde{P}$  remains invariant for  $\tilde{C} = (\tilde{\Phi}, \varphi)$ , we denote by  $\Phi_{\tilde{P}} : \Delta \times \Omega \rightarrow \mathcal{B}(X)$  the map defined by  $\Phi_{\tilde{P}}(m, n, \omega) = \tilde{\Phi}(m, n, \omega)\tilde{P}(n, \omega)$ .

From Definitions 2 and 4, it immediately follows:

**Proposition 1.** *The properties of the map  $\Phi_{\tilde{P}}$  are as follows:*

- (i)  $\Phi_{\tilde{P}}(m, n, \omega) = \tilde{P}(m, \varphi(m, n, \omega))\tilde{\Phi}(m, n, \omega)$ ,  $\forall (m, n, \omega) \in \tilde{\Delta} \times \Omega$ ;
- (ii)  $\Phi_{\tilde{P}}(m, m, \omega) = \tilde{P}(m, \omega)$ ,  $\forall (m, \omega) \in \mathbb{N} \times \Omega$ ;
- (iii)  $\Phi_{\tilde{P}}(m, p, \omega) = \tilde{\Phi}(m, n, \varphi(n, p, \omega))\tilde{\Phi}(n, p, \omega)$ ,  $\forall (m, n, p, \omega) \in \tilde{T} \times \Omega$ .

*Proof.* The properties (i) and (ii) are immediate from Definition 4 and Definition 2. For (iii) we observe that

$$\begin{aligned} \Phi_{\tilde{P}}(m, p, \omega) &= \tilde{\Phi}(m, p, \omega)\tilde{P}(p, \omega) = \tilde{\Phi}(m, n, \varphi(n, p, \omega))\tilde{P}(n, \varphi(n, p, \omega))\tilde{\Phi}(n, p, \omega) \\ &= \tilde{\Phi}(m, n, \varphi(n, p, \omega))\tilde{\Phi}(n, p, \omega), \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$ . □

*Remark 2.* If the projections family  $\tilde{P}$  is invariant to  $\tilde{C} = (\tilde{\Phi}, \varphi)$  then its complementary  $\tilde{Q}(n, \omega) = I - \tilde{P}(n, \omega)$  is also invariant to  $\tilde{C}$ . Thus, for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$  we have that  $\tilde{\Phi}(m, n, \omega)(\text{Range } \tilde{Q}(n, \omega)) \subset \text{Range } \tilde{Q}(m, \varphi(m, n, \omega))$ .

**Definition 5.** The projections family  $\tilde{P} : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  is said to be *strongly invariant* to  $\tilde{C} = (\tilde{\Phi}, \varphi)$  if it is invariant to  $\tilde{C}$  and for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ , the map  $\tilde{\Phi}(m, n, \omega)$  is an isomorphism from  $\text{Range } \tilde{Q}(n, \omega)$  to  $\text{Range } \tilde{Q}(m, \varphi(m, n, \omega))$ .

*Remark 3.* If the projections family  $\tilde{P} : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  is strongly invariant to the discrete-time stochastic skew-evolution semiflow  $\tilde{C} = (\tilde{\Phi}, \varphi)$ , then there exists  $\Psi : \tilde{\Delta} \times \Omega \rightarrow \mathcal{B}(X)$  such that for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ ,  $\Psi(m, n, \omega)$  is an isomorphism from  $\text{Range } \tilde{Q}(m, \varphi(m, n, \omega))$  to  $\text{Range } \tilde{Q}(n, \omega)$ .

We will use the following notation:

$$\Psi_{\tilde{Q}}(m, n, \omega) = \Psi(m, n, \omega) \tilde{Q}(m, \varphi(m, n, \omega)).$$

**Proposition 2.** *If the projections family  $P : \mathbb{N} \times \Omega \rightarrow \mathcal{B}(X)$  is strongly invariant to the discrete-time stochastic skew-evolution semiflow  $\tilde{C} = (\Phi, \varphi)$  then the map  $\Psi_{\tilde{Q}}$  has the following properties:*

- (i)  $\Phi_{\tilde{Q}}(m, n, \omega) \Psi_{\tilde{Q}}(m, n, \omega) = \tilde{Q}(m, \varphi(m, n, \omega))$ , for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ ;
- (ii)  $\Psi_{\tilde{Q}}(m, n, \omega) \Phi_{\tilde{Q}}(m, n, \omega) = \tilde{Q}(n, \omega)$ , for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ ;
- (iii)  $\Psi_{\tilde{Q}}(m, n, \omega) = \tilde{Q}(n, \omega) \Psi_{\tilde{Q}}(m, n, \omega)$ , for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ .
- (iv)  $\Psi_{\tilde{Q}}(m, m, \omega) = \tilde{Q}(m, \omega)$ , for all  $(m, \omega) \in \mathbb{N} \times \Omega$ ;
- (v)  $\Psi_{\tilde{Q}}(m, p, \omega) = \Psi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega))$ , for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$ .

*Proof.* The properties (i) and (ii) are immediate from Definition 5 and Remark 3. To prove (iii), we will use the first two conditions and we have

$$\begin{aligned} \tilde{Q}(n, \omega) \Psi_{\tilde{Q}}(m, n, \omega) &= \Psi_{\tilde{Q}}(m, n, \omega) \Phi_{\tilde{Q}}(m, n, \omega) \Psi_{\tilde{Q}}(m, n, \omega) \\ &= \Psi_{\tilde{Q}}(m, n, \omega) \tilde{Q}(m, \varphi(m, n, \omega)) = \Psi_{\tilde{Q}}(m, n, \omega), \end{aligned}$$

for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ .

The condition (iv) it follows from (i) by taking  $n = m$

$$\Psi_{\tilde{Q}}(m, m, \omega) \Psi_{\tilde{Q}}(m, m, \omega) = \tilde{Q}(m, \varphi(m, m, \omega))$$

which is equivalent with  $\Psi_{\tilde{Q}}(m, m, \omega) = \tilde{Q}(m, \omega)$  and more  $\tilde{Q}(m, \omega) \Psi_{\tilde{Q}}(m, m, \omega) = \tilde{Q}(m, \omega)$ . Follows that  $\Psi_{\tilde{Q}}(m, m, \omega) = \tilde{Q}(m, \omega)$ .

To prove (v), using the properties (i)-(iv) we obtain

$$\begin{aligned} \Psi_{\tilde{Q}}(m, p, \omega) \tilde{Q}(p, \omega) \Psi_{\tilde{Q}}(m, p, \omega) &= \Psi_{\tilde{Q}}(n, p, \omega) \Phi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, p, \omega) \\ &= \Psi_{\tilde{Q}}(n, p, \omega) \tilde{Q}(n, \varphi(n, p, \omega)) \Phi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, p, \omega) \\ &= \Psi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)) \Phi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)) \Phi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, p, \omega) \\ &= \Psi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)) \Phi_{\tilde{Q}}(m, p, \omega) \Psi_{\tilde{Q}}(m, p, \omega) \\ &= \Psi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)) \tilde{Q}(m, \varphi(m, p, \omega)) \\ &= \Psi_{\tilde{Q}}(n, p, \omega) \Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)), \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$ . □

**Definition 6.** A nondecreasing map  $h : \mathbb{N} \rightarrow [1, \infty)$  with  $\lim_{m \rightarrow \infty} h(m) = \infty$  is called a *growth rate*.

Let  $\tilde{C} = (\Phi, \varphi)$  be a strongly measurable discrete-time stochastic skew-evolution semiflow,  $\tilde{P}$  an invariant projections family for  $\tilde{C}$  and  $h : \mathbb{N} \rightarrow [1, \infty)$  a growth rate.

In the following,  $L(\Omega, X, \mu)$  denotes the Banach space of all Bochner-measurable functions  $f : \Omega \rightarrow X$  such that  $\int_{\Omega} \|f(\omega)\| d\mu(\omega) < \infty$ .

**Definition 7.** Let  $\tilde{C} = (\Phi, \varphi)$  be a discrete-time stochastic skew-evolution semi-flow. We say that  $\tilde{C}$  is *strongly measurable* if, for all  $(p, x) \in \mathbb{N} \times L(\Omega, X, \mu)$ , the map  $n \mapsto \int_{\Omega} \|\Phi(n, p, \omega)x(\omega)\| d\mu(\omega)$ , is measurable on  $\mathbb{N}$ , for all  $n \geq p$ .

**Definition 8.** The pair  $(\tilde{C}, \tilde{P})$  is said to be *uniformly h-dichotomic in mean (u.h.d.m.)* if there are some constants  $N \geq 1$  and  $\nu > 0$  such that

(uhd<sub>1</sub>m)

$$\begin{aligned} h(m)^\nu \int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Nh(n)^\nu \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

(uhd<sub>2</sub>m)

$$h(m)^\nu \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq Nh(n)^\nu \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ ;

When we examine the specific cases where  $h(m) = e^m$  and  $h(m) = m + 1$ , we infer the concepts of *uniform exponential dichotomy in mean* and *uniform polynomial dichotomy in mean* respectively.

*Remark 4.* The pair  $(\tilde{C}, \tilde{P})$  is uniformly *h-dichotomic in mean* if and only if there exist  $N \geq 1$  and  $\nu > 0$  with

(uhd'<sub>1</sub>m)

$$\begin{aligned} h(m)^\nu \int_{\Omega} \|\Phi_{\tilde{P}}(m, n, \omega)x(\omega)\| d\mu(\omega) \leq \\ Nh(n)^\nu \int_{\Omega} \|\tilde{P}(n, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

(uhd'<sub>2</sub>m)

$$\begin{aligned} h(m)^\nu \int_{\Omega} \|\tilde{Q}(n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Nh(n)^\nu \int_{\Omega} \|\Phi_{\tilde{Q}}(m, n, \omega)(n, \omega)x(\omega)\| d\mu(\omega), \end{aligned}$$

for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

**Proposition 3.** *The pair  $(\tilde{C}, \tilde{P})$  is uniformly h-dichotomic in mean if and only if there are  $N \geq 1$  and  $\nu > 0$  such that*

**(uhd<sub>1</sub>'')**

$$\begin{aligned} h(m)^\nu \int_{\Omega} \|\Phi_{\tilde{P}}(m, n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Nh(n)^\nu \int_{\Omega} \|\tilde{P}(n, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

**(uhd<sub>2</sub>'')**

$$\begin{aligned} h(m)^\nu \int_{\Omega} \|\Psi_{\tilde{Q}}(m, n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Nh(n)^\nu \int_{\Omega} \|\tilde{Q}(m, \varphi(m, n, \omega))x(\omega)\| d\mu(\omega), \end{aligned}$$

for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

*Proof.* It follows immediately from Remark 4 and Proposition 2.  $\square$

**Theorem 1.** The pair  $(\tilde{C}, \tilde{P})$  is uniformly  $h$ -dichotomic in mean if and only if there exist some constants  $N \geq 1$  and  $\nu > 0$  with

**(uhd<sub>1</sub>'')**

$$\begin{aligned} h(m)^\nu \int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Nh(n)^\nu \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

**(uhd<sub>2</sub>'')**

$$\begin{aligned} h(n)^\nu \int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Nh(p)^\nu \int_{\Omega} \|\Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)) \\ x(\omega)\| d\mu(\omega), \text{ for all } (m, n, p, \omega) \in \tilde{T} \times \Omega \text{ and } x \in L(\Omega, X, \mu). \end{aligned}$$

*Proof.* It arises from Definition 8, Proposition 2 and Proposition 3.  $\square$

**Definition 9.** The pair  $(\tilde{C}, \tilde{P})$  is said to be with *uniform  $h$ -growth in mean (u.h.g.m.)* if there exist constants  $M \geq 1$  and  $\alpha > 0$  such that:

**(uhg<sub>1</sub>)**

$$\begin{aligned} h(n)^\alpha \int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^\alpha \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

**(uhg<sub>2</sub>)**

$$h(n)^\alpha \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)$$

$$\leq Mh(m)^\alpha \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

As specific cases we note that when the growth rate is  $e^m$ , this establishes the concept of *uniform exponential growth in mean* and if the growth rate is  $m + 1$ , then we arrive at the concept of *uniform polynomial growth in mean* respectively.

*Remark 5.* The pair  $(\tilde{C}, \tilde{P})$  has uniform  $h$ -growth in mean if and only if there exist  $M \geq 1$  and  $\alpha > 0$  with

(uhg<sub>1</sub>'m)

$$\begin{aligned} h(n)^\alpha \int_{\Omega} \|\Phi_{\tilde{P}}(m, n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^\alpha \int_{\Omega} \|\tilde{P}(n, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

(uhg<sub>2</sub>'m)

$$\begin{aligned} h(n)^\alpha \int_{\Omega} \|\tilde{Q}(n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^\alpha \int_{\Omega} \|\Phi_{\tilde{Q}}(m, n, \omega)x(\omega)\| d\mu(\omega), \end{aligned}$$

for all  $(m, n, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

**Proposition 4.** *The pair  $(\tilde{C}, \tilde{P})$  is said to be with uniform  $h$ -growth in mean if and only if there exist two constants  $M \geq 1$  and  $\alpha > 0$  such that*

(uhg<sub>1</sub>''m)

$$\begin{aligned} h(n)^\alpha \int_{\Omega} \|\Phi_{\tilde{P}}(m, n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^\alpha \int_{\Omega} \|\tilde{P}(n, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

(uhg<sub>2</sub>''m)

$$\begin{aligned} h(n)^\alpha \int_{\Omega} \|\Psi_{\tilde{Q}}(m, n, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^\alpha \int_{\Omega} \|\tilde{Q}(m, \varphi(m, n, \omega)) \\ x(\omega)\| d\mu(\omega), \text{ for all } (m, n, \omega) \in \tilde{\Delta} \times \Omega \text{ and } x \in L(\Omega, X, \mu). \end{aligned}$$

*Proof.* It follows a similar approach as Proposition 3. □

**Theorem 2.** *The pair  $(\tilde{C}, \tilde{P})$  has uniform  $h$ -growth in mean if and only if there are  $M \geq 1$  and  $\alpha > 0$  with*

(**uhg<sub>1</sub><sup>'''</sup> m**)

$$\begin{aligned} h(n)^\alpha \int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^\alpha \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

(**uhg<sub>2</sub><sup>'''</sup> m**)

$$\begin{aligned} h(p)^\alpha \int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(n)^\alpha \int_{\Omega} \|\Psi_{\tilde{Q}}(m, n, \varphi(n, p, \omega)) \\ x(\omega)\| d\mu(\omega), \text{ for all } (m, n, p, \omega) \in \tilde{T} \times \Omega \text{ and } x \in L(\Omega, X, \mu). \end{aligned}$$

*Proof.* The proof uses the same technique demonstrated in Theorem 1. □

### 3. MAIN RESULTS

We denote by  $\mathcal{H}_1$  the set of all functions  $h : \mathbb{N} \rightarrow [1, \infty)$  with the following properties:

•  $\mathcal{H}_1$  the set of all functions  $h : \mathbb{N} \rightarrow [1, \infty)$  with the property that for all  $\beta < 0$ , exists  $H_1 > 1$  such that  $h(m+1) \leq H_1 h(m)$ , for all  $m \geq 0$  and

$$\sum_{k=n}^{\infty} h(k)^\beta \leq H_1 h(n)^\beta, \quad \text{for all } n \geq 0.$$

•  $\mathcal{H}_2$  the set of functions  $h : \mathbb{N} \rightarrow [1, \infty)$  with the property that for all  $\beta > 0$ , exists  $H_2 > 1$  such that  $h(m+1) \leq H_2 h(m)$  and

$$\sum_{j=n}^m h(j)^\beta \leq H_2 h(m)^\beta, \quad \text{for all } m \geq 0.$$

*Remark 6.* If  $h(m) = e^m$ , then  $h \in \mathcal{H}_\infty$ .

**Theorem 3.** We assume that  $\tilde{C} = (\Phi, \varphi)$  is a strongly measurable discrete-time stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  with uniform  $h$ -growth in mean and  $h \in \mathcal{H}_1$ . The pair  $(\tilde{C}, \tilde{P})$  is uniformly  $h$ -dichotomic in mean if and only if there exist constants  $D \geq 1$  and  $d \in (0, 1)$  such that

$$\begin{aligned} (\mathbf{uhD}_1^1 \mathbf{m}_d) \sum_{k=n}^{\infty} h(k)^d \left( \int_{\Omega} \|\Phi_{\tilde{P}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\ \leq D h(n)^d \left( \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega) \right); \end{aligned}$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

$$\begin{aligned}
 & (\mathbf{uhD}_2^1 m_d) \sum_{k=n}^{\infty} \frac{h(k)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(k, p, \omega)x(\omega)\| d\mu(\omega)} \leq \frac{D h(n)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)}, \\
 & \text{for all } (n, p, \omega) \in \tilde{\Delta} \times \Omega \text{ and } x \in L(\Omega, X, \mu) \\
 & \text{with } \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| \neq 0.
 \end{aligned}$$

*Proof. Necessity.* Let  $d \in (0, \nu)$ . For  $(\mathbf{uhd}_1 m) \implies (\mathbf{uhD}_1^1 m_d)$  we have

$$\begin{aligned}
 & \sum_{k=n}^{\infty} h(k)^d \left( \int_{\Omega} \|\Phi_{\tilde{P}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
 & \leq N \sum_{k=n}^{\infty} \left( \frac{h(k)}{h(n)} \right)^{-\nu} h(k)^d \left( \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
 & = Nh(n)^{\nu} \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega) \sum_{k=n}^{\infty} h(k)^{d-\nu} \\
 & \leq Nh(n)^{\nu} \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega) H_1 h(n)^{d-\nu} \\
 & = NH_1 h(n)^d \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq Dh(n)^d \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega), \\
 & \text{where } D = 1 + NH_1.
 \end{aligned}$$

Analogously, we have to prove  $(\mathbf{uhd}_2 m) \implies (\mathbf{uhD}_2^1 m_d)$

$$\begin{aligned}
 & \sum_{k=n}^{\infty} \frac{h(k)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(k, p, \omega)x(\omega)\| d\mu(\omega)} \\
 & \leq N \sum_{k=n}^{\infty} \left( \frac{h(k)}{h(n)} \right)^{-\nu} \frac{h(k)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)} \\
 & = \frac{Nh(n)^{\nu}}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)} \sum_{k=n}^{\infty} h(k)^{d-\nu} \\
 & \leq \frac{Nh(n)^{\nu}}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)} H_1 h(n)^{d-\nu} \\
 & \leq \frac{Dh(n)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)},
 \end{aligned}$$

where  $D = 1 + NH_1$ , for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ , with  $\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$ .

*Sufficiency.* To establish  $(\mathbf{uhD}_1^1 m_d) \implies (\mathbf{uhd}_1 m)$ , we need to examine the following cases:

Case I.1. If  $m > p + 1$  we have

$$\begin{aligned}
& h(m)^d \left( \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&= h(m)^d \frac{1}{2} \sum_{k=m-1}^m \left( \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&\leq \frac{1}{2} \sum_{k=m-1}^m M h(m)^d \left( \frac{h(m)}{h(k)} \right)^{\alpha} \left( \int_{\Omega} \|\Phi_{\bar{p}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&= \frac{M}{2} \sum_{k=m-1}^m h(k)^d \left( \frac{h(m)}{h(k)} \right)^{\alpha+d} \left( \int_{\Omega} \|\Phi_{\bar{p}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&\leq \frac{M}{2} H_1^{\alpha+d} \sum_{k=n}^{\infty} h(k)^d \left( \int_{\Omega} \|\Phi_{\bar{p}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&\leq \frac{M}{2} D H_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& h(m)^d \left( \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&\leq \frac{M}{2} D H_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega),
\end{aligned}$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

Case I.2. If  $n \in [p, p + 1)$  we have

$$\begin{aligned}
& h(m)^d \left( \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&\leq M h(m)^d \left( \frac{h(m)}{h(n)} \right)^{\alpha} \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega) \\
&= M \left( \frac{h(m)}{h(n)} \right)^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega) \\
&\leq M H_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega)
\end{aligned}$$

Thus,

$$\begin{aligned}
& h(m)^d \left( \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
&\leq M H_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega)
\end{aligned}$$

From Case I.1. and Case I.2. it results that there exists  $N = MDH_1^{\alpha+d} + 1$  with

$$\begin{aligned} h(m)^d \left( \int_{\Omega} \|\Phi_{\tilde{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\ \leq Nh(n)^d \int_{\Omega} \|\Phi_{\tilde{p}}(n, p, \omega)x(\omega)\| d\mu(\omega), \end{aligned}$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

For the second relation  $(uhD_2^1 m_d) \implies (uhD_2 m)$ , we initially consider  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$ . Moreover, there are two cases to be considered:

Case II.1. When  $n \geq p + 1$  we obtain

$$\begin{aligned} \frac{h(m)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} &= \frac{1}{2} \sum_{k=m-1}^m \frac{h(m)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ &\leq \frac{M}{2} \sum_{k=m-1}^m \left( \frac{h(m)}{h(k)} \right)^{\alpha} \frac{h(m)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ &= \frac{M}{2} \sum_{k=m-1}^m \left( \frac{h(m)}{h(k)} \right)^{\alpha+d} \frac{h(k)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ &\leq \frac{M}{2} H_1^{\alpha+d} \sum_{k=n}^{\infty} \frac{h(k)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)} \\ &\leq \frac{M}{2} MDH_1^{\alpha+d} \frac{h(n)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

So,

$$h(m)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq \frac{M}{2} DH_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

Case II.2. If  $n \in [p, p + 1)$  with  $\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$

$$\begin{aligned} h(m)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq Mh(m)^d \left( \frac{h(m)}{h(n)} \right)^{\alpha} \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ = M \left( \frac{h(m)}{h(n)} \right)^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ \leq MH_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \end{aligned}$$

Thus,

$$h(m)^d \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \right) \leq MDH_1^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)$$

From Case II.1. and Case II.2. it follows that there exists  $N = MDH_1^{\alpha+d} + 1$  with

$$h(m)^d \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \right) \leq N_1 h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(m, p, \omega) \in \tilde{\Lambda} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .  $\square$

**Corollary 1.** *We suppose that  $\tilde{C} = (\Phi, \varphi)$  is a strongly measurable discrete-time stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  with uniform exponential growth in mean. The pair  $(\tilde{C}, \tilde{P})$  is uniformly exponentially dichotomic in mean if and only if there exist constants  $D \geq 1$  and  $d \in (0, 1)$  with*

(ueD<sub>1</sub><sup>1</sup>m<sub>d</sub>)

$$\sum_{k=n}^{\infty} e^{dk} \left( \int_{\Omega} \|\Phi_{\tilde{P}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \leq D e^{dn} \left( \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega) \right);$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

$$(ueD_2^1 m_d) \sum_{k=n}^{\infty} \frac{e^{dk}}{\int_{\Omega} \|\Phi_{\tilde{Q}}(k, p, \omega)x(\omega)\| d\mu(\omega)} \leq \frac{D e^{dn}}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)},$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$  with  $\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| \neq 0$ .

*Proof.* It follows from Theorem 3 for  $h(m) = e^m$ .  $\square$

**Theorem 4.** *Consider  $\tilde{C} = (\Phi, \varphi)$  as a strongly measurable discrete-time stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  has uniform  $h$ -growth in mean and  $h \in \mathcal{H}_2$ . The pair  $(\tilde{C}, \tilde{P})$  is uniformly  $h$ -dichotomic in mean if and only if there exist constants  $D \geq 1$  and  $d \in (0, 1)$  such that*

$$(uhD_1^2 m_d) \sum_{j=n}^m \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \leq \frac{D h(m)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)},$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ ,

with  $\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$ .

(uhD<sub>2</sub><sup>2</sup>m<sub>d</sub>)

$$\sum_{j=n}^m h(j)^{-d} \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(j, p, \omega)x(\omega)\| d\mu(\omega) \right)$$

$$\leq D h(m)^{-d} \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right),$$

for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

*Proof. Necessity.* Let  $d \in (0, \nu)$ . For  $(uhd_1m) \implies (uhD_1^2m_d)$  we have

$$\begin{aligned} & \sum_{j=p}^m \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \\ & \leq \sum_{j=p}^m N \left( \frac{h(j)}{h(m)} \right)^{\nu} \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ & = \frac{Nh(m)^{-\nu}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \sum_{j=p}^m h(j)^{\nu-d} \\ & \leq \frac{NH_2h(m)^{-\nu}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)} h(m)^{\nu-d} \\ & = \frac{Dh(m)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

where  $D = 1 + NH_2$ , with  $\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$

For the implication  $(uhd_2m) \implies (uhD_2^2m_d)$  we have

$$\begin{aligned} & \sum_{j=p}^m \frac{\int_{\Omega} \|\Phi_{\tilde{Q}}(j, p, \omega)x(\omega)\| d\mu(\omega)}{h(j)^d} \\ & \leq \sum_{j=p}^m N \left( \frac{h(j)}{h(m)} \right)^{\nu} \frac{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}{h(j)^d} \\ & \leq \frac{N \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}{h(m)^{\nu}} \sum_{j=p}^m h(j)^{\nu-d} \\ & \leq NH_2h(m)^{\nu-d} \cdot \frac{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}{h(m)^{\nu}} \\ & \leq \frac{D \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}{h(m)^d}, \end{aligned}$$

where  $D = 1 + NH_2$ .

*Sufficiency.* For  $(uhD_2^1m'_d) \implies (uhd_1m)$  we have two cases:

Case I.1. If  $n \geq p + 1$  we have

$$\frac{h(n)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j, n, \omega)x(\omega)\| d\mu(\omega)}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{j=p}^{p+1} \frac{h(n)^{-d}}{\int_{\Omega} \|\Phi_{\bar{p}}(j, n, \omega)x(\omega)\| d\mu(\omega)} \\
&\leq \frac{1}{2} \sum_{j=p}^{p+1} h(n)^{-d} M \left( \frac{h(j)}{h(n)} \right)^{\alpha} \frac{h(j)^{-d} h(j)^d}{\int_{\Omega} \|\Phi_{\bar{p}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \\
&= \frac{M}{2} \sum_{j=p}^{p+1} \left( \frac{h(j)}{h(n)} \right)^{\alpha+d} \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\bar{p}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \\
&\leq \frac{M}{2} H_2^{\alpha+d} \sum_{j=p}^m \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\bar{p}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \\
&\leq \frac{M}{2} DH_2^{\alpha+d} \frac{h(m)^{-d}}{\int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega)}
\end{aligned}$$

We obtain

$$h(m)^d \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \leq D \frac{M}{2} H_2^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

Case I.2. If  $m \in [p, p+1)$  and  $\int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$

$$\begin{aligned}
&\int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\
&= \int_{\Omega} \|\Phi(m, n, \varphi(n, p, \omega))\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega) \\
&\leq M \left( \frac{h(m)}{h(n)} \right)^{\alpha} \int_{\Omega} \|\tilde{P}(n, \varphi(n, p, \omega))\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega) \\
&\leq M \left( \frac{h(m)}{h(n)} \right)^{\alpha+d} \left( \frac{h(m)}{h(n)} \right)^{-d} \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega) \\
&\leq MH_2^{\alpha+d} \left( \frac{h(m)}{h(n)} \right)^{-d} \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega),
\end{aligned}$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

From Case I.1. and Case I.2. it follows that there exists  $N = MDH_2^{\alpha+d} + 1$  with

$$h(m)^d \int_{\Omega} \|\Phi_{\bar{p}}(m, p, \omega)x(\omega)\| d\mu(\omega) \leq MDH_2^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\bar{p}}(n, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(n, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

For the relation  $(uhD_2^2 m_d) \implies (uhd_2 m)$  we also consider two cases:

Case II.1. When  $n \geq p+1$  we obtain

$$h(n)^{-d} \int_{\Omega} \|\Phi_{\bar{q}}(n, p, \omega)x(\omega)\| d\mu(\omega)$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{j=p}^{p+1} h(n)^{-d} \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \right) \\
 &\leq \frac{1}{2} \sum_{j=p}^{p+1} h(n)^{-d} M \left( \frac{h(j)}{h(n)} \right)^{\alpha} \int_{\Omega} \|\Phi_{\tilde{Q}}(j, p, \omega)x(\omega)\| d\mu(\omega) \\
 &= \frac{M}{2} \sum_{j=p}^{p+1} \left( \frac{h(j)}{h(n)} \right)^{\alpha+d} h(j)^{-d} \int_{\Omega} \|\Phi_{\tilde{Q}}(j, p, \omega)x(\omega)\| d\mu(\omega) \\
 &\leq \frac{M}{2} H_2^{\alpha+d} \sum_{j=p}^m h(j)^{-d} \int_{\Omega} \|\Phi_{\tilde{Q}}(j, p, \omega)x(\omega)\| d\mu(\omega) \\
 &\leq \frac{M}{2} DH_2^{\alpha+d} h(m)^{-d} \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),
 \end{aligned}$$

So, we obtain

$$h(m)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq \frac{M}{2} DH_2^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

Case II.2. If  $t \in [p, p + 1)$  we have

$$\begin{aligned}
 &\frac{h(m)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\
 &\leq M \left( \frac{h(m)}{h(n)} \right)^{\alpha} \frac{h(m)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)} \\
 &= M \left( \frac{h(m)}{h(n)} \right)^{\alpha+d} \frac{h(n)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)} \\
 &\leq MH_2^{\alpha+d} \frac{h(n)^d}{\int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega)}
 \end{aligned}$$

Therefore,

$$h(m)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq MH_2^{\alpha+d} h(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

From Case II.1. and Case II.2. it follows that there exists  $N = MDH_2^{\alpha+d} + 1$  with

$$h(m)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq Nh(n)^d \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ . □

**Corollary 2.** Let  $\tilde{C} = (\Phi, \varphi)$  as a strongly measurable discrete-time stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  has uniform exponential growth in mean. The pair

$(\tilde{C}, \tilde{P})$  is uniformly exponentially dichotomic in mean if and only if there exist constants  $D \geq 1$  and  $d \in (0, 1)$  with

$$(\mathbf{ueD}_1^2 \mathbf{m}_d) \quad \sum_{j=n}^m \frac{e^{-dj}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \leq \frac{D e^{-dm}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)},$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ , with

$$\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0.$$

$(\mathbf{ueD}_2^2 \mathbf{m}_d)$

$$\begin{aligned} & \sum_{j=n}^m e^{-dj} \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(j, p, \omega)x(\omega)\| d\mu(\omega) \right) \\ & \leq D e^{-dm} \left( \int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right), \end{aligned}$$

for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

*Proof.* It follows from Theorem 4 for  $h(m) = e^m$ .  $\square$

**Theorem 5.** Let  $\tilde{C} = (\Phi, \varphi)$  be a strongly measurable discrete-time stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  with strong uniform  $h$ -growth in mean and  $h \in \mathcal{H}_1$ . The pair  $(\tilde{C}, \tilde{P})$  is uniformly  $h$ -dichotomic in mean if and only if there are two constants  $D \geq 1$  and  $d \in (0, 1)$  such that

$(\mathbf{uhD}_1^3 \mathbf{m}_d)$

$$\begin{aligned} & \sum_{k=p}^{\infty} h(k)^d \left( \int_{\Omega} \|\Phi_{\tilde{P}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\ & \leq Dh(n)^d \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

$(\mathbf{uhD}_2^3 \mathbf{m}_d)$

$$\begin{aligned} & \sum_{k=p}^{\infty} \frac{h(k)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, k, \varphi(k, p, \omega))x(\omega)\| d\mu(\omega)} \\ & \leq \frac{Dh(p)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

*Proof. Necessity.* The relation  $(\mathbf{uhd}_1 m) \implies (\mathbf{uhD}_2^1 m_d)$  is similar with the proof of Theorem 3. To prove the relation  $(\mathbf{uhd}_2 m) \implies (\mathbf{uhD}_2^3 m_d)$  we suppose that  $(\tilde{C}, \tilde{P})$  has u.h.g.m. and we obtain

$$\sum_{k=p}^{\infty} \frac{h(k)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, k, \varphi(k, p, \omega))x(\omega)\| d\mu(\omega)}$$

$$\begin{aligned} &\leq \sum_{k=p}^{\infty} N \left( \frac{h(k)}{h(p)} \right)^{-\nu} \frac{h(k)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ &\leq \frac{Nh(p)^{\nu}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \sum_{k=p}^{\infty} h(k)^{d-\nu} \\ &\leq NH_1 \frac{h(p)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

where  $D = 1 + NH_1$ .

*Sufficiency.* The implication  $(uhD_1^3m'_d) \implies (uhd_1m)$  is similar with the proof of Theorem 3. For the relation  $(uhD_2^3m_d) \implies (uhd_2m)$  we have two cases:

Case I.1.  $n \geq p + 1$  and  $\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$ .

$$\begin{aligned} \frac{1}{\int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega)} &= \frac{1}{2} \sum_{m-1}^m \frac{1}{\int_{\Omega} \|\Phi_{\tilde{Q}}(m, p, \omega)\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ &\leq \frac{1}{2} \sum_{m-1}^m M \left( \frac{h(n)}{h(k)} \right)^{\alpha} \frac{1}{\int_{\Omega} \|\Phi_{\tilde{Q}}(k, p, \omega)\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)} \\ &\leq \frac{M}{2} \sum_{m-1}^m \left( \frac{h(n)}{h(k)} \right)^{\alpha+d} \left( \frac{h(n)}{h(p)} \right)^{-d} \left( \frac{h(k)}{h(p)} \right)^d \frac{1}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, k, \varphi(k, p, \omega))x(\omega)\| d\mu(\omega)} \\ &\leq \frac{M}{2} H_1^{\alpha+d} \left( \frac{h(n)}{h(p)} \right)^{-d} \sum_{k=p}^{\infty} \left( \frac{h(k)}{h(p)} \right)^d \frac{1}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, k, \varphi(k, p, \omega))x(\omega)\| d\mu(\omega)} \\ &\leq \frac{DMH_1^{\alpha+d}}{2} \left( \frac{h(n)}{h(p)} \right)^{-d} \frac{1}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}; \end{aligned}$$

Case I.2.  $n \in [p, p + 1)$ .

$$\begin{aligned} &\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ &\leq M \left( \frac{h(n)}{h(p)} \right)^{\alpha} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) \\ &\leq M \left( \frac{h(n)}{h(p)} \right)^{\alpha+d} \left( \frac{h(n)}{h(p)} \right)^{-d} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) \\ &\leq MH_1^{\alpha+d} \left( \frac{h(n)}{h(p)} \right)^{-d} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega). \end{aligned}$$

Combining Case I.1. with Case I.2., we can conclude that there exist  $N = 1 + MH_1^{\alpha+d}D$  and  $\nu = d$  such that  $(uhd_2m)$  holds for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and all  $x \in L(\Omega, X, \mu)$ . Hence, we have shown that  $(\tilde{C}, \tilde{P})$  is u.h.d.m., completing the proof.  $\square$

**Corollary 3.** *We assume that  $\tilde{C} = (\Phi, \varphi)$  be a strongly measurable discrete-time stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  with strong uniform exponential growth in mean. The pair  $(\tilde{C}, \tilde{P})$  is uniformly exponentially dichotomic in mean if and only if there are two constants  $D \geq 1$  and  $d \in (0, 1)$  with*

$$\begin{aligned} & (\mathbf{ueD}_1^3 \mathbf{m}_d) \\ & \sum_{k=p}^{\infty} e^{dk} \left( \int_{\Omega} \|\Phi_{\tilde{P}}(k, p, \omega)x(\omega)\| d\mu(\omega) \right) \\ & \leq D e^{dn} \int_{\Omega} \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega); \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

$$\begin{aligned} & (\mathbf{ueD}_2^3 \mathbf{m}_d) \\ & \sum_{k=p}^{\infty} \frac{e^{dk}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, k, \varphi(k, p, \omega))x(\omega)\| d\mu(\omega)} \\ & \leq \frac{D e^{dp}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ .

*Proof.* It follows from Theorem 5 for  $h(m) = e^m$ . □

**Theorem 6.** *We assume that  $\tilde{C} = (\Phi, \varphi)$  be a strongly measurable stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  with strong uniform  $h$ -growth in mean and  $h \in \mathcal{H}_2$ . The pair  $(\tilde{C}, \tilde{P})$  is uniformly  $h$ -dichotomic in mean if and only if there exist  $D \geq 1$  and  $d \in (0, 1)$  such that*

$$\begin{aligned} & (\mathbf{uhD}_1^4 \mathbf{m}_d) \\ & \sum_{j=p}^m \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \\ & \leq \frac{D h(m)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ , with

$$\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0.$$

$$\begin{aligned} & (\mathbf{uhD}_2^4 \mathbf{m}_d) \\ & \sum_{j=p}^m \frac{h(j)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, j, \varphi(j, p, \omega))x(\omega)\| d\mu(\omega)} \\ & \leq \frac{D h(m)^d}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)}, \end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ , with  $\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$ .

*Proof. Necessity.* The relation  $(\mathbf{uhd}_1\mathbf{m}) \implies (\mathbf{uhD}_4^1\mathbf{m}_d)$  is similar with the proof of Theorem 4. To prove the relation  $(\mathbf{uhd}_2\mathbf{m}) \implies (\mathbf{uhD}_2^4\mathbf{m}_d)$  we suppose that  $(\tilde{C}, \tilde{P})$  has u.h.g.m. and we obtain

$$\begin{aligned} & \sum_{j=p}^m h(j)^{-d} \left( \int_{\Omega} \|\Psi_{\tilde{Q}}(m, j, \varphi(j, p, \omega))x(\omega)\| d\mu(\omega) \right) \\ & \leq \sum_{j=p}^m h(j)^{-d} N \left( \frac{h(j)}{h(m)} \right)^{\nu} \left( \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) \right) \\ & = Nh(m)^{-\nu} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) \sum_{j=p}^m h(j)^{\nu-d} \\ & \leq Nh(m)^{-\nu} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) H_2 h(m)^{\nu-d} \\ & \leq NH_2 h(m)^{-d} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega), \end{aligned}$$

where  $D = 1 + NH_2$ .

*Sufficiency.* The implication  $(\mathbf{uhD}_1^4\mathbf{m}_d) \implies (\mathbf{uhd}_1\mathbf{m})$  is similar with the proof of Theorem 4. For the relation  $(\mathbf{uhD}_2^4\mathbf{m}_d) \implies (\mathbf{uhd}_2\mathbf{m})$  we have two cases:

Case I.1.  $n \geq p + 1$  and

$$\begin{aligned} & \int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \\ & = \frac{1}{2} \sum_{j=p}^{p+1} \left( \int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega) \right) \\ & \leq \frac{1}{2} \sum_{j=p}^{p+1} M \left( \frac{h(j)}{h(p)} \right)^{\alpha} \left( \int_{\Omega} \|\Psi_{\tilde{Q}}(m, j, \varphi(j, p, \omega))x(\omega)\| d\mu(\omega) \right) \\ & \leq \frac{M}{2} \left( \frac{h(p)}{h(n)} \right)^{d} \sum_{j=p}^{p+1} \left( \frac{h(j)}{h(p)} \right)^{\alpha+d} \left( \frac{h(j)}{h(n)} \right)^{-d} \left( \int_{\Omega} \|\Psi_{\tilde{Q}}(m, j, \varphi(j, p, \omega))x(\omega)\| d\mu(\omega) \right) \\ & \leq \frac{M}{2} H_2^{\alpha+d} \left( \frac{h(p)}{h(n)} \right)^{d} \sum_{j=p}^{p+1} \left( \frac{h(j)}{h(n)} \right)^{-d} \left( \int_{\Omega} \|\Psi_{\tilde{Q}}(m, j, \varphi(j, p, \omega))x(\omega)\| d\mu(\omega) \right) \\ & \leq MDH_2^{\alpha+d} \left( \frac{h(p)}{h(n)} \right)^d \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega). \end{aligned}$$

Case I.2.  $n \in [p, p + 1)$ .

$$\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)$$

$$\begin{aligned}
&\leq M \left( \frac{h(m)}{h(p)} \right)^\alpha \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) \\
&\leq \left( \frac{h(p)}{h(m)} \right)^d \left( \frac{h(n)}{h(p)} \right)^{\alpha+d} \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega) \\
&\leq MH_2^{\alpha+d} \left( \frac{h(p)}{h(m)} \right)^d \int_{\Omega} \|\tilde{Q}(m, \varphi(m, p, \omega))x(\omega)\| d\mu(\omega).
\end{aligned}$$

From Case I.1. with Case I.2., it results that there exist  $N = 1 + MH_2^{\alpha+d}D$  and  $\nu = d$  such that  $(uhd_2m)$  holds for all  $(m, p, \omega) \in \tilde{\Delta} \times \Omega$  and all  $x \in L(\Omega, X, \mu)$ . Hence, we have shown that  $(\tilde{C}, \tilde{P})$  is u.h.d.m., completing the proof.  $\square$

**Corollary 4.** *We suppose that  $\tilde{C} = (\Phi, \varphi)$  be a strongly measurable stochastic skew-evolution semiflow,  $(\tilde{C}, \tilde{P})$  with strong uniform exponential growth in mean. The pair  $(\tilde{C}, \tilde{P})$  is uniformly exponentially dichotomic in mean if and only if there exist  $D \geq 1$  and  $d \in (0, 1)$  with*

$$(\mathbf{ueD}_1^4 \mathbf{m}_d) \sum_{j=p}^m \frac{e^{-dj}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j, p, \omega)x(\omega)\| d\mu(\omega)} \leq \frac{D e^{-dm}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega)},$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ ,

with  $\int_{\Omega} \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \neq 0$ .

$(\mathbf{ueD}_2^4 \mathbf{m}_d)$

$$\begin{aligned}
&\sum_{j=p}^m \frac{e^{dj}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, j, \varphi(j, p, \omega))x(\omega)\| d\mu(\omega)} \\
&\leq \frac{D e^{dm}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega)},
\end{aligned}$$

for all  $(m, n, p, \omega) \in \tilde{T} \times \Omega$  and  $x \in L(\Omega, X, \mu)$ ,

with  $\int_{\Omega} \|\Psi_{\tilde{Q}}(m, p, \omega)x_0(\omega)\| d\mu(\omega) \neq 0$ .

*Proof.* It follows from Theorem 6 for  $h(m) = e^m$ .  $\square$

## CONCLUSIONS

This study broadens several well-known results on uniform exponential dichotomy in mean by applying them to the more general framework of uniform  $h$ -dichotomy in mean. Integral characterizations for this generalized concept are provided for discrete-time stochastic skew-evolution semiflows, considering both invariant projection families and strongly invariant projection families. From these characterizations, necessary and sufficient conditions are derived for the specific case of uniform exponential dichotomy in mean. Future research will focus on unifying the analysis

of discrete and continuous asymptotic properties for stochastic skew-evolution semiflows. Additionally, efforts will be directed toward extending these findings to the nonuniform case.

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#### REFERENCES

- [1] L. Arnold, *Random dynamical systems*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1995, pp. 1–43, doi: [10.1007/BFb0095238](https://doi.org/10.1007/BFb0095238).
- [2] A. M. Atewi, "About bounded solutions of linear stochastic ito systems," *Miskolc Math. Notes*, vol. 3, no. 1, pp. 3–12, 2002.
- [3] L. Barreira and C. Valls, "Polynomial growth rates," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5208–5219, 2009, doi: [10.1016/j.na.2009.04.005](https://doi.org/10.1016/j.na.2009.04.005).
- [4] A. J. Bento and C. Silva, "Stable manifolds for nonuniform polynomial dichotomies," *Journal of Functional Analysis*, vol. 257, no. 1, pp. 122–148, 2009, doi: [doi:10.1016/j.jfa.2009.01.032](https://doi.org/10.1016/j.jfa.2009.01.032).
- [5] R. Boruga, "Polynomial stability in average for cocycles of linear operators," *Theory and Applications of Mathematics & Computer Science*, vol. 9, no. 1, pp. 8–13, 2019.
- [6] R. Boruga, "On uniform dichotomies for the growth rates of linear discrete-time dynamical systems in Banach spaces," in *International Conference on Difference Equations and Applications*. Springer, 2022, pp. 175–188.
- [7] R. Boruga, M. Megan, and D. M.-M. Toth, "Integral characterizations for uniform stability with growth rates in Banach spaces," *Axioms*, vol. 10, no. 3, p. 235, 2021, doi: [10.3390/axioms10030235](https://doi.org/10.3390/axioms10030235).
- [8] T. Caraballo, J. Duany, K. Lu, and B. Schmalfuß, "Invariant manifolds for random and stochastic partial differential equations," *Adv. Nonlinear Stud.*, vol. 10, no. 1, pp. 23–52, 2010, doi: [10.1515/ans-2010-0102](https://doi.org/10.1515/ans-2010-0102).
- [9] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, ser. *Enycl. Math. Appl.* Cambridge etc.: Cambridge University Press, 1992, vol. 44, doi: [10.1017/CBO9780511666223](https://doi.org/10.1017/CBO9780511666223).
- [10] J. L. Daleckiĭ and M. G. Krein, *Stability of solutions of differential equations in Banach space*. Amer. Math. Soc., 2002, no. 43.
- [11] R. Datko, "Uniform asymptotic stability of evolutionary processes in a banach space," *SIAM J. Math. Anal.*, vol. 3, no. 3, pp. 428–445, 1972, doi: [10.1137/0503042](https://doi.org/10.1137/0503042).
- [12] D. Dragičević, A. L. Sasu, and B. Sasu, "Admissibility and polynomial dichotomy of discrete nonautonomous systems," *Carpathian J. Math.*, vol. 38, no. 3, pp. 737–762, 2022, doi: [10.37193/CJM.2022.03.18](https://doi.org/10.37193/CJM.2022.03.18).
- [13] A. Găină, "On uniform h-dichotomy of skew-evolution cocycles in Banach spaces," *An. Univ. Vest Timiș. Ser. Mat.-Inform.*, vol. 58, no. 2, pp. 96–106, 2022, doi: [10.2478/awutm-2022-0020](https://doi.org/10.2478/awutm-2022-0020).
- [14] P. V. Hai, "Polynomial behavior in mean of stochastic skew-evolution semiflows," *arXiv preprint arXiv:1902.04214*, 2019.
- [15] P. V. Hai, "Polynomial stability and polynomial instability for skew-evolution semiflows," *Results in Mathematics*, vol. 74, pp. 1–19, 2019, doi: [10.1007/s00025-019-1099-3](https://doi.org/10.1007/s00025-019-1099-3).

- [16] M. Megan, C. L. Mihit, and R. Lolea, “On splitting with different growth rates for linear discrete-time systems in Banach spaces,” in *Difference equations, discrete dynamical systems and applications, ICDEA 23, Timișoara, Romania, July 24–28, 2017. Proceedings of the 23rd international conference on difference equations and applications*. Cham: Springer, 2019, pp. 351–368, doi: [10.1007/978-3-030-20016-9\\_15](https://doi.org/10.1007/978-3-030-20016-9_15).
- [17] M. Megan and C. Stoica, “Discrete asymptotic behaviors for skew-evolution semiflows on Banach spaces,” *Carpathian J. Math.*, vol. 24, no. 3, pp. 348–355, 2008.
- [18] M. Megan and C. Stoica, “Exponential instability of skew-evolution semiflows in Banach spaces,” *Stud. Univ. Babeș-Bolyai, Math.*, vol. 53, no. 1, pp. 17–24, 2008.
- [19] M. Megan and C. Stoica, “Concepts of dichotomy for skew-evolution semiflows in Banach spaces,” *Ann. Acad. Rom. Sci., Math. Appl.*, vol. 2, no. 2, pp. 125–140, 2010.
- [20] C. L. Mihit and M. Megan, “Integral characterizations for the  $(h, k)$ -splitting of skew-evolution semiflows,” *Stud. Univ. Babeș-Bolyai Math*, vol. 62, no. 3, pp. 353–365, 2017, doi: [10.24193/subbmath.2017.3.08](https://doi.org/10.24193/subbmath.2017.3.08).
- [21] O. Perron, “Die stabilitätsfrage bei differentialgleichungen,” *Math. Z.*, vol. 32, no. 1, pp. 703–728, 1930, doi: [10.1007/BF01194662](https://doi.org/10.1007/BF01194662).
- [22] C. Stoica, “Approaching the discrete dynamical systems by means of skew-evolution semiflows,” *Discrete Dyn. Nat. Soc.*, vol. 2016, p. 10, 2016, id/No 4375069, doi: [10.1155/2016/4375069](https://doi.org/10.1155/2016/4375069).
- [23] D. Stoica and M. Megan, “Concepts of dichotomy for stochastic skew-evolution semiflows in hilbert spaces,” in *Numerical Analysis and Applied Mathematics ICNAAM 2012: International Conference of Numerical Analysis and Applied Mathematics*, vol. 1479, no. 1, doi: [10.1063/1.4756246](https://doi.org/10.1063/1.4756246), 2012, pp. 755–758.
- [24] T. M. Személy Fülöp, “On uniform dichotomy in mean of stochastic skew-evolution semiflows in Banach spaces,” *An. Univ. Vest Timiș., Ser. Mat.-Inform.*, vol. 59, no. 1, pp. 92–104, 2023, doi: [10.2478/awutm-2023-0008](https://doi.org/10.2478/awutm-2023-0008).

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