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FURTHER RESULTS ON THE LEBESGUE-NAGELL EQUATION $dx^2 + p^{2m}q^{2n} = 4y^p$

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Abstract. Let d be a fixed positive integer with $d > 3$ is square-free, and let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Further, let p and q be odd primes such that $p > 3$, $p \neq q$ and $p \nmid h(-d)$. In this paper, we give a sufficient and necessary condition for the Lebesgue-Nagell equation $(*)$ $dx^2 + p^{2m}q^{2n} = 4y^p$ to have positive integer solutions (x,y,m,n) with $\gcd(x,y) = 1$. It can be seen from this condition that if $q \not\equiv \pm 1 \pmod{2p}$, then $(*)$ has no positive integer solutions (x,y,m,n) with $\gcd(x,y) = 1$.

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1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of all integers, positive integers and rational numbers, respectively. Let d be a fixed positive integer with $d > 3$ is square-free, and let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Further, let p and q be distinct odd primes such that $p > 3$ and $p \nmid h(-d)$. As we all know, the Lebesgue-Nagell equation is a class of polynomial-exponential Diophantine equations with a long history and rich content (see [1–5, 8–16, 18, 22–24] and the references of [19]). Recently, K. Chakraborty and A. Hoque [8] discussed in detail a Lebesgue-Nagell equation of the form

$$dx^2 + p^{2m}q^{2n} = 4y^p,\ x, y, m, n \in \mathbb{N},\ \gcd(x,y) = 1. \tag{1.1}$$

They proved that if one of the following conditions is satisfied, then (1.1) has no solutions (x,y,m,n).

(i) $d \equiv 1 \text{ or } 2 \pmod{4}$.

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(ii) $d \equiv 3 \pmod 4$ and $q^n \not\equiv \pm 1 \pmod p$.
(iii) $q = p+2$.
(iv) $h(-d) = 1$, $p > 41$, $q = d+p$ and $n = p$.
(v) $h(-d) \in \{1,2,4,8,16,32\}$, $q = 3$ and $n = p$.

Clearly, if (x,y,m,n) is a solution of (1.1), then the general equation

$$X^2 + dY^2 = 4Z^p,\ X,Y \in \mathbb{Z},\ \gcd(X,Y) = 1,\ Z \in \mathbb{N} \tag{1.2}$$

has a solution

$$(X,Y,Z) = (p^m q^n, x, y) \tag{1.3}$$

with $p \mid X$. In this paper, we first start with (1.2) to prove that

Theorem 1. *If* (X,Y,Z) *is a solution of* (1.2) *with* $p \mid X$, *then*

$$X = \lambda_1 a \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} a^{p-2i-1} \left(\frac{a^2+db^2}{4}\right)^i,$$

$$Y = \lambda_1 \lambda_2 b \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-db^2)^{(p-1)/2-i} \left(\frac{a^2+db^2}{4}\right)^i, \tag{1.4}$$

$$Z = \frac{a^2+db^2}{4},\ a,b \in \mathbb{N},\ \gcd(a,b) = 1,\ 2 \nmid ab,\ p \mid a,\ \lambda_1, \lambda_2 \in \{1,-1\},$$

where

$$\begin{bmatrix} p \\ i \end{bmatrix} = \frac{(p-i-1)!\,p}{(p-2i)!\,i!} \in \mathbb{N},\ i = 0,1,\ldots,\frac{p-1}{2}. \tag{1.5}$$

By Theorem 1, we can obtain the following results from the relation (1.3) for (1.1).

Theorem 2. *A sufficient and necessary condition for* (1.1) *to have solutions* (x,y,m,n) *is that there exist positive integers* b,r,s *which make*

$$pq^s = \left| \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} p^{r(p-2i-1)} \left(\frac{p^{2r}+db^2}{4}\right)^i \right|, \tag{1.6}$$

where $\begin{bmatrix} p \\ i \end{bmatrix}$ *is defined as in* (1.5). *Moreover, if* (1.6) *holds, then* (1.1) *has the solution*

$$(x,y,m,n) = \left(b \left| \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-db^2)^{(p-1)/2-i} \left(\frac{p^{2r}+db^2}{4}\right)^i \right|, \frac{p^{2r}+db^2}{4}, r+1, s \right). \tag{1.7}$$

Corollary 1. *If* $q \not\equiv \pm 1 \pmod{2p}$, *then* (1.1) *has no solutions* (x,y,m,n).

Obviously, Corollary 1 covers and expands on the results $(i)-(iv)$ in [8].

2. PRELIMINARIES

Lemma 1 ([20, Formula 3.76]). *For any positive integer t and any complex numbers ε and $\bar{\varepsilon}$, we have*

$$\varepsilon^t + \bar{\varepsilon}^t = \sum_{i=0}^{[t/2]} (-1)^i \begin{bmatrix} t \\ i \end{bmatrix} (\varepsilon + \bar{\varepsilon})^{t-2i} (\varepsilon\bar{\varepsilon})^i,$$

where

$$\begin{bmatrix} t \\ i \end{bmatrix} = \frac{(t-i-1)!\,t}{(t-2i)!\,i!} \in \mathbb{N},\ i = 0, \dots, \left[\frac{t}{2}\right],$$

$\left[\frac{t}{2}\right]$ *is the integer part of $t/2$.*

Lemma 2 ([17, Theorem 3.3], [21, Chapter 8]). *For any positive integer t, let F_t and L_t denote the t-th Fibonacci and Lucas numbers, respectively. Then we have*

- (i) *F_t and L_t are positive integers satisfying $L_t^2 - 5F_t^2 = (-1)^t 4$.*
- (ii) *The equation*

$$F_k = 5z^2,\ k, z \in \mathbb{N} \tag{2.1}$$

has only one solution $(k, z) = (5, 1)$.

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers, and α/β is not a root of unity, then (α, β) is called a Lucas pair. If $u = \alpha + \beta$ and $w = \alpha\beta$, then we have

$$\alpha = \frac{1}{2}(u + \lambda\sqrt{v}),\ \beta = \frac{1}{2}(u - \lambda\sqrt{v}),\ \lambda \in \{1, -1\}, \tag{2.2}$$

where $v = u^2 - 4w$.

Lemma 3 ([7]). *Let (α, β) be a Lucas pair with (2.2), and let ℓ be an odd prime. If $\ell \mid u$, then $(\alpha^\ell + \beta^\ell)/\ell u$ is a nonzero integer and its prime divisor q satisfies $q \equiv \pm 1 \pmod{2p}$.*

Let $\tilde{\alpha}$ and $\tilde{\beta}$ be algebraic numbers. If $(\tilde{\alpha} + \tilde{\beta})^2$ and $\tilde{\alpha}\tilde{\beta}$ are nonzero coprime integers and $\tilde{\alpha}/\tilde{\beta}$ is not root of unity, then $(\tilde{\alpha}, \tilde{\beta})$ is called a Lehmer pair. If $\tilde{u} = (\alpha + \beta)^2$ and $\tilde{w} = \tilde{u}\tilde{\beta}$, then we have

$$\tilde{\alpha} = \frac{1}{2}(\sqrt{\tilde{u}} + \tilde{\lambda}\sqrt{\tilde{v}}),\ \tilde{\beta} = \frac{1}{2}(\sqrt{\tilde{u}} - \tilde{\lambda}\sqrt{\tilde{v}}),\ \tilde{\lambda} \in \{1, -1\}, \tag{2.3}$$

where $\tilde{v} = \tilde{u} - 4\tilde{w}$. For any positive integer t, let

$$\tilde{L}_t(\tilde{\alpha}, \tilde{\beta}) = \begin{cases} \dfrac{\tilde{\alpha}^t - \tilde{\beta}^t}{\tilde{\alpha} - \tilde{\beta}}, & \text{if } 2 \nmid t, \\[2ex] \dfrac{\tilde{\alpha}^t - \tilde{\beta}^t}{\tilde{\alpha}^2 - \tilde{\beta}^2}, & \text{if } 2 \mid t. \end{cases} \tag{2.4}$$

Then, $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ $(t=1,2,\dots)$ are called the corresponding Lehmer numbers of Lehmer pair $(\tilde{\alpha},\tilde{\beta})$. It is well known that Lehmer numbers are nonzero integers. A prime q is called a primitive divisor of $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ $(t>2)$ if $q \mid \tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ and

$$q \nmid \tilde{u}\tilde{v}\prod_{j=1}^{t-1}\tilde{L}_t(\tilde{\alpha},\tilde{\beta}).$$

Lemma 4 ([6])**.** *For any Lehmer pair* $(\tilde{\alpha},\tilde{\beta})$*, if* $t>30$*, then* $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ *has primitive divisors.*

3. PROOF OF THEOREM 1

Let (X,Y,Z) be a solution of (1.2). Since $d>3$ and $p \nmid h(-d)$, by Lemma 1 of [8], we have

$$\frac{X+Y\sqrt{-d}}{2}=\lambda_1\left(\frac{a+\lambda_2 b\sqrt{-d}}{2}\right)^p,\ \lambda_1,\lambda_2\in\{1,-1\}, \tag{3.1}$$

$$Z=\frac{a^2+db^2}{4},\ a,b\in\mathbb{N},\ \gcd(a,b)=1,\ 2\nmid ab. \tag{3.2}$$

Further, let

$$\varepsilon=\lambda_1\left(\frac{a+\lambda_2 b\sqrt{-d}}{2}\right),\ \bar{\varepsilon}=\lambda_1\left(\frac{a-\lambda_2 b\sqrt{-d}}{2}\right). \tag{3.3}$$

Then we have

$$\varepsilon+\bar{\varepsilon}=\lambda_1 a,\ \varepsilon-\bar{\varepsilon}=\lambda_1\lambda_2 b\sqrt{-d},\ \varepsilon\bar{\varepsilon}=\frac{a^2+db^2}{4}. \tag{3.4}$$

Since

$$\frac{X-Y\sqrt{-d}}{2}=\lambda_1\left(\frac{a-\lambda_2 b\sqrt{-d}}{2}\right)^p \tag{3.5}$$

by (3.1), we get from (3.1), (3.3), (3.4) and (3.5) that

$$X=\varepsilon^p+\bar{\varepsilon}^p \tag{3.6}$$

and

$$Y=\frac{\varepsilon^p-\bar{\varepsilon}^p}{\sqrt{-d}}. \tag{3.7}$$

By Lemma 1, we obtain from (3.4), (3.6) and (3.7) that

$$X=\lambda_1 a\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\ i\end{bmatrix}a^{p-2i-1}\left(\frac{a^2+db^2}{4}\right)^i, \tag{3.8}$$

and

$$Y=\lambda_1\lambda_2 b\sum_{i=0}^{(p-1)/2}\begin{bmatrix}p\\ i\end{bmatrix}(-db^2)^{(p-1)/2-i}\left(\frac{a^2+db^2}{4}\right)^i, \tag{3.9}$$

where $\begin{bmatrix} p \\ i \end{bmatrix}$ is defined as in (1.5). By (1.5), we have

$$\begin{bmatrix} p \\ 0 \end{bmatrix} = 1,\ p \,\left|\, \begin{bmatrix} p \\ j \end{bmatrix}\right.,\ j = 1,\dots,\frac{p-1}{2}. \tag{3.10}$$

When $p \mid X$, by (3.8) and (3.10), we have $0 \equiv X \equiv \lambda_1 a^p \pmod{p}$, whence we get $p \mid a$. Thus, by (3.2), (3.8) and (3.9), we obtain (1.4). The theorem is proved.

4. PROOF OF THEOREM 2

The sufficiency of the theorem is obvious, and we will prove its necessity below. Let (x,y,m,n) be a solution of (1.1). Since $\gcd(x,y)=1$ and d is square free, by (1.1), we have

$$p \nmid d,\ p \nmid x,\ p \nmid y. \tag{4.1}$$

Then, it is well known that (1.2) has the solution (1.3) with $p \mid X$. Hence, by Theorem 1, we get from (1.3) and (1.4) that

$$p^m q^n = a\left|\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix} p \\ i \end{bmatrix}a^{p-2i-1}\left(\frac{a^2+db^2}{4}\right)^i\right|, \tag{4.2}$$

$$x = b\left|\sum_{i=0}^{(p-1)/2}\begin{bmatrix} p \\ i \end{bmatrix}(-db^2)^{(p-1)/2-i}\left(\frac{a^2+db^2}{4}\right)^i\right| \tag{4.3}$$

and

$$y = \frac{a^2+db^2}{4},\ a,b \in \mathbb{N},\ \gcd(a,b)=1,\ 2 \nmid ab,\ p \mid a. \tag{4.4}$$

Since $p \mid a$, we have

$$a = p^r f,\ r,f \in \mathbb{N},\ r \le m,\ p \nmid f. \tag{4.5}$$

Substitute (4.5) into (4.2) yields

$$p^{m-r}q^n = f\left|\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix} p \\ i \end{bmatrix}(p^r f)^{p-2i-1}\left(\frac{p^{2r}f^2+db^2}{4}\right)^i\right|. \tag{4.6}$$

Further, by (1.5), we have

$$\begin{bmatrix} p \\ (p-1)/2 \end{bmatrix} = p, \tag{4.7}$$

and by (4.1), (4.4) and (4.5), we have $p \nmid p^{2r}f^2+db^2$. Hence, by (4.7), we get

$$p \,\Big\|\, \sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix} p \\ i \end{bmatrix}(p^r f)^{p-2i-1}\left(\frac{p^{2r}f^2+db^2}{4}\right)^i. \tag{4.8}$$

Therefore, by (4.5), (4.6) and (4.8), we have $m > 1$,

$$r = m-1 \tag{4.9}$$

and

$$pq^n = f\left|\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\i\end{bmatrix}(p^{m-1}f)^{p-2i-1}\left(\frac{p^{2m-2}f^2+db^2}{4}\right)^i\right|. \tag{4.10}$$

Since $\gcd(a,b)=1$, by (4.1) and (4.5), we have

$$\gcd\left(f, \sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\i\end{bmatrix}(p^{m-1}f)^{p-2i-1}\left(\frac{p^{2m-2}f^2+db^2}{4}\right)^i\right)=1. \tag{4.11}$$

If $f>1$, since $p\nmid f$, then from (4.10) and (4.11) we get $f=q^n$ and

$$p=\left|\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\i\end{bmatrix}(p^{m-1}q^n)^{p-2i-1}\left(\frac{p^{2m-2}q^{2n}+db^2}{4}\right)^i\right|. \tag{4.12}$$

Let

$$\tilde{\alpha}=\frac{\sqrt{-db^2}+\sqrt{p^{2m-2}q^{2n}}}{2},\ \tilde{\beta}=\frac{\sqrt{-db^2}-\sqrt{p^{2m-2}q^{2n}}}{2}. \tag{4.13}$$

Then we have

$$(\tilde{\alpha}+\tilde{\beta})^2=-db^2,\ \tilde{\alpha}-\tilde{\beta}=p^{m-1}q^n,\ \tilde{\alpha}\tilde{\beta}=-db^2-p^{2m-2}q^{2n}. \tag{4.14}$$

We see from (2.3), (4.13) and (4.14) that $(\tilde{\alpha},\tilde{\beta})$ is a Lehmer pair. Further, let $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ $(t=1,2,\dots)$ denote the corresponding Lehmer numbers. By Lemma 1, we get from (2.4), (4.12), (4.13) and (4.14) that

$$|\tilde{L}_p(\tilde{\alpha},\tilde{\beta})|=p. \tag{4.15}$$

Hence, we find from (4.14) and (4.15) that the Lehmer number $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ has no primitive divisor. Therefore, by Lemma 4, we have $p<30$. Based on the results in [25], we can prove that p does not satisfy $6<p\le 30$. So we have $p=5$, by (4.12), we have

$$\left(\frac{-5^{2m-2}q^{2n}+db^2}{2}\right)^2-5(5^{2m-3}q^{2n})^2=\pm 4. \tag{4.16}$$

However, by Lemma 2, (4.16) is false. Thus, we get

$$f=1, \tag{4.17}$$

and by (4.3), (4.4), (4.5), (4.9), (4.10) and (4.17) we obtain (1.6) and (1.7). The theorem is proved.

5. PROOF OF COROLLARY 1

By Theorem 2, if (x,y,m,n) is a solution of (1.1), then it can be expressed as (1.7) with (1.6). Let

$$\alpha=\frac{p^{m-1}+b\sqrt{-d}}{2},\ \beta=\frac{p^{m-1}-b\sqrt{-d}}{2}. \tag{5.1}$$

Then we have

$$\alpha+\beta=p^{m-1},\ \alpha-\beta=b\sqrt{-d},\ \alpha\beta=\frac{p^{2m-2}+db^2}{4}. \tag{5.2}$$

It implies that (α,β) is a Lucas pair. By Lemma 1, we see from (1.6), (1.7), (5.1) and (5.2) that

$$pq^n=\left|\frac{\alpha^p+\beta^p}{\alpha+\beta}\right|. \tag{5.3}$$

Therefore, Lemma 3, we get from (5.2) and (5.3) that $q\equiv\pm1 \pmod{2p}$. Thus, the corollary is proved.

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