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SEMISIMPLE-CONTINUOUS MODULES

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Abstract. An R-module M is said to be a semisimple-continuous, if it is weak CS (i.e., every semisimple submodule of M is essential in a direct summand of M) and semisimple-direct injective (i.e., if A and B are isomorphic semisimple submodules of M such that A is a direct summand of M, then B is a direct summand of M). It is proved that any semisimple-continuous module is decomposed as a direct sum of a semisimple module and a module with square-free socle. We investigate when the finite exchange property implies full exchange property for the former class of modules. Moreover, we explore the notion of the semisimple-continuity for Abelian groups. We also characterize right Noetherian right V-rings in terms of semisimple-continuous modules. Examples are delimit our results.

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1. INTRODUCTION

In this paper, all rings are associative with unity and all modules are unital right R-modules. We use M to denote such a module. Recall that a module is *extending* (or CS), if every submodule is essential in a direct summand, and a module is a $C2$-module, if every submodule isomorphic to a direct summand is a direct summand, in addition a module is a $C3$-module, if the sum of any two direct summands with zero intersection is again a direct summand.

In [5], the “simple” versions of $C2$ and $C3$-modules are introduced and analyzed. The authors in [5] call these modules simple-direct injective. Then M is *simple-direct injective*, if every simple submodule isomorphic to a direct summand is itself a direct summand, or equivalently if the sum of any two simple direct summands with zero intersection is again a direct summand. In [1], the authors explore the “semisimple” version of $C2$ and $C3$-modules. They call a module M *semisimple-direct injective*, if whenever S and T are semisimple submodules of M with $S \cong T$ and T is a direct summand of M, then S is a direct summand of M, or equivalently, for any semisimple direct summands S and T of M with $S \cap T = 0$, $S \oplus T$ is a direct summand of M. The class of simple-direct injective modules properly contains semisimple-direct injective modules. Extending modules play an important role in rings and categories of modules, their generalizations and related modules have been studied extensively. In [19], the concept of simple-continuous modules is defined by a kind of extending condition with simple-direct injectivity. A module is said to be *simple-continuous*, if every simple submodule of M is essential in a direct summand of M, and M is simple-direct injective. This module class has already been known in the literature as min-continuous modules in [16].

The motivation of this paper comes from the studies in [1, 5, 19]. Our goal is to introduce and investigate semisimple-continuous modules. We call a module M *semisimple-continuous*, if it is both weak CS (i.e., every semisimple submodule of M is essential in a direct summand of M [17]) and semisimple-direct injective. The class of semisimple-continuous modules is a proper subclass of simple-continuous modules. We start by presenting some results which are related to the notions of weak CS and semisimple-direct injectivity in Section 2.

In Section 3, we introduce semisimple-continuous modules and provide several examples. Observe that it is unknown whether the direct summand of a weak CS-module is weak CS or not. We examine when the semisimple-continuous module property is inherited by direct summands. As a result of this fact, we underline when the direct summand of weak CS-modules enjoys this property. It is shown that semisimple-continuous modules are not closed under direct sums. Hence, we also deal with when the direct sums of semisimple-continuous modules are semisimple-continuous.

We obtain decomposition results in Section 4. We show that if M is semisimple-continuous, then M can be decomposed as $M = A \oplus B \oplus K$ where $A \cong B$, $A \oplus B$ is semisimple, and K, which is $(A \oplus B)$-injective, has a square-free socle. As a consequence, we show that if M is semisimple-continuous with the finite exchange, then M has the full exchange. Moreover, we characterize right Noetherian right V-rings in terms of semisimple-continuous modules.

For a nonempty subset X of M, $X \leq M$, $X \leq^{ess} M$ and $X \leq^{\oplus} M$ denote X is a right R-submodule of M, X is an essential right R-submodule of M and X is a direct summand of M, respectively. For notation we use $Soc(M)$, $Rad(M)$, $End(M)$ and $E(M)$ the socle, the radical, the endomorphism ring and the injective hull of a module M, respectively. Note that $M_n(R)$, $T_n(R)$ and $l_R(M)$ stand for the n-by-n matrix ring over R, the n-by-n upper triangular matrix ring over R, and left annihilators of M in R, respectively. Other terminology and notation can be found in [13] and [14].

2. More properties of weak CS modules and semisimple-direct injective modules

Recall that a module M is called *weak CS* [17], if every semisimple submodule of M is essential in a direct summand of M. A ring R is right *weak CS* provided that R_R is a right weak CS-module. In this section, we investigate the behavior of weak CS rings with respect to the generalized upper triangular matrix rings. Let R and S be rings with unity and M be a (R,S)-bimodule. Then $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ denotes the generalized upper triangular matrix ring. Recall from [12, Proposition 1.17] that $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix}$ is the right ideal of T such that J_1 is a right ideal of R and J_2 is a right S-submodule of $M \oplus S$ with $J_1 M \subseteq J_2$. For this notation, we refer to [15].

Lemma 1 ([10, Exercise 3]). *Let T be a generalized triangular matrix ring. Then*

$$Soc(T_T) = \begin{pmatrix} Soc(l_R(M)) & Soc(M_S) \\ 0 & Soc(S_S) \end{pmatrix}.$$

Proposition 1. *Let T be a generalized triangular matrix ring. Then T_T is weak CS if and only if for any semisimple right ideal $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix}$ of T, there exists an idempotent $\begin{pmatrix} e & m \\ 0 & f \end{pmatrix}$ of T such that $J_1 \leq^{ess} [eR \cap l_R(M)]$ and $J_2 \leq^{ess} (eM + V)$, where $V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}$.*

Proof. Let T be a right weak CS-ring and $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix} \leq Soc(T_T)$. Then $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix}$ $\leq^{ess} \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} T$ for some idempotent $\begin{pmatrix} e & m \\ 0 & f \end{pmatrix}$ of T. Now by [15, Lemma 2.2], $J_2 \leq^{ess} (eM + V)$ as right S-modules, where $V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}$ and $[J_1 \cap l_R(M)]$ $\leq^{ess} [eR \cap l_R(M)]$ as right R-modules. However, by Lemma 1, $[J_1 \cap l_R(M)] = J_1$, so as desired. The converse is clear. □

Corollary 1. *Let T be a generalized triangular matrix ring and ${}_RM$ is faithful. Then T_T is a weak CS-ring if and only if for any right S-module*

$$\begin{pmatrix} 0 & \\ & J_2 \\ 0 & \end{pmatrix} \leq \begin{pmatrix} 0 & Soc(M_S) \\ 0 & Soc(S_S) \end{pmatrix},$$

there exists an idempotent $\begin{pmatrix} 0 & m \\ 0 & f \end{pmatrix}$ *of* T *such that* $J_2 \leq^{ess} V$*, where*

$$V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}.$$

Let M and N be right R-modules. Recall from [2] that M is called *socle-N-injective*, denoted soc-N-injective, if any R-homomorphism $f\colon Soc(N) \to M$ extends to N. M is called *soc-injective*, if M is soc-R-injective. In case, M is soc-N-injective and N is soc-M-injective, then M and N are called *relatively soc-injective*.

Proposition 2. *Let T be a generalized triangular matrix ring. If T is a right weak CS-ring, then the followings hold:*

(i) *S_S is weak CS, and if $l_R(M) = R$, then R_R is weak CS.*
(ii) *M_S is a soc-injective module.*
(iii) *$l_R(M)$ (without identity) is a weak CS-module.*

Proof. (i) If $l_R(M) = R$, then by Proposition 1, R is a right weak CS-ring. To see that S is right weak CS, let $X \leq Soc(S_S)$. Then $\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \leq Soc(T_T)$, so there exist idempotent $\alpha = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ of T such that $(0 \oplus X) \leq^{ess} \alpha T$. Then for $x \in X$, write $\bar{x} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$, so we have $\alpha\bar{x} = \bar{x}$, this implies that $x = sx$. Thus $X \leq sS$, where $s^2 = s$. Let $0 \neq ss_1 \in sS$. Then $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} 0 & ms_1 \\ 0 & ss_1 \end{pmatrix} \neq 0$. Thus $(0 \oplus X)$ is essential in αT, so there exists $\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} \in T$ such that $0 \neq \begin{pmatrix} 0 & ms_1 \\ 0 & ss_1 \end{pmatrix} \begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} \in (0 \oplus X)$. Hence, $\begin{pmatrix} 0 & ms_1s_2 \\ 0 & ss_1s_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x' \end{pmatrix}$ for some $x' \in X$, and so $0 \neq ss_1s_2 = x' \in X$. It shows that $X_S \leq^{ess} sS_S$.

(ii) Let $X \leq Soc(S_S)$. Consider the S-linear map $\phi\colon X \to M$. Write

$$F = \left\{ \begin{pmatrix} 0 & \phi(x) \\ 0 & x \end{pmatrix} : x \in X \right\}.$$

Then $F \leq Soc(T_T)$. Therefore, $F \leq^{ess} tT$ for some $t^2 = t = \begin{pmatrix} 0 & m \\ 0 & s \end{pmatrix}$ of T by [18, Lemma 3.87]. Hence, for each $\alpha \in F$, $\alpha = \begin{pmatrix} 0 & \phi(x) \\ 0 & x \end{pmatrix}$ for some $x \in X$. Thus $\alpha = t\alpha$, so $\phi(x) = mx$, as required.

(iii) Let $B = l_R(M)$ and $X \leq Soc(B_B)$. Then $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \leq Soc(T_T)$. So there exists an element $\alpha^2 = \alpha = \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} \in T$ such that $X \leq^{ess} \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} T_T$. Observe that $f = 0$. Assume $0 \neq k \in M$. Then $0 \neq \begin{pmatrix} e & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} T$. So there is $\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \in T$ such that $0 \neq \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} 0 & ks_1 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$, a contradiction. So, $k = 0$. Then $X \leq^{ess} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} T_T$. By Proposition 1, $X \leq^{ess} (eR \cap B)$ and $eM = 0$. It follows that $e \in B$, and hence $eR \cap B = eB$, so $X_B \leq^{ess} eB_B$. Therefore B_B is weak CS. □

Proposition 3. *If T is a right weak CS-ring, then $(M \oplus S)_S$ is a weak CS-module.*

Proof. Let $X \leq Soc(M \oplus S)_S$. Then $\begin{pmatrix} 0 & \\ & X \\ 0 & \end{pmatrix} \leq Soc(T_T)$. Hence there exists $\alpha^2 = \alpha = \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} \in T$ such that $\begin{pmatrix} 0 & \\ & X \\ 0 & \end{pmatrix} \leq^{ess} \alpha T$. Note that $T = \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} T \oplus \begin{pmatrix} (1-e) & -m \\ 0 & (1-f) \end{pmatrix} T$ and $\alpha T = \begin{bmatrix} eR & eM \\ 0 & 0 \end{bmatrix} + K$, where $K = \left\{ \begin{pmatrix} 0 & ms \\ 0 & fs \end{pmatrix} : \forall s \in S \right\}$. Now by [15, Lemma 2.2], $X \leq^{ess} (eM + V)$ where $V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}$. Hence $(eM + V) \oplus [(1-e)M \oplus U] = (M \oplus S)_S$, where $U = \left\{ \begin{pmatrix} -ms \\ (1-f)s \end{pmatrix} : \forall s \in S \right\}$. Thus $(M \oplus S)_S$ is weak CS. □

Recall from [9] that a ring extension T of R is said to be *right intrinsic over R*, if $X \cap R \neq 0$ for each nonzero right ideal X of T, denoted by $R \leq_r^{int} T$.

Proposition 4. *If $R \leq_r^{int} S$ and R_R is a weak CS-module, then S_S is a weak CS-module.*

Proof. Let Y be a semisimple submodule of S and $X = R \cap Y$. Then X is a semisimple submodule R_R. Thus there is $e = e^2 \in R$ such that $X_R \leq^{ess} eR_R$. Let $y \in Y$. Then $y = ey + (1-e)y$. If $(1-e)y \neq 0$, then there exists $r \in R$ such that $0 \neq (1-e)yr \in R \cap Y = X \subseteq eR$, a contradiction. Therefore $(1-e)y = 0$. So $Y \leq eS$. Let $0 \neq es \in eS$. Then $0 \neq esr_1 \in R \cap eS \leq eR$, for some $r_1 \in R$. Hence $R \cap Y \leq R \cap eS \leq eR$. It follows that $0 \neq es(r_1 r_2) \in R \cap Y \leq Y$, for some $r_2 \in R$. Thus $Y_S \leq^{ess} eS_S$ which gives that S_S is a weak CS-module. □

Corollary 2. *Let S be an essential overring of a ring R. If R_R is weak CS, then so is S_R.*

Proof. It is a consequence of Proposition 4. □

Theorem 1. (i) *Let R be a ring such that $R = ReR$ and $S = eRe$ for some $e^2 = e \in R$. Then M_R is weak CS if and only if the right S-module Me is weak CS.*

(ii) *Let R be a ring such that $R = ReR$ for some $e^2 = e \in R$. Then R_R is weak CS if and only if the right eRe-module Re is weak CS.*

Proof. It is clear from [18, Propositions 2.77, 2.78]. □

Corollary 3. *$M_n(R)$ is a right weak CS-ring if and only if the free right R-module R^n is weak CS.*

Proof. Note that $M_n(R) = M_n(R)eM_n(R)$, where e is the matrix unit with 1 in the (1,1)th position and zero elsewhere. Then the result follows from Theorem 1. □

Observe that for any ring R, the polynomial ring $R[x]$ has zero socle. Thus $M_n(R[x])$ is also a weak CS-module by Corollary 3.

Corollary 4. (i) *If $T_n(R)$ is weak CS, then so is $M_n(R)$.*

(ii) *If $T_n(R)$ is weak CS, then the free right R-module R^n is weak CS.*

Proof. (i) Note that $T_n(R)$ is an essential overring of $M_n(R)$. Thus Corollary 2 yields the result.

(ii) It follows from part (i) and Corollary 3. □

In the rest of this section, we deal with some structural properties of semisimple-direct injective modules.

Proposition 5. *Let $M = \bigoplus_{i\in I} M_i$ for some $M_i \leq M$, and consider one of the following conditions are satisfied:*

- (i) *For each semisimple direct summand D of M, $D \subseteq M_i$ for some $i \in I$.*
- (ii) *For each direct summand D of M, $D = \bigoplus_{i\in I}(D \cap M_i)$.*
- (iii) *Each direct summand D of M is fully invariant.*

Then M is semisimple-direct injective if and only if M_i is semisimple-direct injective for all $i \in I$.

Proof. Assume M is semisimple-direct injective and let U and V be semisimple submodules of M_i such that $U \cong V \leq^{\oplus} M_i$. Then we have $V \leq^{\oplus} M$ and hence $U \leq^{\oplus} M$. Then $U \leq^{\oplus} M_i$, so M_i is semisimple-direct injective. Conversely, suppose M_i is semisimple-direct injective for all $i \in I$. Observe that (iii) implies (ii), and (ii) implies (i). Thus it is enough to complete the proof for condition (i). Now, let A and B be semisimple direct summands of M with $A \cap B = 0$. By (i), $A \subseteq M_i$ and $B \subseteq M_j$ for some $j,k \in I$. Note that $A \leq^{\oplus} M_j$ and $B \leq^{\oplus} M_k$. Assume $j \neq k$. Then $A \oplus B \leq^{\oplus} M_j \oplus M_k \leq^{\oplus} M$. If $j = k$, then $A \oplus B \leq^{\oplus} M_j$, as M_j is semisimple-direct injective. Hence $A \oplus B \leq^{\oplus} M$. Therefore M is semisimple-direct injective by [1, Proposition 2.1]. □

The conditions (i), (ii) and (iii) in Proposition 5 are not superfluous. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}(p^\infty)$ for a prime p. Since M is not simple-direct injective by [11, Lemma 2.5], M is not semisimple-direct injective. Let $X = (1,a)\mathbb{Z}$, where $0 \neq a \in \mathbb{Z}(p^\infty)$ such that $pa = 0$. Then $X \oplus \mathbb{Z}(p^\infty) = M$. But $X \nsubseteq \mathbb{Z}_p \oplus 0$ and $X \nsubseteq 0 \oplus \mathbb{Z}(p^\infty)$. Note that $T_p(M) = \{x \in M \,|\, xp^k = 0 \text{ for some non-negative integer } k\}$ is a submodule of M which is called the *p-primary component* of M. It is well-known that every torsion R-module is a direct sum of its p-primary components.

Corollary 5. *Let R be a Dedekind domain and T a torsion module over R. Then T is semisimple-direct injective if and only if the $T_p(M)$ is semisimple-direct injective for each prime p.*

Proof. Note that $T_p(M)$ is fully invariant. Now, it follows from Proposition 5. □

Proposition 6. *Let M be an R-module. If $M \oplus M$ is a semisimple-direct injective module, then M is a semisimple-direct injective module.*

Proof. Let $M \oplus M$ be a semisimple-direct injective module and S be a semisimple submodule of M such that $S \cong S' \leq^{\oplus} M$. Clearly, S' is a semisimple submodule of M. We need to show that $S \leq^{\oplus} M$. Write $M = S' \oplus T$ for some $T \leq M$. Since $M \oplus M = (S' \oplus T) \oplus M = S' \oplus (M \oplus T)$ is a semisimple-direct injective module, and if we take $\tau\colon S \to S'$ as the preceding isomorphism, $\tau^{-1}\colon S' \to M = (S' \oplus T)$ splits by [1, Proposition 2.1(4)]. Hence $S \leq^{\oplus} M$. □

Theorem 2. *Let $M = A \oplus B$, where A and B are relatively soc-injective. Then M is a semisimple-direct injective module.*

Proof. Let X and Y be any two semisimple direct summands of M with $X \cap Y = 0$. We will show that $X \oplus Y \leq^{\oplus} M$. We have submodules X', Y' of M such that $M = X \oplus X'$ and $M = Y \oplus Y'$. By the hypothesis, X and X' are relatively soc-injective and so also Y and Y'. Hence by [2, Theorem 2.2 (4)], X (respectively, Y) is soc-$(X \oplus X')$-injective (respectively, soc-$(Y \oplus Y')$-injective). Then the semisimple module $X \oplus Y$ is soc-M-injective by [2, Theorem 2.2 (1)]. Therefore $X \oplus Y \leq^{\oplus} M$. □

3. Semisimple-continuous modules

In this section, the class of semisimple-continuous modules is introduced and investigated. Module theoretical properties such as direct summands and direct sums of the former class of modules are examined and examples are given to illustrate the results.

Definition 1. A module M is called *semisimple-continuous*, if M is both weak CS and semisimple-direct injective. A ring R is called *right semisimple-continuous* if the module R_R is semisimple-continuous.

Example 1. (i) Semisimple, uniform and (quasi-)continuous modules are semisimple-continuous modules.

(ii) Semisimple-continuous modules are simple-continuous, but not vice versa:

(1) Consider $R =< \oplus_{i=1}^{\infty} F_i, 1_{\prod_{i=1}^{\infty} F_i} >$ as a subring of $\prod_{i=1}^{\infty} F_i$ generated by $\oplus_{i=1}^{\infty} F_i$ and $1_{\prod_{i=1}^{\infty} F_i}$, where $F_i = \mathbb{Z}_2$ for any $i \in \mathbb{N}$. Then R is a commutative, non self-injective V-ring and $Soc(R)$ is essential in R. It implies that R is not Noetherian. Therefore we can infer from Theorem 6 or [19, Theorem 3.1] that there exists a simple-continuous module over R which is not semisimple-continuous.

(2) Let V be an infinite-dimensional vector space over a field F. Let $Q = End_F(V)$, $J = \{x \in Q : dim_F(xV) < \infty\}$ and $R = F + J$. Then R is a right V-ring and R is not right Noetherian. By Theorem 6 and [19, Theorem 3.1], there is a simple-continuous right R-module which is not semisimple-continuous.

(3) Let $L = End(V_F)$ be the full right linear ring of an infinite dimensional right vector space V over a field F, let S be the ideal consisting of linear transformations of finite rank, and let $R = S + F$ be the subring generated by S and the subring F consisting of scalar transformations (sending every $v \longrightarrow va$ for some $a \in F$). Then R is a right V-ring such that $R/Soc(R)$ is a field, hence a simple right R-module. Now it is easy to see that R_R is not weak CS. For this write $Soc(R) = A \oplus B$, where A and B are infinitely generated semisimple right ideals. If R_R were weak CS, A and B would be essential in some direct summands A' and B', respectively. It is clear that A and B are proper submodules of A' and B'. But then, $\frac{A' \oplus B'}{A \oplus B} \subseteq \frac{R}{Soc(R)}$, a contradiction since $\frac{A' \oplus B'}{A \oplus B}$ is not simple. Note that R_R is simple-continuous by [19, Theorem 3.1].

As an application of Proposition 1, we construct the following example.

Example 2. Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a generalized triangular matrix ring, where $R = M = \mathbb{Z}_2$ and $S = \mathbb{Z}$. Then ${}_RM$ is a faithful module and $Soc(T_T) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$. Clearly, $Soc(T_T)$ essential in αT, where $\alpha^2 = \alpha = \begin{pmatrix} \bar{1} & \bar{0} \\ 0 & 0 \end{pmatrix} \in T$. Hence T_T is right weak CS by Corollary 1. Moreover, T_T is right semisimple-direct injective by [1, Theorem 2.21], so T_T is right semisimple-continuous.

Proposition 7. *M is a semisimple-continuous module if and only if the followings hold:*

(i) Let $X \leq Soc(M)$. Then there exists a closed submodule K in M such that $X \leq^{ess} K$ and any homomorphism $\phi \colon K \oplus L \to M$ can be extended to M for some complement L of K.

(ii) For $X, Y \leq Soc(M)$ with $X \cong Y \leq^{\oplus} M$, every homomorphism $\theta \colon X \to M$ can be extended to M.

Proof. Let M be a semisimple-continuous module. Then (i) and (ii) hold clearly. For converse, condition (i) implies that K is a direct summand of M by [18, Lemma 3.97]. Hence $X \leq^{ess} K \leq^{\oplus} M$ for all $X \leq Soc(M)$. Thus M is weak CS. By condition

(ii), X is a direct summand of M by virtue of [18, Proposition 2.85]. Hence M is semisimple-direct injective, so M is semisimple-continuous. □

Proposition 8. *Let R be a Dedekind domain. If the component $T_p(M)$ is semisimple-direct injective for every prime p, then every finitely generated R-module M is a semisimple-continuous module.*

Proof. Let M be a finitely generated module, then by [17, Theorem 1.16] M is a weak CS-module. To see that M is semisimple-direct injective, note that M has a decomposition $M = M_1 \oplus M_2$ for some torsion module M_1 and torsion-free module M_2. Note that, if S is any semisimple submodule of M, then $S \subseteq M_1$. Hence M is a semisimple-direct injective module by Corollary 5. □

Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_8 \oplus \mathbb{Z}_2$. It is clear that M is finitely generated module over Dedekind domain. But it is not a semisimple-continuous module by Example 3(1). The following results are related to the direct summand of semisimple-continuous modules.

Proposition 9. *Let M be a semisimple-continuous module module and X a fully invariant direct summand of M. Then X is semisimple-continuous.*

Proof. Let X be a fully invariant direct summand of a semisimple-continuous module M. Then X is semisimple-direct injective by Proposition 5. Assume A is a semisimple submodule of X. Then by hypothesis, $A \leq^{ess} Y \leq^{\oplus} M$ for some $Y \leq M$. Then A is essential in $Y \cap X$ and clearly, $Y \cap X$ is a direct summand of X. Thus X is weak CS. Therefore X is semisimple-continuous. □

Recall that M is called a *UC-module*, if every submodule of M has a unique essential closure.

Proposition 10. (i) *Let M be a semisimple-continuous and UC module. Then any direct summand of M is semisimple-continuous.*

(ii) *Let M be a semisimple-continuous module satisfying $C3$. Then any direct summand of M is semisimple-continuous.*

(iii) *Any direct summand of a nonsingular semisimple-continuous module is semisimple-continuous.*

Proof. The proof follows from Proposition 5 and [18, Propositions 4.7-4.9, Corollary 4.8]. □

Corollary 6. *Let M be a semisimple-continuous module. If M is a duo, or M has an Abelian endomorphism ring, then every direct summand of M is semisimple-continuous.*

Proof. It follows from [7, Theorem 4.4] and Proposition 10. □

Proposition 11. *Let R be an Artian serial ring with $J(R)^2 = 0$. Then any direct summand of semisimple-continuous R-module is semisimple-continuous.*

Proof. By Proposition 5, any direct summand of semisimple-direct injective R-module is semisimple-direct injective. Hence [1, Theorem 2.10] implies that every semisimple-continuous R-module has $C3$. Thus the proof follows from Proposition 10. □

Proposition 12. *Let $M = M_1 \oplus M_2$ be a semisimple-continuous module and M_1 be a semisimple fully invariant submodule of M. Then M_1 and M_2 are semisimple-continuous.*

Proof. M_1 is a semisimple-continuous module by Proposition 9. Let X be a semisimple submodule of M_2. Then $M_1 \oplus X$ is a semisimple submodule of M. Then there exists a direct summand N of M such that $M_1 \oplus X \leq^{ess} N$. Note that $M_1 \subseteq M_1 \oplus X \subseteq N$, so by modular law, $N = N \cap (M_1 \oplus M_2) = M_1 \oplus (N \cap M_2)$. Since $M_1 \oplus X \subseteq N$, we have $M_2 \cap (M_1 \oplus X) \leq^{ess} N \cap M_2$. Notice that $X = M_2 \cap (M_1 \oplus X)$ by modular law. Thus, $X \leq^{ess} N \cap M_2$. It is enough to show that $N \cap M_2$ is a direct summand of M_2. Observe that $M = N \oplus W = M_1 \oplus (N \cap M_2) \oplus W$, and hence $M_2 = M_2 \cap M = M_2 \cap (M_1 \oplus (N \cap M_2) \oplus W) = (N \cap M_2) \oplus [M_2 \cap (M_1 \oplus W]$. Therefore $N \cap M_2$ is a direct summand of M_2. Thus M_2 is a weak CS-module. Note that M_2 is semisimple-direct injective by Proposition 5. Therefore M_2 is semisimple-continuous. □

Recall that a module M has the *summand intersection property*, *SIP*, in case the intersection of any two direct summands is again a direct summand of M.

Proposition 13. *Assume M is semisimple-continuous with SIP. Then any direct summand of M is semisimple-continuous.*

Proof. Assume $M = D \oplus D'$ for some $D, D' \leq M$. To see D is weak CS, let $A \leq Soc(D)$. Then $A \oplus Soc(D') \leq Soc(M)$. By hypothesis, $A \oplus Soc(D') \leq^{ess} K$, for some direct summand K of M. Now clearly, $D \cap (A \oplus Soc(D')) \leq^{ess} D \cap K$. By modular law, $D \cap (A \oplus Soc(D')) = A$. Hence $A \leq^{ess} D \cap K$. But then M has SIP and so $D \cap K$ a direct summand of M. Observe that $D \cap K$ is a direct summand of D. Thus D is weak CS. Note that SIP condition implies semisimple-direct injectivity, hence M is semisimple-continuous. □

Proposition 14. *Let M be an R-module. If $M \oplus M$ is a semisimple-continuous module, then M is semisimple-continuous.*

Proof. In view of Proposition 6, we only show that M is a weak CS-module. Let S be a semisimple submodule of M. Then $S' = S \oplus 0$ is semisimple in $M \oplus M$. So by hypothesis, $S' \leq^{ess} D$, where $D \leq^{\oplus} (M \oplus M)$. Consider $T = \pi_1(D)$ where $\pi_1 : M \oplus M \to M$ is the first projection on M. Then T is a summand of M and clearly, S is essential in T. □

The next example shows that the direct sum of any two semisimple-continuous modules may not be semisimple-continuous.

Example 3. (1) Let $M=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z}$ be a $\mathbb{Z}$-module. Clearly, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ are semisimple-continuous $\mathbb{Z}$-modules. However, M is not a semisimple continuous $\mathbb{Z}$-module, because the simple non-summand $0\oplus\mathbb{Z}(4+8\mathbb{Z})$ is isomorphic to the simple summand $\mathbb{Z}/2\mathbb{Z}\oplus 0$.

(2) Let R be the trivial extension of $\mathbb{Z}_4$ with the $\mathbb{Z}_4$-module $2\mathbb{Z}_4$, such that

$$R=\left\{\begin{pmatrix} a & 2b \\ 0 & a\end{pmatrix} : a,b\in\mathbb{Z}_4\right\}.$$

Let $X=2\mathbb{Z}_4\oplus 0$, $Y=0\oplus 2\mathbb{Z}_4$, and $A=R/X$ and $B=R/Y$. Then A and B are uniform, and hence semisimple-continuous. Since $A\oplus B$ is not weak *CS* from [18, Example 4.3], $A\oplus B$ is not semisimple-continuous.

Proposition 15. *Let $M=A\oplus B$, where A is weak CS-module such that $Soc(A)\leq Rad(A)$ and B is fully invariant semisimple submodule of M. Then M is a semisimple-continuous module.*

Proof. M is a weak CS-module, by [17, Lemma 1.10]. We show that M is a semisimple-direct injective module. Consider semisimple submodules X and Y of M such that $X\cong Y$ and $Y\leq^{\oplus} M$. Then Y is not contained in A because each simple module of A is small in A. This shows that $\pi_2(Y)\neq 0$, where $\pi_2\colon M\to B$ is the projection. Since B is semisimple, $\pi_2(Y)$ is a direct summand of B. So $X\cong Y\cong\pi_2(Y)\leq^{\oplus} B$ which implies that there exists $f\in End(M_R)$ such that $f(\pi_2(Y))=X$. Since $Soc(B)$ is fully invariant in M, then $X\subseteq Soc(B)$. As above X is not small in B, $X\leq^{\oplus} B$. Hence $X\leq^{\oplus} M$. □

The fully invariant condition in Proposition 15 is not superflous. For example, let $M_{\mathbb{Z}}=A\oplus B$, where $A=\mathbb{Z}/8\mathbb{Z}$ and $B=\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}$-modules. Then A is a weak CS-module such that $4\mathbb{Z}/8\mathbb{Z}=Soc(A)\leq Rad(B)=2\mathbb{Z}/8\mathbb{Z}$ and B is a semisimple module but B is not fully invariant in M. Indeed, take $\theta=i\circ f\circ\pi$, where $i\colon(4\mathbb{Z}/8\mathbb{Z}\oplus 0)\to M$ is the inclusion map, $f\colon(0\oplus\mathbb{Z}/2\mathbb{Z})\to(4\mathbb{Z}/8\mathbb{Z}\oplus 0)$ is the isomorphism and $\pi\colon M\to(0\oplus\mathbb{Z}/2\mathbb{Z})$ is the natural projection map. Then clearly $\theta\in End(M)$ and $\theta(B)=(4\mathbb{Z}/8\mathbb{Z}\oplus 0)\nleq B$. By Example 3, M is not a semisimple-continuous module.

Theorem 3. *Let $M=\bigoplus_{i\in I}X_i$, where each X_i is a fully invariant submodule of M for all $i\in I$, and all semisimple submodules of M are fully invariant in M. Then X_i is semisimple-continuous for all $i\in I$ if and only if M is a semisimple-continuous module.*

Proof. $(\Leftarrow)$ It is clear from Proposition 9.

$(\Rightarrow)$ By Proposition 5, M is semisimple-direct injective. Now we show that M is a weak CS-module. Let S be a semisimple submodule of M. By hypothesis, S is fully invariant. Then $S=\oplus_{i\in I}(S\cap X_i)$. As X_i is weak CS, there is $D_i\leq^{\oplus} X_i$ such that $(S\cap X_i)\leq^{ess} D_i$ for all $i\in I$. Hence $S=\oplus_{i\in I}(S\cap X_i)\leq^{ess}(\oplus_{i\in I}D_i)\leq^{\oplus}(\oplus_{i\in I}X_i)=M$. □

Theorem 4. *Let $M = A \oplus B$ be a UC-module such that $Soc(A) \leq^{ess} A$ and $Soc(B) = 0$. Then M is semisimple-continuous if and only if A and B are semisimple-continuous.*

Proof. ($\Rightarrow$) Assume M is semisimple-continuous then both A and B are semisimple-continuous by Proposition 10.

($\Leftarrow$) Let X, Y be semisimple submodules of M with $X \cong Y \leq^{\oplus} M$. Then Y is not contained in B because $Soc(B) = 0$. Then $Y \leq A$, so $Y \leq^{\oplus} A$ but then A is semisimple-direct injective and so $X \leq^{\oplus} A$ which in turn implies that $X \leq^{\oplus} M$. Thus M is semisimple-direct injective. [6, Corollary 1.1] yields the result. □

4. Decompositions

Recall that any submodules A and B of M, A is *superspective* to B, if for any submodule $X \leq M$, $M = A \oplus X$ if and only if $M = B \oplus X$. Two modules are *orthogonal*, if they have no non-zero isomorphic submodules. For any class $\mathcal{K}$ of modules, $\mathcal{K}^{\perp}$ denotes the class of modules orthogonal to all members of $\mathcal{K}$. A pair of classes $\mathcal{A}$ and $\mathcal{B}$ are called an *orthogonal pair*, if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{B}^{\perp} = \mathcal{A}$.

Proposition 16. *Let $\mathcal{A}$ and $\mathcal{B}$ be an orthogonal pair of classes of modules:*

(1) *If M is weak CS which is closed under direct summand, then $M = A \oplus B$ with semisimple $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In fact, in this case, M is necessarily semisimple.*
(2) *Every semisimple-continuous module M which is closed under direct summand, has a decomposition such that $M = A \oplus B$ with semisimple $A \in \mathcal{A}$ and $B \in \mathcal{B}$ unique up to superspectivity.*

Proof. (1) By Zorn's Lemma, M has a semisimple submodule A maximal w.r.t. $A \in \mathcal{A}$. Since $\mathcal{A}$ is closed under essential extensions, A is a closed submodule of M; hence $A \leq^{\oplus} M$, as M is weak CS. Then $M = A \oplus B$ for some $B \leq M$. Applying the same argument to B, we get $B = C \oplus D$ where C is maximal semisimple submodule such that $C \in \mathcal{B}$. Assume that $D \neq 0$. Since $D \notin \mathcal{B}$, D contains a non-zero semisimple submodule $Z \in \mathcal{A}$; which is a contradiction to the maximality of $A \in \mathcal{A}$. Hence $D = 0$ and so $M = A \oplus B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(2) Let $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ for all $i = 1, 2$. Assume that $M = A_1 \oplus X$. Then $X \cong B_1$, hence $X \in \mathcal{B}$ and therefore $A_2 \cap X = 0$. By hypothesis, $A_2 \oplus X \leq^{\oplus} M$, and so $M = A_2 \oplus X \oplus Y$. Then $A_2 \oplus Y \cong A_1$ and $X \oplus Y \cong B_2$. Consequently $Y \in \mathcal{A} \cap \mathcal{B} = 0$, so $M = A_2 \oplus X$. This proves that A_1 and A_2 are superspective. Similarly one can prove that B_1 and B_2 are superspective. □

We provide the decomposition theorem for the semisimple-continuous modules.

Theorem 5. *If M is a semisimple-continuous module, then $M = A \oplus B \oplus K$, where*

(1) *$A \cong B$,*
(2) *$A \oplus B$ is semisimple,*
(3) *$Soc(K)$ is a square-free module, and*

(4) *K is $(A\oplus B)$-injective.*

Proof. Let $F=\{(A,B,f)\mid A,B$ are semisimple closed submodules in M such that $A\cap B=0$, and $A\cong^{f} B\}$. Define an order on F as follows:

$$(A,B,f)\leq(A_1,B_1,f_1)\Leftrightarrow A\leq A_1, B\leq B_1, \text{ and } f_1 \text{ extends } f.$$

Clearly, F is a non-empty partially ordered set and every chain of elements of F has an upper bound in F. Then by Zorn's Lemma, F has a maximal member, say, (A,B,f). Since M is weak CS, there exist $A_1,B_1\leq M$ such that $A\leq^{ess}A_1\leq^{\oplus}M$, $B\leq^{ess}B_1\leq^{\oplus}M$. Note that $A_1\cap B_1=0$. But A and B are closed in M, so $(A,B,f)=(A_1,B_1,f)$. By semisimple-direct injectivity, we have $A\oplus B\leq^{\oplus}M$. Write $M=(A\oplus B)\oplus K$ for some $K\leq M$. Since $A\oplus B$ is semisimple with $A\cong B$, we prove $(1),(2)$ and (4). To show that condition (3) holds, let X and Y are submodules of $Soc(K)$ such that $X\cong^{\sigma}Y$ and $X\cap Y=0$. Then $(A,B,f)\leq(A\oplus X,B\oplus Y,f\oplus\sigma)$. By maximality of (A,B,f), we have $(A,B,f)=(A\oplus X,B\oplus Y,f\oplus\sigma)$. Hence $A=A\oplus X$ and $B=B\oplus Y$ which yield $X=Y=0$. Thus $Soc(K)$ is a square-free module. □

Proposition 17. *Let G be an Abelian group. Then G is semisimple-continuous if and only if $G=T\oplus F$, where F is torsion-free and T is a torsion Abelian group with each p-component, $T_p(M)$, a direct sum of a bounded Abelian group and a divisible Abelian group. Moreover, $T_p(M)$ is semisimple or $Soc(T_p(M))\subseteq Rad(T_p(M))$, for each prime p.*

Proof. $(\Rightarrow)$ Assume G is semisimple-continuous. Then $G=T\oplus F$, where $Soc(G)\leq^{ess}T$ such that F is torsion-free and T is a torsion Abelian group with each p-component, T_p, a direct sum of a bounded Abelian group and a divisible Abelian group by [18, Corollary 5.98]. Note that $Soc(G)=Soc(T)$ and $Soc(F)=0$. Now T is semisimple-direct injective and so T is simple-direct injective, then by [3, Theorem 2 (iv)], for each prime p, T_p is semisimple, or $Soc(T_p)\subseteq Rad(T_p(M))$.

$(\Leftarrow)$ Suppose $G=T\oplus F$, where F is torsion-free and T is a torsion Abelian group with each p-component, T_p, a direct sum of a bounded Abelian group and a divisible Abelian group. Then G is weak CS by [18, Corollary 5.98]. Now we will show that G is semisimple-direct injective. Firstly, if for each prime p, T_p is semisimple, then T is semisimple-direct injective by Corollary 5. Also, $Soc(F)=0$ so it is straightforward to see that $T\oplus F$ is semisimple-direct injective. Now if $Soc(T_p)\leq Rad(T_p(M))$, then $Soc(T)\leq Rad(T)$ and $Soc(F)\cap Rad(F)=0$. Let A, B be semisimple submodules of G with $A\cong B\leq^{\oplus}G$. Then B is not contained in T because each simple module of T is small in T. This shows that $\pi(B)\neq 0$, where $\pi\colon T\oplus F\to F$ is the projection map. But $Soc(F)=0$, so $\pi(B)=0$, a contradiction and hence $A=B=0$. This shows that G is semisimple-direct injective. □

Recall that an R-module M is said to be a *C4-module* [8], if $B\cong A\leq^{\oplus}M$, $B\leq M$ and $A\cap B=0$, then $B\leq^{\oplus}M$. Clearly, $C4$ condition implies semisimple-direct injective, but not vice versa. For instance, let $M_{\mathbb{Z}}=\mathbb{Z}\oplus 2\mathbb{Z}$. Since $Soc(M)=0$, M is semisimple-direct injective. However it is not a $C4$-module. Note that a module M is called *pseudo- N-injective* if every monomorphism $f\colon K \to M$, where $K \leq N$, can be extended to a homomorphism from N into M.

Proposition 18. *Let $M = A \oplus B$ be a weak CS and C4-module, then A is pseudo-Soc(B)-injective and B is pseudo-Soc(A)-injective.*

Proof. We only show that B is pseudo- Soc(A)-injective. In a similar manner, it can be shown that A is pseudo- Soc(B)-injective. Let $A' \leq Soc(A)$ and consider a monomorphism $\phi\colon A' \to B$. Then $K = \{a' - \phi(a')\colon a' \in A'\} \leq Soc(M)$. Also $K \cap A = K \cap B = 0$. Hence $K \leq^{ess} D \leq^{\oplus} M$ by weak-CS condition. Now consider $\pi\colon M \to A$. Then $B \oplus D = B \oplus \pi(D)$. Hence we have $D \cong \pi(D)$ and $D \cap \pi(D) = 0$. Since M has C4, $\pi(D) \leq^{\oplus} M$. So $\pi(D) \leq^{\oplus} A$. Write $A = \pi(D) \oplus D'$ for some $D' \leq A$. Then $M = B \oplus A = B \oplus \pi(D) \oplus D' = B \oplus (D \oplus D')$. Now take the canonical projection $\pi\colon B \oplus (D \oplus D') \to B$. Then we have the restriction map $\pi|_A\colon A \to B$. Hence $\pi(a') = \pi(a' - \phi(a')) + \pi(\phi(a')) = \phi(a')$. This shows that $\pi|_A$ extends ϕ. □

Corollary 7. (i) *Let $M = A \oplus B$ be a weak CS and C4-module, then A is min-B-injective and B is min-A-injective.*

(ii) *Let R be a commutative ring. If $R \oplus R$ is semisimple-continuous as an R-module, then R is mininjective.*

Proof. It follows from Proposition 18. □

Proposition 19. *If M is weak CS and a C4-module and $A \leq^{\oplus} M$, then A is pseudo-Soc(B)-injective for any submodule B of M with $A \cap B = 0$.*

Proof. Write $M = A \oplus A'$ for a submodule $A' \leq M$ and consider the natural projection $\gamma\colon M \to A'$. Clearly, the restriction of γ on B is a monomorphism and so $B \cong \gamma(B) \leq A'$. Since M is weak CS and C4, A is pseudo-soc-A'-injective by Proposition 18. Consequently, A is pseudo-soc-$\gamma(B)$- injective which in turn gives that A is pseudo-soc-B-injective. □

In the following result, we examine when the finite exchange property implies full exchange property for the related module classes.

Proposition 20. (i) *Let M be a weak CS module and every submodule of M is a C4-module. If M has the finite exchange property, then M has the full exchange property.*

(ii) *Let R be a semiartinian ring and M be an R-module. If M is a semisimple-continuous module with the finite exchange property, then M has the full exchange property.*

Proof. (i) Let M be a weak CS module and every submodule of M be a C4-module. Clearly, M is semisimple-continuous. Then, by Proposition 5, $M = (A \oplus B) \oplus K$, where $A \oplus B$ is semisimple and K has square-free socle. [8, Proposition 2.20] yields that K is square-free. Since direct summands and finite direct sums of modules with the (finite) exchange property also have the (finite) exchange property, M has the full exchange property.

(ii) By Proposition 5, we have $M = (A \oplus B) \oplus K$, where $A \oplus B$ is semisimple and K has square-free socle. Now by [8, Proposition 2.21], K is square-free. Hence the proof follows from similar arguments in the proof of part (i). $\square$

Lemma 2. *If M is semisimple and $M \oplus E(M)$ is semisimple-continuous, then M is injective.*

Proof. Let M be semisimple and $i\colon M \to E(M)$ be the inclusion map. Hence $i(M) = M \leq^{\oplus} E(M)$ by [1, Proposition 2.1(4)]. Therefore $M = E(M)$ and M is injective. $\square$

Proposition 21. *The following assertions are equivalent for a ring R:*

(i) *Every projective semisimple right R-module is injective.*

(ii) *Every nonsingular right R-module is semisimple-continuous.*

Proof. $(i) \Rightarrow (ii)$ It is known from [10, Corollary 1.25] that any nonsingular semisimple right R-module is projective. Let M be a nonsingular module with semisimple direct summands $A, B \leq M$ such that $A \cap B = 0$. Then A and B are injective by (i). Hence $A \oplus B \leq^{\oplus} M$. Thus M is semisimple-direct injective. To show that M is weak CS, let S be a semisimple submodule of M. Then S is also injective by hypothesis. Therefore $S \leq^{\oplus} M$, so clearly M is weak CS. Thus, (ii) holds.

$(ii) \Rightarrow (i)$ Let P be a projective semisimple right R-module. By [13, p. 269], P is nonsingular and hence $E(P)$ and $P \oplus E(P)$ both are nonsingular. By assertion (ii), we have $P \oplus E(P)$ is semisimple-continuous. Thus, P is injective by Lemma 2. $\square$

Theorem 6. *The following conditions are equivalent for a ring R.*

(i) *R is a right Noetherian right V-ring.*

(ii) *Every right R-module is a semisimple-continuous module.*

(iii) *The direct sum of any two semisimple-continuous right R-modules is semisimple-continuous.*

(iv) *Every semisimple-continuous module is strongly soc-injective.*

Proof. $(i) \Rightarrow (ii)$ It is clear because R is right Noetherian right V-ring if and only if every semisimple R-module is injective by [4, Proposition 1].

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ Let M be a semisimple module. By the hypothesis, $M \oplus E(M)$ is a semisimple-continuous module. By Lemma 2, M is injective. Therefore, R is a right Noetherian right V-ring.

$(i) \Leftrightarrow (iv)$ This part follows from [1, Proposition 2.15]. $\square$

Corollary 8. *A ring R is semisimple Artinian if and only if all semisimple-continuous right R-modules are injective.*

Corollary 9. *Let R be a right V-ring. Then R is right Noetherian if and only if every simple-continuous right R-module is semisimple-continuous.*

Proposition 22. *The followings are equivalent for a regular ring R:*

(i) *R is a right V-ring.*
(ii) *Every cyclic right R-module is semisimple-continuous.*
(iii) *Every cyclic right R-module is simple-continuous.*

Proof. It follows from Example 1(ii) and [19, Proposition 3.3]. □

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