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SOME PROPERTIES OF r-SMALL ELEMENTS IN LATTICES

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Abstract. In this paper, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice, $a \in L$ and $a \leq r(L)$. If $a \ll r(L)/0$, then a is called an r-small (or r-superfluous) element of L and denoted by $a \ll_r L$. In this work, some properties of these elements are investigated. This concept is a generalization of an r-small submodule of any module. It is clear that every r-small element is small. But the converse of this statement is not true in general. Let L be a lattice, $a \in r(L)/0$ and $r(L)$ be a supplement element in L. Then $a \ll L$ if and only if $a \ll_r L$. Let L be a lattice, $a \in L$ and $b \ll_r L$. Then $a \ll_r L$ if and only if $a \vee b \ll_r 1/b$.

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1. INTRODUCTION

In this paper, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice and $a,b \in L$ with $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice* and denoted by b/a. Let L be a lattice and $a,b \in L$. If $a \vee b = 1$ and $a \wedge b = 0$, then a is called a *complement* of b in L and denoted by $1 = a \oplus b$ (here we also call a and b are *direct summands* of L). L is called a *complemented* lattice if every element of L has at least one complement in L. Let L be a lattice and $a \in L$. a is called a *compact* element of L if every subset X of L with $a \leq \vee X$ there exists a finite subset F of X such that $a \leq \vee F$. L is said to be *compact* if 1 is compact in L. L is said to be *compactly generated* if each element of L is a join of compact elements in L. Let L be a lattice and $a \in L$. If $b = 1$ for every $b \in L$ with $a \vee b = 1$, then a is called a *small* (or *superfluous*) element of L and denoted by $a \ll L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If L have no maximal $(\neq 1)$ elements, then the radical of L is defined by $r(L) = 1$. Let L be a lattice and $a,b \in L$. If a is minimal for $1 = b \vee a$, then a is called a *supplement* of b in L. a is a supplement of b in a lattice L if and only if $1 = b \vee a$ and $b \wedge a \ll a/0$. A lattice L is said to be *supplemented* if every element of L has at least one supplement in L. L is said to be *hollow* if every element with distinct from 1 is small in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement x in L with $x \leq b$. L is said to be *amply supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by $x\beta_* y$ with $x, y \in L$ if and only if for each $t \in L$ such that $x \vee t = 1$ then $y \vee t = 1$ and for each $k \in L$ such that $y \vee k = 1$ then $x \vee k = 1$. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$.

More information about (amply) supplemented lattices are in [1,5,7]. More results about (amply) supplemented modules are in [4, 6, 12]. The definition of β_* relation on lattices and some properties of this relation are in [8]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [3].

Lemma 1. *Let L be a lattice and $a, b, c, d \in L$. Then the followings hold.*

(i) *If $a \leq b$ and $b \ll L$, then $a \ll L$.*
(ii) *Let $a \leq b$. If $a \ll L$ and $b \ll 1/a$, then $b \ll L$.*
(iii) *If $a \ll b/0$, then $a \ll t/0$ for every $t \in L$ with $b \leq t$.*
(iv) *Let $a \leq b$ and b be a supplement element in L. Then $a \ll b/0$ if and only if $a \ll L$.*
(v) *If $a \ll b/0$, then $a \vee c \ll (b \vee c)/c$.*
(vi) *If $a \ll L$, then $a \vee b \ll 1/b$.*
(vii) *If $a \ll b/0$ and $c \ll d/0$, then $a \vee c \ll (b \vee d)/0$.*
$(viii)$ *If $a \ll L$ and $b \ll L$, then $a \vee b \ll L$.*
(ix) *If $a \ll L$, then $a \leq r(L)$.*
(x) *If a is compact in L and $a \leq r(L)$, then $a \ll L$.*
(xi) *If L is compactly generated, then $r(L) = \bigvee_{x \ll L} x$.*
(xii) *If L is compact, then $r(L) \ll L$.*

Proof. See [5]. □

2. r-SMALL ELEMENTS IN LATTICES

Definition 1. Let L be a lattice, $a \in L$ and $a \leq r(L)$. If $a \ll r(L)/0$, then a is called an r-small (or r-superfluous) element of L and denoted by $a \ll_r L$. (See also [2]).

This concept is a generalization of an r-small submodule of any module. The definition of r-small submodules and some properties of these submodules are in [9–11].

Proposition 1. *Let L be a lattice, $a \in L$ and $r(L) = 1$. Then $a \ll_r L$ if and only if $a \ll L$.*

Proof. Clear from definitions. □

Lemma 2. *Let L be a lattice and $a \in L$. If $a \ll_r L$, then $a \ll L$.*

Proof. Since $a \ll_r L$, $a \ll r(L)/0$. Then by Lemma 1, $a \ll L$, as desired. □

The converse of this lemma is not true in general. (See Example 1, Example 2 and Example 3).

Lemma 3. *Let L be a lattice and $a \in L$. If $a \ll L$ and $r(L)$ is a supplement element in L, then $a \ll_r L$.*

Proof. Since $a \ll L$, by Lemma 1, $a \leq r(L)$. Since $a \leq r(L)$ and $r(L)$ is a supplement element in L, by Lemma 1, $a \ll r(L)/0$ and $a \ll_r L$, as desired. □

Corollary 1. *Let L be a lattice, $a \in r(L)/0$ and $r(L)$ be a supplement element in L. Then $a \ll L$ if and only if $a \ll_r L$.*

Proof. $(\Longrightarrow)$ Clear from Lemma 3.
$(\Longleftarrow)$ Clear from Lemma 2. □

Corollary 2. *Let L be a lattice, $a \in r(L)/0$ and $r(L)$ be a direct summand of L. Then $a \ll L$ if and only if $a \ll_r L$.*

Proof. Clear from Corollary 1. □

Proposition 2. *Let L be a lattice and $a\beta_* b$ in $r(L)/0$. If $b \ll_r L$, then $a \ll_r L$.*

Proof. Since $b \ll_r L$, then $b \ll r(L)/0$. Since $a\beta_* b$ in $r(L)/0$, by [8, Theorem 2], $a \ll r(L)/0$. Hence $a \ll_r L$, as desired. □

Lemma 4. *Let L be a lattice, $a \ll L$ and $r(L)$ be a supplement element in L. If $b \in r(L)/0$ and $a\beta_* b$ in L, then $b \ll_r L$.*

Proof. Since $a\beta_* b$ in L and $a \ll L$, by [8, Theorem 2], $b \ll L$. Since $b \leq r(L)$ and $r(L)$ is a supplement element in L, by Corollary 1, $b \ll_r L$. □

Corollary 3. *Let L be a lattice, $a \ll L$ and $r(L)$ be a direct summand of L. If $b \in r(L)/0$ and $a\beta_* b$ in L, then $b \ll_r L$.*

Proof. Clear from Lemma 4. □

Proposition 3. *Let L be a lattice and $a \ll_r L$. Then $a \ll c/0$ for every maximal $(\neq 1)$ element c of L.*

Proof. Let c be a maximal $(\neq 1)$ element of L. Since $a \ll_r L$, $a \ll r(L)/0$ and since $r(L) \leq c$, by Lemma 1, $a \ll c/0$, as desired. □

Lemma 5. *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll_r b/0$, then $a \ll_r L$.*

Proof. Since $a \ll_r b/0$, $a \ll r(b/0)/0$. Then because of $r(b/0) \leq r(L)$, by Lemma 1, $a \ll r(L)/0$. Hence $a \ll_r L$, as desired. □

Proposition 4. *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $b \ll_r L$, then $a \ll_r L$.*

Proof. Since $b \ll_r L$, $b \ll r(L)/0$. Since $a \leq b$, by Lemma 1, $a \ll r(L)/0$. Hence $a \ll_r L$, as desired. □

Lemma 6. *Let L be a lattice and $a, b \in L$. If $a \ll_r L$, then $a \vee b \ll_r 1/b$.*

Proof. Since $a \ll_r L$, $a \ll r(L)/0$ and by Lemma 1, $a \vee b \ll (r(L) \vee b)/b$. Then by $r(L) \vee b \leq r(1/b)$ and Lemma 1, $a \vee b \ll r(1/b)/b$. Hence $a \vee b \ll_r 1/b$, as desired. □

Lemma 7. *Let L be a lattice and $a, b \in L$. If $b \ll_r L$ and $a \vee b \ll_r 1/b$, then $a \ll_r L$.*

Proof. Since $b \ll_r L$, $b \ll r(L)/0$. Then $b \leq r(L)$. Here we can see that $r(1/b) = r(L)$. Since $a \vee b \ll_r 1/b$, $a \vee b \ll r(1/b)/b = r(L)/b$. Let $a \vee t = r(L)$ for $t \in r(L)/0$. Here $(a \vee b) \vee (b \vee t) = r(L)$ and since $a \vee b \ll r(L)/b$, $b \vee t = r(L)$. Since $b \ll r(L)/0$, $t = r(L)$. Hence $a \ll r(L)/0$ and $a \ll_r L$. □

Corollary 4. *Let L be a lattice, $a \in L$ and $b \ll_r L$. Then $a \ll_r L$ if and only if $a \vee b \ll_r 1/b$.*

Proof. Clear from Lemma 6 and Lemma 7. □

Lemma 8. *Let L be a lattice and $a, b, c, d \in L$. If $a \ll_r b/0$ and $c \ll_r d/0$, then $a \vee c \ll_r (b \vee d)/0$.*

Proof. Since $a \ll_r b/0$ and $b \leq b \vee d$, by Lemma 5, $a \ll_r (b \vee d)/0$ and $a \ll r((b \vee d)/0)/0$. Similarly we can see that $c \ll r((b \vee d)/0)/0$. Since $a \ll r((b \vee d)/0)/0$ and $c \ll r((b \vee d)/0)/0$, by Lemma 1, $a \vee c \ll r((b \vee d)/0)/0$ and $a \vee c \ll_r (b \vee d)/0$, as desired. □

Corollary 5. *Let L be a lattice and $a, b \in L$. If $a \ll_r L$ and $b \ll_r L$, then $a \vee b \ll_r L$.*

Proof. Clear from Lemma 8. □

Corollary 6. *Let L be a lattice and $a_1, a_2, \ldots, a_n \in L$. If $a_i \ll_r L$ for every $i = 1, 2, \ldots, n$, then $a_1 \vee a_2 \vee \cdots \vee a_n \ll_r L$.*

Proof. Clear from Corollary 5. □

Proposition 5. *Let L be a lattice and $a \ll_r L$. Then $a \leq r(r(L)/0)$.*

Proof. Since $a \ll_r L$, $a \ll r(L)/0$. Then by Lemma 1, $a \leq r(r(L)/0)$, as desired. □

Lemma 9. *Let L be a lattice and $a \leq r(r(L)/0)$. If a is compact in $r(L)/0$, then $a \ll_r L$.*

Proof. Since a is compact in $r(L)/0$ and $a \leq r(r(L)/0)$, by Lemma 1, $a \ll r(L)/0$. Then $a \ll_r L$, as desired. □

Corollary 7. *Let L be a lattice and $a \leq r(r(L)/0)$. If a is compact in L, then $a \ll_r L$.*

Proof. Clear from Lemma 9. □

Lemma 10. *Let L be a lattice and $r(L)/0$ is compactly generated. Then*

$$r(r(L)/0) = \bigvee_{a \ll_r L} a.$$

Proof. Since $r(L)/0$ is compactly generated, by Lemma 1(xi),

$$r(r(L)/0) = \bigvee_{a \ll r(L)/0} a.$$

Hence $r(r(L)/0) = \bigvee_{a \ll_r L} a$, as desired. □

Corollary 8. *Let L be a compactly generated lattice. Then $r(r(L)/0) = \bigvee_{a \ll_r L} a$.*

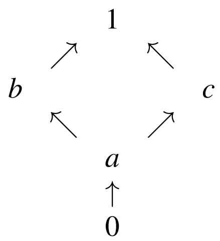
Proof. Clear from Lemma 10. □

Proposition 6. *Let L be a lattice. If $r(L)$ is compact, then $r(r(L)/0) \ll_r L$.*

Proof. Clear from Lemma 1(xii). □

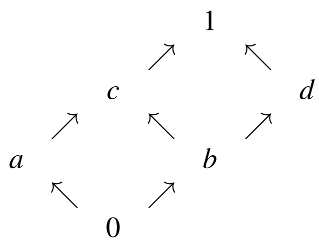
Remark 1. Let L be a compact lattice and $r(L) \neq 0$. Then $r(L) \ll L$, but not $r(L) \ll_r L$.

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



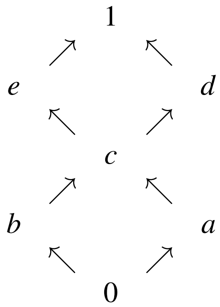
Here $a \ll L$, but not $a \ll_r L$.

Example 2. Consider the lattice $L = \{0, a, b, c, d, 1\}$ given by the following diagram.



Here $r(L) = b \ll L$, but not $b \ll_r L$.

Example 3. Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ given by the following diagram.



In this lattice $r(L) = c$. Here $a \ll L$, but not $a \ll_r L$. Here also $b \ll L$, but not $b \ll_r L$.

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