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STUDY ON A SUBCLASS OF HOLOMORPHIC FUNCTIONS ASSOCIATED TO THE q-ANALOGUE MULTIPLIER TRANSFORMATION DEFINED IN A JANOWSKI DOMAIN

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Abstract. The present study uses differential subordination in conjunction with Janowski-type functions to establish the particular class $\mathcal{F}(m,n,\lambda,q,D,E)$ of holomorphic functions in the open unit disk. This class is associated with the q-analogue multiplier transformation. Using both the Keogh-Merkes and Ma-Minda's inequalities and the well-known Carathéodory's inequality for functions with positive real parts, an upper bound for the first two initial coefficients of the Taylor-Maclaurin power series expansion is derived. Also, for the functions in this family, an upper bound on the Fekete-Szegö functional is provided. Furthermore, for the function $\mathcal{G}^{-1}$, a similar conclusion is derived for the Fekete-Szegö inequality and the first two coefficients when $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,D,E)$. Properties regarding partial sums, necessary and sufficient conditions for functions to be part of $\mathcal{F}(m,n,\lambda,q,D,E)$, radii of close-to-convexity and starlikeness for this class, as well as distortion bounds are also established. The novelty of the results consists in the investigation of the basic properties of the new class of functions using simple methods, and the fact that the class is connected with the new above-mentioned q-operator and the Janowski functions.

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1. INTRODUCTION

The set of functions $\mathcal{G}$ of the type

$$\mathcal{G}(\eta) = \eta + \sum_{n=2}^{\infty} a_n \eta^n, \tag{1.1}$$

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That are holomorphic on the open unit disk $\mathcal{U} := \{\eta \in \mathbb{C} : |\eta| < 1\}$ of the complex plane are denoted by $\mathcal{A}$, and the subclass $\mathcal{S}$ of $\mathcal{A}$ refers to functions that are *univalent* in $\mathcal{U}$.

Given $\mathcal{G}, \mathcal{F} \in \mathcal{A}$, the function $\mathcal{G}$ is referred to as subordinate to $\mathcal{F}$ if there is a function $w(\eta) \in \mathcal{U}$ satisfying $w(0) = 0$, $|w(\eta)| < 1$, $\eta \in \mathcal{U}$, known as *Schwarz function*, and $\mathcal{G}(\eta) = \mathcal{F}(w(\eta))$ for all $\eta \in \mathcal{U}$. The mathematical sign used for subordination is:

$$\mathcal{G} \prec \mathcal{F} \quad \text{or} \quad \mathcal{G}(\eta) \prec \mathcal{F}(\eta).$$

The following inclusion equivalency holds if the function $\mathcal{F} \in \mathcal{S}$:

$$\mathcal{G}(\eta) \prec \mathcal{F}(\eta) \Leftrightarrow \mathcal{G}(0) = \mathcal{F}(0) \quad \text{and} \quad \mathcal{G}(\mathcal{U}) \subset \mathcal{F}(\mathcal{U}).$$

The *starlike* and *convex* functions in $\mathcal{U}$, respectively, are the subfamilies of $\mathcal{S}$:

$$\mathcal{S}^* := \left\{ \mathcal{G} \in \mathcal{A} : \operatorname{Re}\left\{ \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \right\} > 0,\ \eta \in \mathcal{U} \right\}$$

and

$$\mathcal{K} := \left\{ \mathcal{G} \in \mathcal{A} : \operatorname{Re}\left\{ \frac{(\eta \mathcal{G}'(\eta))'}{\mathcal{G}'(\eta)} \right\} > 0,\ \eta \in \mathcal{U} \right\},$$

respectively. Similarly,

$$\mathcal{S}^*(\varphi) = \left\{ \mathcal{G} \in \mathcal{A} : \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \prec \varphi(\eta) \right\}, \quad \mathcal{K}(\varphi) = \left\{ \mathcal{G} \in \mathcal{A} : \frac{(\eta \mathcal{G}'(\eta))'}{\mathcal{G}'(\eta)} \prec \varphi(\eta) \right\},$$

where

$$\varphi(\eta) = \frac{1+\eta}{1-\eta}.$$

Janowski defined in [19] the *Janwoski class of functions* $\mathfrak{S}^*[\mathcal{D}, \mathcal{E}]$, an extended function family of starlike functions. A function $\mathcal{G} \in \mathcal{A}$ is said to be in the family $\mathfrak{S}^*[\mathcal{D}, \mathcal{E}]$ if

$$\frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \prec \frac{1+\mathcal{D}\eta}{1-\mathcal{E}\eta} \quad (-1 \leq \mathcal{E} < \mathcal{D} \leq 1).$$

We mention that the above subordination could be written as:

$$\frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} = \frac{(\mathcal{D}+1)\,p(\eta) - (\mathcal{D}-1)}{(\mathcal{E}+1)\,p(\eta) - (\mathcal{E}-1)} \quad (-1 \leq \mathcal{E} < \mathcal{D} \leq 1),$$

where $p(\eta)$ is an analytical function with a positive real part in $\mathcal{U}$.

The Janowski convex and Janowski starlike functions are obtained by reducing the above-described classes to the requirement $-1 \leq \mathcal{E} < \mathcal{D} \leq 1$. The starlike and convex functions of order ϑ $(0 \leq \vartheta < 1)$ are obtained for the special cases when

$\mathcal{D} := 1-2\vartheta$ and $\mathcal{E} := -1$, where $0 \leq \vartheta < 1$. These functions were earlier defined by Robertson in [26], and were considered as:

$$\mathcal{S}^*(\vartheta) := \left\{ \mathcal{G} \in \mathcal{A} : \operatorname{Re}\left\{ \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \right\} > \vartheta,\ \eta \in \mathcal{U} \right\},$$

$$\mathcal{K}(\vartheta) := \left\{ \mathcal{G} \in \mathcal{A} : \operatorname{Re}\left\{ \frac{(\eta \mathcal{G}'(\eta))'}{\mathcal{G}'(\eta)} \right\} > \vartheta,\ \eta \in \mathcal{U} \right\}.$$

Considering the well-known inclusions $\mathcal{S}^*(\vartheta) \subset \mathcal{S}$ and $\mathcal{K}(\vartheta) \subset \mathcal{S}$, it follows from the familiar Alexander's duality relation that $\mathcal{G} \in \mathcal{K}(\vartheta)$ if and only if $\eta \mathcal{G}'(\eta) \in \mathcal{S}^*(\vartheta)$ for each $0 \leq \vartheta < 1$. Geometric Function Theory (GFT) has been developed substantially based on the aforementioned families, and several key properties of $\mathcal{S}$ have been examined considering various perspectives.

When $0 < q < 1$, $[n]_q!$ represents the q-factorial described as (see [18]):

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [2]_q [1]_q, & \text{if } n = 1,2,3,\ldots, \\ \\ 1, & \text{if } n = 0, \end{cases}$$

where $[n]_q$, named the q-analogue of $n \in \mathbb{N}$, is given by:

$$[n]_q = \frac{1-q^n}{1-q} \quad \text{for } n \in \mathbb{N},$$

The applications of q-calculus across various mathematical fields, as well as in physics and engineering, are widely recognized.

Jackson [18] proposed the q-derivative operator $\mathfrak{D}_q$ of a function $\mathcal{G}$ as follows:

$$\mathfrak{D}_q \mathcal{G}(\eta) = \frac{\mathcal{G}(\eta) - \mathcal{G}(q\eta)}{(1-q)\eta} \quad (0 < q < 1;\ \eta \neq 0).$$

It is clear that

$$\lim_{q \to 1-} \mathfrak{D}_q \mathcal{G}(\eta) = \mathcal{G}'(\eta) \quad \text{and} \quad \mathfrak{D}_q \mathcal{G}(0) = \mathcal{G}'(0).$$

For additional information on the q-derivative operator's theory $\mathfrak{D}_q$, one can refer to [11–13].

The relationship between q-calculus and the theory of univalent functions was made clear by Ismail et al. [17] who introduced and investigated a specific class of q-stalike functions. It was Srivastava who established the foundational principles for the applications of q-calculus within the realm of geometric function theory, as presented in the book chapter that appeared in 1989 [29]. A recent study [1] highlights some aspects of the application of quantum calculus in geometric function theory, while Srivastava's review from 2020 [30] highlights other breakthroughs.

The introduction of new q-analogue operators led to a multitude of applications of q-calculus on univalent functions. Convolution was used by Kanas and Răducanu [20] to establish the q-analogue of the Ruscheweyh differential operator. Further research on the use of this differential operator was conducted by Mahmood and Sokół [24] and Mohammed and Darus [6]. The same pattern led to the emergence of the q-analogue of the Sălăgean differential operator [14], which sparked a number of applications among very recent ones being [16, 22]. Other interesting very recent studies involving q-analogue operators can be seen in [4, 5, 7].

Recently, Shah and Noor [27] defined the q-analogue multiplier transformation $I_q^{m,\lambda}\mathcal{G} : \mathcal{A} \to \mathcal{A}$ by

$$I_q^{m,\lambda}\mathcal{G}(\eta) := \eta + \sum_{n=2}^{\infty} [\Phi_n(\lambda, q)]^m a_n \eta^n, \quad \eta \in \mathcal{U}, \tag{1.2}$$

where

$$\Phi_n(\lambda, q) := \frac{[n+\lambda]_q}{[1+\lambda]_q}, \qquad \lambda > -1,\ q \in (0,1),\ m \in \mathbb{R},\ \eta \in \mathcal{U}. \tag{1.3}$$

Motivated by the recent new studies involving q-analogue operators like [2, 15, 32, 33], in this article, using the q-analogue multiplier transformation defined in (1.2), we define a new subclass of $\mathcal{A}$ given by:

$$\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E}) = \left\{ \mathcal{G} \in \mathcal{A} : \frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} \prec \frac{1+\mathcal{D}\eta}{1+\mathcal{E}\eta} \right\}, \tag{1.4}$$

where $-1 \leq E < D \leq 1$, $\lambda > -1$, $q \in (0,1)$ and $m \in \mathbb{R}$.

Specializing the parameters $\mathcal{D}$ and $\mathcal{E}$, one can obtain the particular cases

$$\mathcal{F}(m,n,\lambda,q,1-2\alpha,-1) =: \mathcal{F}(m,n,\lambda,q,\alpha),$$

and

$$\mathcal{F}(m,n,\lambda,q,1,-1) =: \mathcal{F}(m,n,\lambda,q,\varphi).$$

The study described in this paper focuses on investigating several coefficient properties of this subclass. The study starts in the next section with the assessment of the Fekete-Szegö problem. In addition, the following sections establish the outcomes of partial sums, specific properties, and coefficient estimates.

2. The Fekete-Szegö functional bounds for the class $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$

In order to assess the Fekete-Szegö type inequality for $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ the next already established results will be employed (the first part is due to Carathéodory [9]):

Lemma 1 ([21, 23]). *If $P(\eta) = 1 + p_1\eta + p_2\eta^2 + \cdots \in \mathcal{P}$ where $\mathcal{P}$ the class of functions $P \in \mathcal{A}$ with* Re $P(\eta) > 0$ *and $P(0) = 1$, then*

$$|p_n| \leq 1,\ n \geq 1, \tag{2.1}$$

and for $\hbar \in \mathbb{C}$ we have

$$\left|p_2 - \hbar p_1^2\right| \leq 2\max\{1; |1-2\hbar|\}. \tag{2.2}$$

If $\hbar \in \mathbb{R}$, then

$$\left|p_2 - \hbar p_1^2\right| \leq \begin{cases} -4\hbar + 2, & \text{if } \hbar \leq 0, \\ 2, & \text{if } 0 \leq \hbar \leq 1, \\ 4\hbar - 2, & \text{if } \hbar \geq 1. \end{cases} \tag{2.3}$$

If $\hbar > 1$ or $\hbar < 0$, (2.3) remains valid if and only if $P_1(\eta) = \dfrac{1+\eta}{1-\eta}$ or one of its rotations.

When $0 < \hbar < 1$, then the equality (2.3) remains valid if and only if $P_2(\eta) = \dfrac{1+\eta^2}{1-\eta^2}$ or one of its rotations.

When $\hbar = 0$, equality equality (2.3) remains valid if and only if

$$P_3(\eta) = \left(\frac{1+c}{2}\right)\frac{1+\eta}{-\eta+1} + \left(\frac{1-c}{2}\right)\frac{-\eta+1}{1+\eta} \quad (0 \leq c \leq 1)$$

or one of its rotations.

Theorem 1. *If $\mathcal{G} \in \mathcal{A}$ has the form (1.1) and $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then*

$$|a_2| \leq \frac{|\mathcal{D}-\mathcal{E}|}{[\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n}, \tag{2.4}$$

$$|a_3| \leq \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)} \tag{2.5}$$

$$\times \max\left\{1; \left|\frac{1+2\mathcal{E}-\mathcal{D}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m} - [\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2}\right|\right\},$$

and for a complex number τ, we have

$$\left|a_3 - \tau a_2^2\right| \leq \frac{2(\mathcal{D}-\mathcal{E})}{[\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n}\max\{1; |\Theta(\tau,\mathcal{D},\mathcal{E})|\}, \tag{2.6}$$

where

$$\Theta(\tau,\mathcal{D},\mathcal{E}) := \frac{1+2\mathcal{E}-\mathcal{D}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m} - [\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2}$$
$$+ \frac{2\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2},$$

and $\Phi_n(\lambda,q)$ *is given by* (1.3).

Proof. Our aim is to demonstrate that the relations (2.4), (2.5) and (2.6) remain valid for $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. Considering $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, we have:

$$\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)}\prec\frac{1+\mathcal{D}\eta}{1+\mathcal{E}\eta},$$

which yields

$$\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)}=\frac{1+\mathcal{D}w(\eta)}{1+\mathcal{E}w(\eta)}=G(w(\eta)),\quad(-1\le\mathcal{E}<\mathcal{D}\le 1).$$

Since $w(\eta)$ can be written as:

$$w(\eta)=\frac{1-h(\eta)}{1+h(\eta)}=\frac{p_1\eta+p_2\eta^2+p_3\eta^3+\cdots}{2+p_1\eta+p_2\eta^2+p_3\eta^3+\cdots},$$

we get:

$$G(w(\eta))=1+\frac{1}{2}(\mathcal{D}-\mathcal{E})p_1\eta+\frac{1}{4}\left(2(\mathcal{D}-\mathcal{E})p_2-(1+\mathcal{E})p_1^2\right)\eta^2+\cdots,\tag{2.7}$$

and therefore

$$\begin{aligned}\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)}&=1+\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)a_2\eta\\&\quad+\left(\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)a_3\right.\\&\quad\left.-\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)a_2^2\right)\eta^2+\cdots.\end{aligned}\tag{2.8}$$

If we compare the first coefficients of (2.7) and (2.8) we have:

$$a_2=\frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)}p_1,\tag{2.9}$$

$$\begin{aligned}a_3&=\frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\\&\quad\times\left(p_2-\frac{p_1^2}{2}\left[\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}-\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\right)\right]\right),\end{aligned}\tag{2.10}$$

and by using (2.1) in (2.9) and (2.2) in (2.10) we get:

$$|a_2|\le\frac{|\mathcal{D}-\mathcal{E}|}{2[\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n},$$

$$|a_3|\le\frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}$$

$$\times \max\left\{1; \left|\frac{1+2\mathcal{E}-\mathcal{D}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\right|\right\}.$$

In addition, from (2.9) and (2.10), we have

$$\left|a_3-\tau a_2^2\right| = \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\left|p_2-p_1^2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\right|, \tag{2.11}$$

where

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E}) := \frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2} + \frac{\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}. \tag{2.12}$$

The necessary results are now obtained if we apply (2.2) in (2.11). Furthermore, we obtain our inequality for real τ by applying (2.3) to the previously mentioned (2.11). □

Theorem 2. *If $\mathcal{G}\in\mathcal{A}$ is given by (1.1) and $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then for any $\tau\in\mathbb{R}$ we obtain:*

$$\left|a_3-\tau a_2^2\right| \le \frac{|\mathcal{D}-\mathcal{E}|}{\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\begin{cases}1-2\mathcal{K}(\tau,\mathcal{D},\mathcal{E}), & \text{if } \tau\le\sigma_1,\\ 1, & \text{if } \sigma_1\le\tau\le\sigma_2,\\ 2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})-1, & \text{if } \tau\ge\sigma_2,\end{cases}$$

where $\mathcal{K}(\tau,\mathcal{D},\mathcal{E})$ defined by (2.12) and

$$\sigma_1 = \frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)} \times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2} - \frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}\right),$$

and

$$\sigma_2 = \frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)} \times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2} + \frac{\mathcal{D}-2\mathcal{E}-1}{\mathcal{D}-\mathcal{E}}\right).$$

Proof. For real τ, using Lemma 1 and equation (2.12), we get:

$$\left|a_3-\tau a_2^2\right| = \frac{|\mathcal{D}-\mathcal{E}|}{2\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\left|p_2-p_1^2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\right|$$

$$\leq \frac{|\mathcal{D}-\mathcal{E}|}{2\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\begin{cases}-4\mathcal{K}(\tau,\mathcal{D},\mathcal{E})+2, & \text{if } \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 0,\\ 2 & \text{if } 0\leq \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 1,\\ 4\mathcal{K}(\tau,\mathcal{D},\mathcal{E})-2 & \text{if } \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\geq 1.\end{cases}$$

$$\leq \frac{|\mathcal{D}-\mathcal{E}|}{\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\begin{cases}-2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})+1, & \text{if } \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 0,\\ 1 & \text{if } 0\leq \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 1,\\ 2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})-1 & \text{if } \hbar\geq \mathcal{K}(\tau,\mathcal{D},\mathcal{E}).\end{cases}$$

where $\mathcal{K}(\tau,\mathcal{D},\mathcal{E})$ defined by (2.12).

If

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 0,$$

then

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E}):=\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}-\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\leq 0,$$

and we have

$$\tau\leq\frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}-\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}\right)=\sigma_1.$$

If

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\geq 1,$$

then

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E}):=\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}-\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\geq 1,$$

and we get

$$\tau\geq\frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\mathcal{D}-2\mathcal{E}-1}{\mathcal{D}-\mathcal{E}}\right)=\sigma_2.$$

□

3. Characterization properties

We will introduce some characteristic properties of the functions $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ by using the techniques that Silverman introduced in [28]. These properties include partial sums results, necessary and sufficient conditions for functions to be part of $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, radii of close-to-convexity and starlikeness for this class as well as distortion bounds.

Theorem 3. *If $\mathcal{G} \in \mathcal{A}$ has the form (1.1) and $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then*

$$\sum_{j=2}^{\infty} \left((1-\mathcal{E}) \left[\Phi_j(\lambda,q) \right]^m + (\mathcal{D}-1) \left[\Phi_j(\lambda,q) \right]^n \right) |a_n| \leq |\mathcal{D}-\mathcal{E}|, \tag{3.1}$$

where $\Phi_j(\lambda,q)$ is given by (1.3).

Proof. Letting $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, by (1.4) we deduce that

$$\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} = \frac{1+\mathcal{D}w(\eta)}{\mathcal{E}w(\eta)+1}, \quad \eta \in \mathcal{U},$$

with $w(\eta)$ a Schwarz function, meaning that:

$$\left| \frac{I_q^{m,\lambda}\mathcal{G}(\eta) - I_q^{n,\lambda}\mathcal{G}(\eta)}{\mathcal{D}I_q^{n,\lambda}\mathcal{G}(\eta) - \mathcal{E}I_q^{m,\lambda}\mathcal{G}(\eta)} \right| < 1, \quad \eta \in \mathcal{U}.$$

Thus, the above relation leads us to

$$\begin{aligned} &\left| \frac{I_q^{m,\lambda}\mathcal{G}(\eta) - I_q^{n,\lambda}\mathcal{G}(\eta)}{\mathcal{D}I_q^{n,\lambda}\mathcal{G}(\eta) - \mathcal{E}I_q^{m,\lambda}\mathcal{G}(\eta)} \right| \\ &= \left| \frac{\sum_{j=2}^{\infty} \left([\Phi_j(\lambda,q)]^m - [\Phi_j(\lambda,q)]^n \right) a_j \eta^j}{(\mathcal{D}-\mathcal{E})\eta + \sum_{j=2}^{\infty} \left(\mathcal{D}[\Phi_j(\lambda,q)]^n - \mathcal{E}[\Phi_j(\lambda,q)]^m \right) a_j \eta^j} \right| \\ &\leq \frac{\sum_{j=2}^{\infty} \left([\Phi_j(\lambda,q)]^m - [\Phi_j(\lambda,q)]^n \right) |a_j| k^{j-1}}{|\mathcal{D}-\mathcal{E}| - \sum_{j=2}^{\infty} \left(\mathcal{D}[\Phi_j(\lambda,q)]^n - \mathcal{E}[\Phi_j(\lambda,q)]^m \right) |a_j| k^{j-1}} < 1, \end{aligned}$$

and taking $k \to 1^-$, a simple computation yields (3.1). □

Example 1. For

$$\mathcal{G}(\eta) = \eta + \sum_{j=2}^{\infty} \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E})[\Phi_j(\lambda,q)]^m + (\mathcal{D}-1)[\Phi_j(\lambda,q)]^n} \ell_j \eta^j, \ \eta \in \Omega,$$

such that $\sum\limits_{j=2}^{\infty} \ell_j = 1$, we get

$$\sum_{j=2}^{\infty} \left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right) |a_j|$$

$$= \sum_{j=2}^{\infty} \left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right)$$
$$\times \frac{|\mathcal{D}-\mathcal{E}|}{\left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right)} \ell_j = |\mathcal{D}-\mathcal{E}| \sum_{j=2}^{\infty} \ell_j = |\mathcal{D}-\mathcal{E}|.$$

Then $\mathcal{G}(\eta) \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$.

Corollary 1. *Consider $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ described by (1.1). We have:*

$$|a_j| \leq \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n}, \quad \text{for } j \geq 2,$$

where $\Phi_j(\lambda,q)$ is defined by (1.3).

Theorem 4. *For $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then*

$$r - \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_2(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_2(\lambda,q)]^n} r^2 \leq |\mathcal{G}(\eta)|$$
$$\leq r + \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_2(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_2(\lambda,q)]^n} r^2.$$

For the function defined by

$$\widehat{\mathcal{G}}(\eta) := \eta - \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_2(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_2(\lambda,q)]^n} \eta^2, \quad |\eta| = r < 1, \tag{3.2}$$

the approximation is sharp.

Proof. For $|\eta| = r < 1$ we have

$$|\mathcal{G}(\eta)| = \left| \eta + \sum_{j=2}^{\infty} a_j \eta^j \right| \leq |\eta| + \sum_{j=2}^{\infty} a_j |\eta|^j = r + \sum_{j=2}^{\infty} a_j |r|^j.$$

Moreover, since for $|\eta| = r < 1$ we get $r^j < r^2$ for all $j \geq 2$, the above relation implies that

$$|\mathcal{G}(\eta)| \leq r + r^2 \sum_{j=2}^{\infty} |a_j|. \tag{3.3}$$

Similarly, we get

$$|\mathcal{G}(\eta)| \geq r - r^2 \sum_{j=2}^{\infty} |a_j|. \tag{3.4}$$

From the relation (3.1) we have

$$\sum_{j=2}^{\infty} \left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right) |a_j| \leq |\mathcal{D}-\mathcal{E}|,$$

but

$$\left((1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n\right)\sum_{j=2}^{\infty}\left|a_j\right|$$

$$\leq\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)|a_n|\leq|\mathcal{D}-\mathcal{E}|\,.$$

Therefore,

$$\sum_{j=2}^{\infty}a_j\leq\frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n},\tag{3.5}$$

and using (3.5) in (3.3) and (3.4) we get the desired result. □

The next distortion theorem for the family $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ could be similarly obtained:

Theorem 5. *If $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, we have:*

$$1-\frac{2|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n}k\leq\left|\mathcal{G}'(\eta)\right|$$

$$\leq 1+\frac{2|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n}k.$$

The equality holds if the function is $\widehat{\mathcal{G}}$ given by (3.2).

Proof. We shall skip the proof since it closely resembles the arguments presented in Theorem 4. □

The next result deals with the fact that a convex combination of functions from the class $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ belongs to the same class, as follows:

Theorem 6. *Consider $\mathcal{G}_i\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ having the form:*

$$\mathcal{G}_i(\eta)=\eta+\sum_{j=2}^{\infty}a_{i,j}\eta^j,\quad i=1,2,3,\ldots,m.\tag{3.6}$$

Then $H\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, where

$$H(\eta):=\sum_{i=1}^{m}c_i\mathcal{G}_i(\eta)\,,\quad \textit{and}\quad \sum_{i=1}^{m}c_i=1.$$

Proof. Applying the outcome of Theorem 3 we write:

$$\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_j\right|\leq|\mathcal{D}-\mathcal{E}|,$$

and, in addition,

$$H(\eta)=\sum_{i=1}^{m}c_i\left(\eta+\sum_{j=2}^{\infty}a_{i,j}\eta^j\right)=\eta+\sum_{j=2}^{\infty}\left(\sum_{i=1}^{m}c_ia_{i,j}\right)\eta^j.$$

Therefore

$$\begin{aligned}&\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|\sum_{i=1}^{m}c_i a_{i,j}\right|\\&\leq\sum_{i=1}^{m}\left[\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_{i,j}\right|\right]c_i\\&=\sum_{i=1}^{m}|\mathcal{D}-\mathcal{E}|c_i=|\mathcal{D}-\mathcal{E}|\sum_{i=1}^{m}c_i=|\mathcal{D}-\mathcal{E}|,\end{aligned}$$

thus $H(\eta)\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. □

Regarding the arithmetic means of the functions of the family $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ the next result holds:

Theorem 7. *When $\mathcal{G}_i\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ have the form seen in (3.6), we obtain:*

$$\mathcal{G}(\eta):=\eta+\frac{1}{m}\sum_{j=2}^{\infty}\left(\sum_{i=1}^{m}a_{i,j}\eta^j\right)\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E}),\tag{3.7}$$

with function $\mathcal{G}$ being the arithmetic mean of $\mathcal{G}_i$, $i=1,2,3,\ldots,m$.

Proof. Using (3.7) we get

$$\mathcal{G}(\eta)=\frac{1}{m}\sum_{i=1}^{m}f_i(\eta)\,\frac{1}{m}\sum_{i=1}^{m}\left(\eta+\sum_{j=2}^{\infty}a_{i,j}\eta^j\right)\eta+\sum_{j=2}^{\infty}\left(\frac{1}{m}\sum_{i=1}^{m}a_{i,j}\right)\eta^j,$$

and to prove that $\mathcal{G}(\eta)\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, according to the Theorem 3, it suffices to demonstrate that:

$$\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left(\frac{1}{m}\sum_{i=1}^{m}\left|a_{i,j}\right|\right)\leq|\mathcal{D}-\mathcal{E}|.$$

A quick calculation reveals that:

$$\begin{aligned}&\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left(\frac{1}{m}\sum_{i=1}^{m}\left|a_{i,j}\right|\right)\\&=\frac{1}{m}\sum_{i=1}^{m}\left(\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_{i,j}\right|\right)\\&\leq\frac{1}{m}\sum_{i=1}^{m}|\mathcal{D}-\mathcal{E}|=|\mathcal{D}-\mathcal{E}|,\end{aligned}$$

therefore $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. □

Theorem 8. *When $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, we have that $\mathcal{G} \in \mathcal{S}^*(\vartheta)$ $(0 \leq \vartheta < 1)$, $|\eta| < k_1^*$,*

$$k_1^* = \inf_{j\geq 2}\left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right]}{(j-\vartheta)\left|\mathcal{D}-\mathcal{E}\right|}\right)^{\frac{1}{j-1}}.$$

Proof. Consider $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. Then $\mathcal{G} \in \mathcal{S}^*(\vartheta)$ if:

$$\left|\frac{\eta\mathcal{G}'(\eta)}{\mathcal{G}(\eta)} - 1\right| < 1-\vartheta.$$

By applying a simple calculation, we deduce:

$$\sum_{j=2}^{\infty}\left(\frac{j-\vartheta}{1-\vartheta}\right)\left|a_j\right|\left|\eta\right|^{j-1} < 1. \tag{3.8}$$

Since $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, considering (3.1) we have:

$$\sum_{j=2}^{\infty}\frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{\left|\mathcal{D}-\mathcal{E}\right|}\left|a_j\right| < 1.$$

Inequality (3.8) is true when:

$$\sum_{j=2}^{\infty}\left(\frac{j-\vartheta}{1-\vartheta}\right)\left|a_j\right|\left|\eta\right|^{j-1} < \sum_{j=2}^{\infty}\frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{\left|\mathcal{D}-\mathcal{E}\right|}\left|a_j\right|,$$

which implies that

$$\left|\eta\right|^{j-1} < \left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)}{\left|\mathcal{D}-\mathcal{E}\right|(j-\vartheta)}\right),$$

or, equivalently

$$\left|\eta\right| < \left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)}{\left|\mathcal{D}-\mathcal{E}\right|(j-\vartheta)}\right)^{\frac{1}{j-1}},$$

hence, the family is starlike. □

Theorem 9. *Any function $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, is a close-to-convex function of order ϑ $(0 \leq \vartheta < 1)$, $|\eta| < k_2^*$,*

$$k_2^* = \inf_{j\geq 2}\left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)}{j\left|\mathcal{D}-\mathcal{E}\right|}\right)^{\frac{1}{j-1}}.$$

Proof. Let $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. If $\mathcal{G}$ is a close-to-convex function of order ϑ, then we can write:

$$\left|\mathcal{G}'(\eta) - 1\right| < 1-\vartheta,$$

equivalently written,

$$\sum_{j=2}^{\infty} \frac{j}{1-\vartheta} \left|a_j\right| |\eta|^{j-1} < 1.$$

Since $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, using (3.1) we obtain:

$$\sum_{j=2}^{\infty} \frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{|\mathcal{D}-\mathcal{E}|} \left|a_j\right| < 1.$$

Inequality (3.8) is true when:

$$\sum_{j=2}^{\infty} \frac{j}{1-\vartheta} \left|a_j\right| |\eta|^{j-1} < \sum_{j=2}^{\infty} \frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{|\mathcal{D}-\mathcal{E}|} \left|a_j\right|,$$

which implies that

$$|\eta|^{j-1} < \left(\frac{(1-\vartheta)\left[(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right]}{j|\mathcal{D}-\mathcal{E}|} \right),$$

or, equivalently

$$|\eta| < \left(\frac{(1-\vartheta)\left[(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right]}{j|\mathcal{D}-\mathcal{E}|} \right)^{\frac{1}{j-1}},$$

which yields the desired result. □

4. THE COEFFICIENT INEQUALITIES FOR $\mathcal{G}^{-1} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$

According to the "Koebe one quarter theorem" [10], there is a disk with radius $\frac{1}{4}$ in the image of $\mathcal{U}$ through any function $\mathcal{G} \in \mathcal{S}$. Consequently, an inverse function $\mathcal{G}^{-1}$ for each $\mathcal{G} \in \mathcal{S}$ exists and satisfies:

$$\mathcal{G}^{-1}(\mathcal{G}(\eta)) = \eta, \ (\eta \in \Omega) \quad \text{and} \quad \mathcal{G}\left(\mathcal{G}^{-1}(w)\right) = w, \ \left(|w| < r_0(\mathcal{G}), \ r_0(\mathcal{G}) \geq \frac{1}{4}\right).$$

When both $\mathcal{G}$ and $\mathcal{G}^{-1}$ are univalent in $\mathcal{U}$, a function $\mathcal{G} \in \mathcal{A}$ is referred to as bi-univalent in $\mathcal{U}$. It is important to remember that the set of bi-univalent functions is not empty. The bi-univalent function family includes, for instance, the functions η, $\frac{\eta}{1-\eta}$, $-\log(1-\eta)$, and $\frac{1}{2}\log\frac{1+\eta}{1-\eta}$ but the Koebe function is not included.

Theorem 10. *Considering $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ and $\mathcal{G}^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$, we have:*

$$|d_2| = \frac{|\mathcal{D}-\mathcal{E}|}{\left[\Phi_2(\lambda,q)\right]^m - \left[\Phi_2(\lambda,q)\right]^n},$$

$$|d_3| = \frac{|\mathcal{D}-\mathcal{E}|}{\left[\Phi_3(\lambda,q)\right]^m - \left[\Phi_3(\lambda,q)\right]^n} \max\{1; |\mathcal{K}(2,\mathcal{D},\mathcal{E})-1|\},$$

and for any $\hbar \in \mathbb{C}$, *obtain:*

$$\left|d_3 - \hbar d_2^2\right| \leq \frac{|\mathcal{D}-\mathcal{E}|}{[\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n} \times \max\left[1; \left|\mathcal{K}(2,\mathcal{D},\mathcal{E}) + \hbar\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2} - 1\right|\right],$$

where

$$\mathcal{K}(2,\mathcal{D},\mathcal{E}) := \frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m} - [\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2} + \frac{2(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2}.$$

Proof. Since

$$\mathcal{G}^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n,$$

is the inverse of the function $\mathcal{G}$, it can be seen that

$$\eta = \mathcal{G}^{-1}(\mathcal{G}(\eta)) = \mathcal{G}\left(\mathcal{G}^{-1}(\eta)\right), \quad |\eta| < r_0(\mathcal{G}). \tag{4.1}$$

From (1.1) and (4.1), we obtain that

$$\eta = \mathcal{G}^{-1}\left(\eta + \sum_{n=2}^{\infty} a_n \eta^n\right), \quad |\eta| < r_0(\mathcal{G}), \tag{4.2}$$

therefore from (4.1) and (4.2) we get:

$$\eta + (a_2 + d_2)\eta^2 + (a_3 + 2a_2 d_2 + d_3)\eta^3 + \cdots = \eta, \quad |\eta| < r_0(\mathcal{G}). \tag{4.3}$$

Equating the corresponding coefficients of the relation (4.3), we conclude that

$$d_2 = -a_2, \tag{4.4}$$

$$d_3 = 2a_2^2 - a_3. \tag{4.5}$$

First, from the relations (2.9) and (4.4) we have

$$d_2 = -\frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)} p_1.$$

To find $|d_3|$, from (4.5) we write:

$$|d_3| = |a_3 - 2a_2^2|,$$

hence by using (2.11) for real $\tau = 2$ we deduce that:

$$|d_3| = \left|a_3 - 2a_2^2\right| = \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)} \left|p_2 - \frac{p_1^2}{2}\mathcal{K}(2,\mathcal{D},\mathcal{E})\right|$$

$$= \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)} \max\left\{1; |\mathcal{K}(2,\mathcal{D},\mathcal{E})-1|\right\},$$

where

$$\mathcal{K}(2,\mathcal{D},\mathcal{E}) := \frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2} + \frac{2(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}.$$

For any complex number $\hbar$, a simple computation gives us that:

$$\begin{aligned} d_3-\hbar d_2^2 &= \frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\left(p_2-\frac{p_1^2}{2}\mathcal{K}(2,\mathcal{D},\mathcal{E})\right) \\ &\quad -\hbar\frac{(\mathcal{D}-\mathcal{E})^2}{4\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}p_1^2 \\ &= \frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)} \\ &\quad \times\left(p_2-\frac{p_1^2}{2}\left[\mathcal{K}(2,\mathcal{D},\mathcal{E})+\hbar\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\right]\right). \end{aligned} \tag{4.6}$$

After using Lemma 1 and (2.1), and considering modulus on both sides of (4.6), we determine:

$$\begin{aligned} \left|d_3-\hbar d_2^2\right| &\le \frac{|\mathcal{D}-\mathcal{E}|}{[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n} \\ &\quad \times\max\left\{1;\left|\mathcal{K}(2,\mathcal{D},\mathcal{E})+\hbar\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}-1\right|\right\} \end{aligned}$$

and this completes our proof. □

5. Conclusions

The study on the generalized differential operator $I_q^{m,\lambda}$ given by (1.2), used for introducing the new subclass of $\mathcal{A}$ given by (1.4), contains information that may serve as a basis for future research efforts on introducing other new classes of analytic functions.

Furthermore, we hope that this study will inspire other researchers to develop this concept further for new families that can be obtained by applying the concept of subordination with connection to specific probability distribution series [3, 31] or with generalized telephone numbers[8, 25].

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