Miskolc Mathematical Notes
Vol. 27 (2026), No. 1, pp. 359–371

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2026.5096

FURTHER RESULTS ON I-CONVERGENCE OF MULTISET SEQUENCES

SUMIT SOM AND ARGHA GHOSH

Received 25 June, 2024

Abstract. The notions of multiset and multiset sequences play an important role in the theory of computation and information sciences. To study convergence properties of multiset sequences, notions of statistical convergence, statistical limit points, and cluster points for multiset sequences were introduced by Debnath and Debnath [5]. Later, in [6], Demir and Gümüş introduced the notions of I-convergence and I^*-convergence for multiset sequences to generalize the results in [5] and investigated the connections between these two notions. In this paper, we extend the results in [6] and introduce and study the notion of I-limit points and I-cluster points for multiset sequences. Further, we introduce and study the notions of I-Cauchy and I^*-Cauchy multiset sequences, and establish relationships with the notions of I-convergence and I^*-convergence of multiset sequences.

2010 *Mathematics Subject Classification:* 03E99; 40D05; 40D99

Keywords: multiset, multiset sequences, ideal Cauchy, ideal convergence, ideal limit point, ideal cluster point

1. INTRODUCTION

According to classical set theory, a set is a well-defined collection of distinct objects, and there is no ordering of the objects in the set. Each object occurs only once in a set. But, in many situations, multiple occurrences of a particular object play an important role in our daily lives. For example, in the case of a cellphone number, one digit can occur multiple times. Also, we know that duplicates could appear at different points in the information retrieval process. In such cases, the notion of multisets comes into play.

Let X be a non-empty set. A multiset M with elements from the set X contains elements $x \in X$ with the multiplicity $C(x)$, where $C : X \to \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. The positive integer $C(x)$ represents the multiplicity of the element x. Consider the set $\{3,3,8,8,8,8,7,7\}$. This is a multiset as 3 occurs 2 times, 8 occurs 4 times, and 7 occurs 2 times. In this case, we represent this multiset as

$\{3|2,8|4,7|2\}$. Here, $C(3)=2, C(8)=4, C(7)=2$. Multiset theory can be employed in instances where classical set theory is inadequate. In 1980, Hickman [10] studied algebraic operations on multisets. In 1981, Knuth studied multisets related to the computer programming [11]. Bender [1] investigated the partitions of multisets. In 1976, Lake [14] gave axiomatization of the theory of multisets. In [15], Majumdar introduced the notion of soft multisets, and studied distance, and similarity between two soft multisets. The theory of multiset has many applications in computer and information sciences, interested readers can see [1, 2, 20, 22, 23], and many other references therein. In 2021, Pachilangode and John [17] introduced the notion of distance $d_M(x,y)=d(x,y)+|C(x)-C(y)|$ on a multiset M whose elements are from a metric space (Z,d), and in addition, they introduced and studied the notions of Wijsman convergence, and Hausdorff convergence in the realm of multisets.

On the other hand, the idea of convergence of a sequence of real numbers has been extended to statistical convergence by Fast [8], and Steinhaus [21] independently, and later on re-introduced by Schoenberg [19], and is based on the notion of asymptotic density of the subset of natural numbers. Let $K \subseteq \mathbb{N}$. The asymptotic density of K is defined as $d(K)=\lim_{n\to\infty}\frac{1}{n}|\{k\leq n : k\in K\}|$ if the limit exists. In 1980, Šalát [18] has considered the set of all statistically convergent sequences in l_∞ over the sup norm and showed that the set is dense in l_∞. In 1985, Fridy [9] has defined the notion of statistically Cauchy sequence and investigated the relationships between statistical convergence and statistically Cauchy sequence. In 2000, Kostyrko et al. [12] generalized the notion of statistical convergence of sequences of real numbers by introducing the notion of I-convergence (I is an ideal on the set of natural numbers $\mathbb{N}$) for sequence of elements in a metric space (Z,d). In addition, they introduced and studied the notions of I-limit points and I-cluster points for sequence of elements in a metric space (Z,d). In [7], Dems introduced and studied the notion of I-Cauchy sequences of real numbers and established its relationships with the notion of I-convergence of sequence of real numbers. Later in 2007, Nabiev et al. [16] introduced and studied the notions of I-Cauchy sequence and I^*-Cauchy sequence of elements in a metric space (Z,d). Moreover, they have established relationships between the notions of I-convergence and I-Cauchy. For further studies in the direction, one can see [3,4,13], and references therein.

In 2021, Debnath and Debnath [5] introduced the notion of statistical convergence for multiset sequences, and established various properties of this new convergence. Moreover, they defined the notion of statistically boundedness in case of multiset sequences, and established the relation between statistically boundedness and statistical convergence of multiset sequences. Later in 2023, Demir and Gümüş [6] introduced and discussed the notion of ideal convergence for multiset sequences.

The notion of multisets plays an important role in the realm of computer and information sciences. For example, some domain-specific language, such as Structured Query Language (SQL), operates on multisets and displays identical data. To see some applications of multisets, we refer interested readers to [20]. In recent years one can see a surge in studies on the notion of multisets. Naturally, some people desire to investigate further in the area of multisets. In this paper, we plan to do so. The significance of this work lies in the fact that further investigation on multisets may lead to advancements in the realm of computer science and information sciences. We note that this work advances the work done in [6]. This paper is organized as follows: In section 2, we discuss some preliminary notions about ideal and multiset sequence. In section 3, we introduce the notions of I-limits and I-cluster points for multiset sequences and study the concepts. We also investigate the between ideal convergence and I-limit points, and I-cluster points. In section 4, we introduce the notions of I-Cauchy and I^*-Cauchy multiset sequences. We prove that a multiset sequence is I-convergent if and only if it is I-Cauchy. Also, we show that every I^*-Cauchy multiset sequence is I-Cauchy. We provide an example to show that there exists an ideal I for which the notions of I-Cauchy and I^*-Cauchy are different. Furthermore, we establish that the notions of I-Cauchy, and I^*-Cauchy are equivalent if I satisfies weakly additive property.

2. Preliminaries

In this section, we consider some basic notions and results that help us to understand the paper thoroughly.

Definition 1 ([12])**.** Let Z be a non-empty set. A non-void family I of subsets of Z is said to be an ideal on Z if the following conditions hold:

(1) $A,B \in I \Longrightarrow A \cup B \in I$;
(2) $A \in I, B \subseteq A \Longrightarrow B \in I$.

Definition 2 ([12])**.** Let I be an ideal on Z. Then, I is said to be admissible if $\{z\} \in I$ for each $z \in Z$.

Clearly, $\varnothing \in I$, so, I is always non-empty. If $Z \notin I$ and $I \neq \{\varnothing\}$, then I will be called a non-trivial proper ideal on Z. From now on, we always consider an ideal to be non-trivial, proper, and admissible unless otherwise stated.

Definition 3 ([12])**.** Let Z be a non-empty set. A family $\mathcal{F}$ of subsets of Z is said to be a filter on Z if the following conditions hold:

(1) $\varnothing \notin \mathcal{F}$;
(2) $A,B \in \mathcal{F} \Longrightarrow A \cap B \in \mathcal{F}$;
(3) $A \in \mathcal{F}, A \subseteq B \Longrightarrow B \in \mathcal{F}$.

Let I be an ideal on Z. Then, the family $\mathcal{F}(I) = \{A \subset Z : Z \setminus A \in I\}$ is a filter on Z. We say that $\mathcal{F}(I)$ is the filter associated with ideal I.

Definition 4 ([12])**.** admissible ideal I on $\mathbb{N}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets in I, there is a sequence

$\{B_1, B_2, \dots\}$ of sets such that for each $i \in \mathbb{N}$, the symmetric difference $A_i \Delta B_i$ is finite and $\bigcup_i A_i = \bigcup_i B_i \in I$.

Definition 5 ([12])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I-convergent to $a \in Z$ if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : d(z_n, a) \geq \varepsilon\} \in I$.

Definition 6 ([12])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I^*-convergent to $a \in Z$ if a set $M = \{k_1 < k_2 < \dots\} \in \mathcal{F}(I)$ such that (z_{k_n}) is convergent to a.

Definition 7 ([12])**.** Let (Z,d) be a metric space, I be an admissible ideal on $\mathbb{N}$, and (z_n) be a sequence in Z.

(1) An element $a \in Z$ is said to be an I-limit point of (z_n) if there exists a set $M = \{k_1 < k_2 < \dots\} \notin I$ such that (z_{k_n}) is convergent to a.
(2) An element $a \in Z$ is said to be an I-cluster point of (z_n) if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : d(z_n, a) < \varepsilon\} \notin I$.

Definition 8 ([16])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I-Cauchy if for each $\varepsilon > 0$ there exists a $k = k(\varepsilon) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : d(z_n, z_k) \geq \varepsilon\} \in I.$$

Definition 9 ([16])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I^*-Cauchy if there exists a set $M = \{k_1 < k_2 < \dots\} \in \mathcal{F}(I)$ such that (z_{k_n}) is a Cauchy subsequence of (z_n).

Proposition 1 ([16])**.** *Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. If (z_n) is I^*-Cauchy, then (z_n) is I-Cauchy. In addition, if we consider the ideal I with property (AP), then the concepts I-Cauchy sequence and I^*-Cauchy sequence coincide.*

Definition 10 ([5])**.** A set of real numbers where repetition of real numbers is allowed, is called a multiset of real numbers, denoted by $m\mathbb{R}$. Thus,

$$m\mathbb{R} = \{x|c : x \in \mathbb{R} \wedge c \in \mathbb{N}_0\}.$$

Definition 11 ([5])**.** A function whose domain is the set $\mathbb{N}$ of natural numbers and co-domain $m\mathbb{R}$ is said to be a multiset sequence of $m\mathbb{R}$ (in short, multisequence). We denote a multiset sequence by $H = \{x_n|c_n\}$, where $x_n \in \mathbb{R}$ and $c_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.

Let (Z,d) be a metric space. On a multiset M whose elements are from Z one can define different types of metric, for more details see [17]. In this paper, we consider the metric

$$d_M(x_n|c_n, y_n|d_n) = \sqrt{(x_n - y_n)^2 + (c_n - d_n)^2}$$

on M.

Definition 12 ([6])**.** Let I be an ideal on $\mathbb{N}$. A multiset sequence by $H = \{x_n|c_n\}$ is said to be I-convergent to $l|c$ if for each $\varepsilon > 0$,

$$\{n \in \mathbb{N} : d_M(x_n|c_n, l|c) \geq \varepsilon\} \in I.$$

Definition 13 ([6])**.** Let I be an ideal on $\mathbb{N}$. A multiset sequence by $H = \{x_n|c_n\}$ is said to be I^*-convergent to $l|c$ if there exists a set $M = \{m_1 < m_2 < \dots\} \in \mathcal{F}(I)$ such that the multisubsequence $\{x_{m_i}|c_{m_i}\}$ is convergent to $l|c$.

Throughout the paper, whenever we use the term multiset sequence, we mean multiset sequence of $m\mathbb{R}$ unless otherwise stated.

3. I-LIMIT POINTS AND I-CLUSTER POINTS

We introduce the notions of I-limit points and I-cluster points for a sequence of multisets as follows:

Definition 14. Let $H = \{x_n|c_n\}$ be a multiset sequence. An element $\{x|c\}$ is said to be an I-limit point of H if there exists a set $T = \{h_1 < h_2 < \dots\} \subset \mathbb{N}, T \notin I$ such that the multiset sequence $\{x_{h_i}|c_{h_i}\}$ is convergent to $x|c$; i.e, $\forall \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_M(x_{h_i}|c_{h_i}, x|c) < \varepsilon$ for all $i \geq n_0$. The collection of all I-limit point of H is denoted by $I(\Lambda_H^m)$.

Definition 15. Let $H = \{x_n|c_n\}$ be a multiset sequence. An element $\{x|c\}$ is said to be an I-cluster point of H if for all $\varepsilon > 0$, $\{n \in \mathbb{N} : d_M(x_n|c_n, x|c) < \varepsilon\} \notin I$. The collection of all I-cluster point of H is denoted by $I(\Gamma_H^m)$.

Theorem 1. *Let I be any proper nontrivial admissible ideal and $H = \{x_n|c_n\}$ be a multiset sequence. Then, $I(\Lambda_H^m) \subset I(\Gamma_H^m)$.*

Proof. Let $x|c$ be an I-limit point of H. Then, there exists a set $T = \{h_1 < h_2 < \dots\} \subset \mathbb{N}, T \notin I$ such that $d_M(x_{h_i}|c_{h_i}, x|c) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. So, there exists $n_0 \in \mathbb{N}$ such that $d_M(x_{h_i}|c_{h_i}, x|c) < \varepsilon$ for all $i > n_0$. We have $T \setminus \{h_1, h_2, \dots, h_{n_0}\} \subseteq \{i \in \mathbb{N} : d_M(x_i|c_i, x|c) < \varepsilon\}$. But, if $\{i \in \mathbb{N} : d_M(x_i|c_i, x|c) < \varepsilon\} \in I$, then this will imply $T \in I$. So, $\{i \in \mathbb{N} : d_M(x_i|c_i, x|c) < \varepsilon\} \notin I$. So, $x|c$ be an I-cluster point of H. □

But, converse of Theorem 1 is not true. Let $\mathfrak{I} = \{J_1, J_2, J_3, \dots\}$ be mutually disjoint partition of $\mathbb{N}$ such that each of J_i is infinite. If we take

$$I = \{A \subset \mathbb{N} : \text{ there exist } l_1, l_2, \dots, l_p \text{ such that } A \cap J_{l_i} \neq \phi \text{ for all } i = 1, 2, \dots, p\},$$

then I becomes a proper nontrivial admissible ideal. Define a multiset sequence H with same multiplicity p by

$$x_n = \begin{cases} \frac{1}{i}, & \text{if } n \in J_i \text{ and } i \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

Now, consider the multiset $\{0|p\}$. Then, for any $\varepsilon > 0$, we have $\{i \in \mathbb{N} : d_M(x_i|p, 0|p) < \varepsilon\} \notin I$. So, $0|p$ is an I-cluster point of H. Also, it can be easily verified that $0|p$ is not an I-limit point of H.

Theorem 2. *Let $H = \{x_n|c_n\}$ be a multiset sequence and I be a proper nontrivial admissible ideal in $\mathbb{N}$. Let H is I-convergent to $x|c$. Then, $x|c$ is an I-cluster point of H.*

Proof. Since H is I-convergent to $x|c$, for $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \sqrt{(x_n - x)^2 + (c_n - c)^2} \geq \varepsilon\right\} \in I.$$

Therefore, $\left\{n \in \mathbb{N} : \sqrt{(x_n - x)^2 + (c_n - c)^2} < \varepsilon\right\} \notin I$. Otherwise, $\mathbb{N} \in I$ which is not possible. This shows that $x|c$ is an I-cluster point of H. □

Theorem 3. *Let $H = \{x_n|c_n\}$ be a multiset sequence and I be a proper nontrivial admissible ideal in $\mathbb{N}$. Let H is I^*-convergent to $x|c$. Then, $x|c$ is an I-limit point of H.*

Theorem 4. *Let $H = \{x_n|c_n\}$ be a multiset sequence and $I, \mathcal{J}$ be two ideals in $\mathbb{N}$ with $\mathcal{J} \subset I$. If $\{x|c\}$ is an I-limit point (I-cluster point) of H, then $\{x|c\}$ is an $\mathcal{J}$-limit point ($\mathcal{J}$-cluster point) of H.*

Proof. Let $H = \{x_n|c_n\}$ be a multiset sequence and $I, \mathcal{J}$ be two ideals in $\mathbb{N}$ with $\mathcal{J} \subset I$. Let $\{x|c\}$ is an I-limit point of H. Then, there exists a set $T = \{t_1 < t_2 < \dots\} \notin I$ such that the multisubsequence $\{x_{t_i}|c_{t_i}\}$ is convergent to $\{x|c\}$. Since $T = \{t_1 < t_2 < \dots\} \notin \mathcal{J}$, so, $\{x|c\}$ is also, an $\mathcal{J}$-limit point of H. Similarly, we can show that $\{x|c\}$ is an $\mathcal{J}$-cluster point of H. □

Now, we introduce the notion of $I - \limsup$ and $I - \liminf$ for a multiset sequence H with respect to a proper nontrivial admissible ideal I. The introduced notion will improve the corresponding notion in [5] as the results will become a particular case for the density zero ideal I_d. We start with a multiset sequence $H = \{x_n|c_n\}$. Consider the two sets,

$$B_H = \left\{x|c : \left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} > \sqrt{x^2 + (c-1)^2}\right\} \notin I\right\}$$

and

$$A_H = \left\{x|c : \left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} < \sqrt{x^2 + (c-1)^2}\right\} \notin I\right\}.$$

$\sup B_H$ and $\inf A_H$ is defined in the same way as in [5].

Definition 16. Let $H = \{x_n|c_n\}$ be a multiset sequence and I be a proper nontrivial admissible ideal in $\mathbb{N}$. We define

$$I - \limsup H = \begin{cases} \sup B_H, & \text{if } B_H \neq \phi, \\ -\infty, & \text{if } B_H = \phi. \end{cases}$$

$$I-\liminf H=\begin{cases}\inf A_H, & \text{if } A_H\neq\phi,\\ +\infty, & \text{if } A_H=\phi.\end{cases}$$

Example 1. Consider the ideal $I=I_f$ of all finite subsets of $\mathbb{N}$. Let $H=\{x_n|c_n\}$ be defined by

$$(x_i|c_i)=\begin{cases}0|4, \text{ if } i \text{ is even},\\ 1|3 \text{ if } i \text{ is odd}.\end{cases}$$

Here, it can be shown that

$$B_H=\left\{[-2,2]|1,[-2,2]|2,[-2,2]|3,[-\sqrt{2},\sqrt{2}]|1,[-\sqrt{2},\sqrt{2}]|2\right\}.$$

So, $I-\limsup H=2|3$. Also,

$$A_H=\left\{(-\infty,-\sqrt{5}]|4,\dots,[\sqrt{5},\infty)|4,[\sqrt{5},\infty)|5,\dots,(-\infty,-\sqrt{3}]|3,\dots\right\}.$$

So, $I-\liminf H=\sqrt{5}|4$.

Theorem 5. *Let $H=\{x_n|c_n\}$ be a multiset sequence and I be a nontrivial proper admissible ideal on $\mathbb{N}$. Then, $I-\limsup H$ and $I-\liminf H$ are unique.*

Proof. Proof is straightforward, so omitted. □

Theorem 6. *Let $H=\{x_n|c_n\}$ be a multiset sequence. If $I-\limsup H=p|r$, then for every $\varepsilon>0$,*

(1) $\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p-\varepsilon)^2+(r-1)^2}\right\}\notin I.$

(2) $\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p+\varepsilon)^2+(r-1)^2}\right\}\in I.$

Proof. Since $I-\limsup H=p|r$. Then, r is the greatest multiplicities in B_H and p is the supremum of all real numbers whose multiplicity is r. Let $\varepsilon>0$. So, there exists a real number q with $(p-\varepsilon)<q$ and $q|r\in B_H$. Since $\sqrt{(p-\varepsilon)^2+(r-1)^2}<\sqrt{q^2+(r-1)^2}$, $\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{q^2+(r-1)^2}\right\}\subset\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p-\varepsilon)^2+(r-1)^2}\Big\}$. This shows that

$$\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p-\varepsilon)^2+(r-1)^2}\right\}\notin I.$$

On the other hand for $\varepsilon>,0$ if

$$\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p+\varepsilon)^2+(r-1)^2}\right\}\notin I,$$

then $(p+\varepsilon)|r\in B_H$, and this will contradict the fact that p is the supremum of all real numbers whose multiplicity is r. So,

$$\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p+\varepsilon)^2+(r-1)^2}\right\}\in I.$$

□

Theorem 7. *Let $H = \{x_n|c_n\}$ be a multiset sequence. If $I - \liminf H = p|r$, then for every $\varepsilon > 0$,*

(1) $\left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} > \sqrt{(p+\varepsilon)^2 + (r-1)^2}\right\} \notin I$.

(2) $\left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} > \sqrt{(p-\varepsilon)^2 + (r-1)^2}\right\} \in I$.

Proof. Proof is similar to Theorem 6, so omitted. □

4. I-Cauchy and I^*-Cauchy multiset sequences

In this section, we introduce and study the notions of I-Cauchy and I^*-Cauchy sequences. Furthermore, we establish relationships between the notions of I-convergence, I^*-convergence, I-Cauchy, and I^*-Cauchy multiset sequences.

Definition 17. Let $H = \{x_n|c_n\}$ be a multiset sequence. Then, $H = \{x_n|c_n\}$ is said to be I-Cauchy if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\{n \in \mathbb{N} : \sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \geq \varepsilon\right\} \in I.$$

Example 2. Let I be an admissible ideal on $\mathbb{N}$ such that $A \in I$, where A is an infinite subset of $\mathbb{N}$. Let $\{x_n\}$ be any I-Cauchy sequence of real numbers. Define a sequence of positive integers $\{c_n\}$ as follows:

$$c_n = \begin{cases} n, & \text{if } n \in A \\ 10, & \text{otherwise.} \end{cases}$$

Clearly, the multiset sequence $H = \{x_n|c_n\}$ is I-Cauchy.

Remark 1. Since I is an admissible ideal, every multiset Cauchy sequence is I-Cauchy. In particular, if I_f is the ideal of all finite subsets of $\mathbb{N}$, then the notions of multiset Cauchy sequences and I_f-Cauchy sequences are equivalent. Let I_d be the collection of all density zero subsets of $\mathbb{N}$. Then, I_d is a non-trivial admissible ideal on $\mathbb{N}$. A multiset sequence is said to be statistically Cauchy if and only if it is I_d-Cauchy.

Theorem 8. *A multiset sequence $H = \{x_n|c_n\}$ is I-convergent if and only if it is I-Cauchy.*

Proof. Let $H = \{x_n|c_n\}$ is convergent to $l|c$. Then, for any $\varepsilon > 0$,

$$A\left(\frac{\varepsilon}{2}\right) = \left\{n \in \mathbb{N} : \sqrt{(x_n - l)^2 + (c_n - c)^2} \geq \frac{\varepsilon}{2}\right\} \in I.$$

Choose $N \notin A\left(\frac{\varepsilon}{2}\right)$ and fix it. Then, $\sqrt{(x_N-l)^2+(c_N-c)^2} < \frac{\varepsilon}{2}$. Also, for all $n \notin A\left(\frac{\varepsilon}{2}\right)$, we have $\sqrt{(x_n-l)^2+(c_n-c)^2} < \frac{\varepsilon}{2}$. Then, by the property (triangle inequality) of the usual norm in $\mathbb{R}^2$, we have

$$\sqrt{(x_n-x_N)^2+(c_n-c_N)^2} \le \sqrt{(x_n-l)^2+(c_n-c)^2}+\sqrt{(x_N-l)^2+(c_N-c)^2} < \varepsilon$$

for all $n \notin A\left(\frac{\varepsilon}{2}\right)$. Consequently, $\left\{n \in \mathbb{N} : \sqrt{(x_n-x_N)^2+(c_n-c_N)^2} \ge \varepsilon\right\} \subset A\left(\frac{\varepsilon}{2}\right)$. Since $A\left(\frac{\varepsilon}{2}\right) \in I$, $\left\{n \in \mathbb{N} : \sqrt{(x_n-x_N)^2+(c_n-c_N)^2} \ge \varepsilon\right\} \in I$. Hence, $H = \{x_n|c_n\}$ is I-Cauchy.

Conversely, let $H = \{x_n|c_n\}$ be I-Cauchy. Then, for any $\varepsilon > 0$, there exists a positive integer k_ε such that

$$A(\varepsilon) = \left\{n \in \mathbb{N} : \sqrt{(x_n-x_{k_\varepsilon})^2+(c_n-c_{k_\varepsilon})^2} \ge \varepsilon\right\} \in I.$$

Set $\varepsilon_m = \frac{1}{2^m}$ for $m \in \mathbb{N}$. Then, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$A(m) = \left\{n \in \mathbb{N} : \sqrt{(x_n-x_{k_m})^2+(c_n-c_{k_m})^2} \ge \frac{\varepsilon_m}{2}\right\} \in I.$$

Define recursively, $B_1 = \operatorname{cl} B(x_{k_1}, \frac{\varepsilon_1}{2})$, $E_1 = \operatorname{cl} B(c_{k_1}, \frac{\varepsilon_1}{2})$,

$$B_{m+1} = B_m \cap \operatorname{cl} B(x_{k_{m+1}}, \frac{\varepsilon_{m+1}}{2}),$$

and

$$E_{m+1} = E_m \cap \operatorname{cl} B(c_{k_{m+1}}, \frac{\varepsilon_{m+1}}{2})$$

for $m \in \mathbb{N}$. We claim that both B_m and E_m are nonempty for each $m \in \mathbb{N}$.

Indeed, since $A(1) \in I$, for each $n \notin A(1)$ we have $\sqrt{(x_n-x_{k_1})^2+(c_n-c_{k_1})^2} < \frac{\varepsilon_1}{2}$. Since both $|x_n-x_{k_1}|$ and $|c_n-c_{k_1}|$ are less than or equal to $\sqrt{(x_n-x_{k_1})^2+(c_n-c_{k_1})^2}$, $x_n \in B_1$ and $c_n \in E_1$. Similarly, $x_n \in \operatorname{cl} B(x_{k_{m+1}, \frac{\varepsilon_{m+1}}{2}})$ and $c_n \in \operatorname{cl} B(c_{k_{m+1}, \frac{\varepsilon_{m+1}}{2}})$ for all $n \notin A(m+1)$ and $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $C \in I$ such that $x_n \in B_m$ and $c_n \in E_m$ for each $n \notin C$. Then, $x_n \in B_{m+1}$ and $c_n \in E_{m+1}$ for all $n \notin C \cup A(m+1)$. Furthermore, since $\{B_m\}$ and $\{E_m\}$ are nested sequences of closed sets of real numbers with diameters tend to zero, there exist $l, c \in \mathbb{R}$ such that $\bigcap_m B_m = \{l\}$ and $\bigcap_m E_m = \{c\}$. We show that $H = \{x_n|c_n\}$ is I-convergent to $l|c$. Let $\delta > 0$ be given. Then, there exists $m \in \mathbb{N}$ such that $\varepsilon_m < \frac{\delta}{4}$. Let

$$B(\delta) = \left\{n \in \mathbb{N} : \sqrt{(x_n-l)^2+(c_n-c)^2} \ge \delta\right\}.$$

If $B(\delta)$ is empty, then $B(\delta) \in I$. Let $B(\delta) \neq \varnothing$. Then, for all $n \in B(\delta)$, we have $\sqrt{(x_n-l)^2+(c_n-c)^2} \ge \delta$. Since

$$\sqrt{(x_n-l)^2+(c_n-c)^2} \le \sqrt{(x_n-x_{k_m})^2+(c_n-c_{k_m})^2}+\sqrt{(l-x_{k_m})^2+(c-c_{k_m})^2},$$

either $\sqrt{(x_n - x_{k_m})^2 + (c_n - c_{k_m})^2} \geq \frac{\delta}{2} > 2\varepsilon_m$ or $\sqrt{(l - x_{k_m})^2 + (c - c_{k_m})^2} \geq \frac{\delta}{2} > 2\varepsilon_m$. But $l \in \mathrm{cl}\,B(x_{k_m}, \frac{\varepsilon_m}{2})$ and $c \in \mathrm{cl}\,B(c_{k_m}, \frac{\varepsilon_m}{2})$, that is, $|x_{k_m} - l| < \varepsilon_m$ and $|c_{k_m} - c| < \varepsilon_m$. Therefore, $\sqrt{(l - x_{k_m})^2 + (c - c_{k_m})^2} \leq |x_{k_m} - l| + |c_{k_m} - c| < 2\varepsilon_m$. Thus,

$$B(\delta) \subset \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_{k_m})^2 + (c_n - c_{k_m})^2} > 2\varepsilon_m \right\} \subset A_m \in I.$$

Hence, the multiset sequence $H = \{x_n|c_n\}$ is I-convergent to $l|c$. □

Definition 18. A multiset sequence $H = \{x_n|c_n\}$ is said to be I^*-Cauchy if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(I)$ such that the submultiset sequence $\{x_{m_k}|c_{m_k}\}$ is Cauchy.

Proposition 2. *A multiset sequence $H = \{x_n|c_n\}$ is I^*-convergent if and only if it is I^*-Cauchy.*

Theorem 9. *If a multiset sequence $H = \{x_n|c_n\}$ is I^*-Cauchy, then it is I-Cauchy.*

Proof. Since $H = \{x_n|c_n\}$ is I^*-Cauchy, there exists a set

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(I)$$

such that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $k > n_\varepsilon$, we have

$$\sqrt{(x_{m_k} - x_{m_{n_\varepsilon}})^2 + (c_{m_k} - c_{m_{n_\varepsilon}})^2} < \varepsilon.$$

Set $N = m_{n_\varepsilon} + 1$ and

$$A(\varepsilon) = \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \geq \varepsilon \right\}.$$

Clearly, $A(\varepsilon) \subset (\mathbb{N} \setminus M) \cup \{m_1, m_2, \cdots, m_{n_\varepsilon}\}$. Since the latter set is in I, so does $A(\varepsilon)$. Hence, $H = \{x_n|c_n\}$ is I-Cauchy. □

The following example shows that the notions of I-Cauchy and I^*-Cauchy of multiset sequences are not equivalent in general:

Example 3. Let $\mathbb{N} = \bigcup_j A_j$, be a decomposition of $\mathbb{N}$ such that each A_j is infinite and $A_i \cap A_j = \varnothing$ for $i \neq j$. Let I be the class of all subsets A of $\mathbb{N}$ which intersects at most finite number of A_j's. It is easy to verify that I is a non-trivial admissible ideal of $\mathbb{N}$. Construct a multiset sequence $H = \{x_n|c_n\}$ as follows: $x_n = \frac{1}{j}$ if $n \in A_j$ and $c_n = 10$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then, there exists $j \in \mathbb{N}$ such that $\frac{2}{j+1} < \varepsilon$. Let m_0 be the least positive integer such that $m_0 \in A_{j+1}$. Let $C(\varepsilon) = \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_{m_0})^2 + (c_n - c_{m_0})^2} \geq \varepsilon \right\}$. Clearly, $C(\varepsilon) \subset A_1 \cup A_2 \cup \cdots \cup A_j$. Since the latter set belongs to I, so does $C(\varepsilon)$. Therefore, the multiset sequence $H = \{x_n|c_n\}$ is I-Cauchy.

Now we show that $H = \{x_n|c_n\}$ is not I^*-Cauchy. Suppose, on contrary $H = \{x_n|c_n\}$ is not I^*-Cauchy. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in$

$\mathcal{F}(I)$ such that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $k > n_\varepsilon$, we have $\sqrt{(x_{m_k} - x_{m_{n_\varepsilon}})^2 + (c_{m_k} - c_{m_{n_\varepsilon}})^2} < \varepsilon$. From the definition of I, we have $\mathbb{N} \setminus M \subset A_1 \cup A_2 \cup \cdots \cup A_l$ for some $l \in \mathbb{N}$. Thus, $A_i \subset M$ for all $i > l$. Therefore, there are infinitely many terms of the sequence $\{x_{m_k}\}$ equal to $\frac{1}{l+1}$ and $\frac{1}{l+2}$. Let $0 < \varepsilon_0 < \frac{1}{(l+1)(l+2)}$. Then, there does not exit any $n_{\varepsilon_0} \in \mathbb{N}$ such that $\sqrt{(x_{m_k} - x_{m_{n_{\varepsilon_0}}})^2 + (c_{m_k} - c_{m_{n_{\varepsilon_0}}})^2} < \varepsilon_0$ holds for all $k > n_{\varepsilon_0}$, which is a contradiction. Hence, $H = \{x_n|c_n\}$ is not I^*-Cauchy.

We now introduce the following definition to provide a sufficient condition for a multiset I-Cauchy sequence to be an I^*-Cauchy.

Definition 19. An admissible ideal I on $\mathbb{N}$ is said to have the weakly additive property (WAP) if for every sequence of mutually disjoint sets $\{P_i\}$ in I, there exists a set $P \in \mathcal{F}(I)$ such that $P \setminus P_i$ is finite for all $i \in \mathbb{N}$.

From [16, Lemma 4], we observe that if an admissible ideal has the property (AP), then it has the property (WAP).

Theorem 10. *Let I be an admissible ideal and it has the property (WAP). If a multiset sequence $H = \{x_n|c_n\}$ is I-Cauchy, then it is I^*-Cauchy.*

Proof. Since $H = \{x_n|c_n\}$ is I-Cauchy, for any $\varepsilon > 0$, there exists a positive integer m_ε such that

$$A(\varepsilon) = \left\{ n : \mathbb{N} : \sqrt{(x_n - x_{m_\varepsilon})^2 + (c_n - c_{m_\varepsilon})^2} \geq \varepsilon \right\} \in I.$$

Set $P_i = \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_{m_i})^2 + (c_n - c_{m_i})^2} < \frac{1}{i} \right\}$ for all $i \in \mathbb{N}$. Clearly, $P_i \in \mathcal{F}(I)$ for all $i \in \mathbb{N}$. Since I has the property (WAP), we have a set $P \in \mathcal{F}(I)$ such that $P \setminus P_i$ is finite for all $i \in \mathbb{N}$. Let $\delta > 0$ be given. Then, there exists $j \in \mathbb{N}$ such that $\frac{1}{j} < \frac{\delta}{2}$. Since $P \setminus P_j$ is finite, there exists a positive integer k_j such that $m, n \in P_j$ for all $m, n \in P$ and $m, n > k_j$. Therefore,

$$\sqrt{(x_m - x_{m_j})^2 + (c_m - c_{m_j})^2} < \frac{1}{j}$$

and

$$\sqrt{(x_n - x_{m_j})^2 + (c_n - c_{m_j})^2} < \frac{1}{j}$$

for all $m, n \in P$ and $m, n > k_j$. Hence, by a property (triangle property) of the usual norm of $\mathbb{R}^2$, we have $\sqrt{(x_m - x_n)^2 + (c_m - c_n)^2} < \frac{2}{j} < \delta$ for all $m, n \in P$ and $m, n > k_j$. This proves that $H = \{x_n|c_n\}$ is I^*-Cauchy. □

Corollary 1. *Let I be an admissible ideal and it has the property (WAP). Then, the notions of I-Cauchy, I-convergent, I^*-Cauchy, and I^*-convergent of multiset sequences are equivalent.*

CONCLUSION

In this paper, we introduced and studied the notions of I-limit points and I-cluster points for multiset sequences. Also, we introduced and studied the notions of I-Cauchy and I^*-Cauchy multiset sequences. We proved some basic results. We observed that if I is an admissible ideal and has weak additive property (WAP), then the notions of I-Cauchy, I-convergent, I^*-Cauchy, and I^*-convergent of multiset sequences are equivalent. Let (Z,d) be a metric space. On a multiset M whose elements are from Z, in this paper, we considered the metric

$$d_M(x_n|c_n, y_n|d_n) = \sqrt{(x_n - y_n)^2 + (c_n - d_n)^2}$$

on M. However, one may start with a probabilistic metric space and consider a multiset whose elements are from the probabilistic metric space. Now the question is, can we define a probabilistic metric on the multiset? In addition, we leave the following question to the interested reader: If the notions of I-Cauchy, I-convergent, I^*-Cauchy, and I^*-convergent of multiset sequences are equivalent for an admissible ideal I, then can we say that I has the property (WAP)?

ACKNOWLEDGEMENTS

The authors thank the referees for their valuable feedback.

REFERENCES

[1] E. A. Bender, “Partitions of multisets,” *Discrete Mathematics*, vol. 9, no. 4, pp. 301–311, 1974, doi: 10.1016/0012-365X(74)90076-4.
[2] C. S. Calude, G. Paun, G. Rozenberg, and A. Salomaa, *Multiset Processing LNCS 2235*. Springer Verlag, 2001.
[3] P. Das, “Some further results on ideal convergence in topological spaces,” *Topology and its Applications*, vol. 159, no. 10-11, pp. 2621–2626, 2012, doi: 10.1016/j.topol.2012.04.007.
[4] P. Das and S. K. Ghosal, “Some further results on I-cauchy sequences and condition (AP),” *Computers & Mathematics with Applications*, vol. 59, no. 8, pp. 2597–2600, 2010, doi: 10.1016/j.camwa.2010.01.027.
[5] S. Debnath and A. Debnath, “Statistical convergence of multisequences on $\mathbb{R}$,” *Applied Sciences*, vol. 23, pp. 17–28, 2021.
[6] N. Demir and H. Gümüş, “Ideal convergence of multiset sequences,” *Filomat*, vol. 37, no. 30, pp. 10 199–10 207, 2023, doi: 10.2298/FIL2330199D.
[7] K. Dems, “On I-Cauchy sequences.” *Real analysis exchange*, vol. 30, no. 1, pp. 123–128, 2005, doi: 10.14321/realanalexch.30.1.0123.
[8] H. Fast, “Sur la convergence statistique,” in *Colloquium mathematicae*, vol. 2, no. 3-4, doi: 10.4064/cm-2-3-4-241-244, 1951, pp. 241–244.
[9] J. A. Fridy, “On statistical convergence,” *Analysis*, vol. 5, no. 4, pp. 301–314, 1985, doi: 10.1524/anly.1985.5.4.301.
[10] J. L. Hickman, “A note on the concept of multiset,” *Bulletin of the Australian Mathematical society*, vol. 22, no. 2, pp. 211–217, 1980, doi: 10.1017/S000497270000650X.
[11] D. E. Knuth, *The art of computer programming*. Pearson Education, 2005.

[12] P. Kostyrko, W. Wilczyński, and T. Šalát, "I-convergence," *Real Anal. Exchange*, vol. 26, no. 2, pp. 669–685, 2001.

[13] B. K. Lahiri and P. Das, "I and I^*-convergence in topological spaces," *Mathematica Bohemica*, vol. 130, no. 2, pp. 153–160, 2005, doi: 10.21136/MB.2005.134133.

[14] J. Lake, "Sets, fuzzy sets, multisets and functions," *Journal of the London Mathematical Society*, vol. 12, no. 3, pp. 323–326, 1976, doi: 10.1112/jlms/s2-12.3.323.

[15] P. Majumdar, "Soft multisets," *J. Math. Comput. Sci.*, vol. 2, no. 6, pp. 1700–1711, 2012.

[16] A. Nabiev, S. Pehlivan, and M. Gürdal, "On I-Cauchy sequences," *Taiwanese Journal of Mathematics*, vol. 11, no. 2, pp. 569–576, 2007, doi: 10.11650/twjm/1500404709.

[17] S. Pachilangode and S. J. John, "Convergence of multiset sequences," *Journal of New Theory*, no. 34, pp. 20–27, 2021.

[18] T. Šalát, "On statistically convergent sequences of real numbers," *Mathematica slovaca*, vol. 30, no. 2, pp. 139–150, 1980.

[19] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *The American mathematical monthly*, vol. 66, no. 5, pp. 361–375, 1959, doi: 10.2307/2308747.

[20] D. Singh, A. Ibrahim, T. Yohanna, and J. Singh, "An overview of the applications of multisets," *Novi Sad Journal of Mathematics*, vol. 37, no. 2, pp. 73–92, 2007.

[21] H. Steinhaus, "Sur la convergence ordinaire et la convergence asymptotique," in *Colloq. math*, vol. 2, no. 1, 1951, pp. 73–74.

[22] A. Syropoulos, *Mathematics of Multisets*. WMC 2000. Lecture Notes in Computer Science, Springer, Berlin, Heidelberg 2235, 2001.

[23] N. Wildberger, "A new look at multisets," *School of mathematics, UNSW Sydney*, vol. 2052, pp. 1–21, 2003.

Authors' addresses

Sumit Som

Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University, Bhopal-Indore Highway, Kothrikalan, Madhya Pradesh-466114, India
and Department of Mathematics, Dinabandhu Mahavidyalaya, Bongaon-743235, West Bengal, India

E-mail address: somkakdwip@gmail.com

Argha Ghosh

(**Corresponding author**) Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India.

E-mail address: buagbu@yahoo.co.in