



## SOME PROPERTIES OF WEAKLY G-SUPPLEMENTED LATTICES

BERNA KOŞAR, CELIL NEBIYEV, AND HASAN HÜSEYİN ÖKTEN

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*Abstract.* In this work, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let  $L$  be a lattice and  $a, b \in L$ . If  $a \vee b = 1$  and  $a \wedge b \ll_g L$ , then  $b$  is called a weak  $g$ -supplement of  $a$  in  $L$ . If every element of  $L$  has a weak  $g$ -supplement in  $L$ , then  $L$  is called a weakly  $g$ -supplemented lattice. In this work, some properties of these lattices are investigated. Let  $L$  be a lattice and  $a, b, c \in L$ . If  $c \ll L$ , then  $b$  is a weak  $g$ -supplement of  $a$  in  $L$  if and only if  $b$  is a weak  $g$ -supplement of  $a \vee c$  in  $L$ . Let  $L$  be a lattice and  $1 = a_1 \vee a_2 \vee \dots \vee a_n$  with  $a_i \in L$  ( $1 \leq i \leq n$ ). If  $a_i/0$  is weakly  $g$ -supplemented for every  $i = 1, 2, \dots, n$ , then  $L$  is also weakly  $g$ -supplemented. Let  $L$  be a weakly  $g$ -supplemented lattice. Then  $1/a$  is weakly  $g$ -supplemented for every  $a \in L$ . If  $L$  is a weakly  $g$ -supplemented lattice, then  $1/r_g(L)$  is complemented. Let  $L$  be a lattice. Then  $L$  is weakly  $g$ -supplemented if and only if for every  $a, b \in L$  with  $1 = a \vee b$ ,  $a$  has a weak  $g$ -supplement  $c$  in  $L$  with  $c \leq b$ . Let  $L$  be a lattice and  $a\beta_*b$  in  $L$ . If  $a$  and  $b$  have weak  $g$ -supplements in  $L$ , then they have the same weak  $g$ -supplements in  $L$ .

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### 1. INTRODUCTION

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let  $L$  be a lattice,  $x, y \in L$  and  $x \leq y$ . A sublattice  $\{a \in L \mid x \leq a \leq y\}$  is called a *quotient sublattice* and denoted by  $y/x$ . An element  $y$  of a lattice  $L$  is called a *complement* of  $x$  in  $L$  if  $x \wedge y = 0$  and  $x \vee y = 1$ , this case we denote  $1 = x \oplus y$  (in this case we call  $x$  and  $y$  are *direct summands* of  $L$ ).  $L$  is said to be *complemented* if each element of  $L$  has at least one complement in  $L$ . An element  $x$  of  $L$  is said to be *small* or *superfluous* and denoted by  $x \ll L$  if  $y = 1$  for every  $y \in L$  such that  $x \vee y = 1$ . The meet of all maximal elements of the poset  $L - \{1\}$  is called the *radical* of  $L$  and denoted by  $r(L)$ . If  $L - \{1\}$  have not any maximal elements, then the radical of  $L$  is defined by  $r(L) = 1$ . An element  $a$  of  $L$  is called a *supplement* of  $b$  in  $L$  if it is minimal for  $a \vee b = 1$ .  $a$  is a supplement of  $b$  in a lattice  $L$  if and only if  $a \vee b = 1$  and  $a \wedge b \ll a/0$ . A lattice  $L$  is called a *supplemented* lattice if every

element of  $L$  has a supplement in  $L$ .  $L$  is said to be  $\oplus$ -supplemented if every element of  $L$  has a supplement that is a direct summand in  $L$ . Let  $L$  be a lattice and  $a, b \in L$ . If  $a \vee b = 1$  and  $a \wedge b \ll L$ , then  $a$  is called a *weak supplement* of  $b$  in  $L$ .  $L$  is said to be *weakly supplemented* if every element of  $L$  has a weak supplement in  $L$ . We say that an element  $y$  of  $L$  lies above an element  $x$  of  $L$  if  $x \leq y$  and  $y \ll 1/x$ .  $L$  is said to be *hollow* if every element distinct from 1 is superfluous in  $L$ , and  $L$  is said to be *local* if  $L - \{1\}$  has the greatest element. We say an element  $x \in L$  has *ample supplements* in  $L$  if for every  $y \in L$  with  $x \vee y = 1$ ,  $x$  has a supplement  $z$  in  $L$  with  $z \leq y$ .  $L$  is said to be *amply supplemented* if every element of  $L$  has ample supplements in  $L$ . It is clear that every amply supplemented lattice is supplemented. Let  $L$  be a lattice and  $k \in L$ . If  $t = 0$  for every  $t \in L$  with  $k \wedge t = 0$ , then  $k$  is called an *essential* element of  $L$  and denoted by  $k \trianglelefteq L$ . Let  $L$  be a lattice and  $a \in L$ . If  $b = 1$  for every  $b \trianglelefteq t$  with  $a \vee b = 1$ , then  $a$  is called a *generalized small* (briefly, *g-small*) element of  $L$  and denoted by  $a \ll_g L$ . Let  $L$  be a lattice and  $a, b \in L$ . If  $1 = a \vee b$  and  $1 = a \vee t$  with  $t \trianglelefteq b/0$  implies that  $t = b$ , then  $b$  is called a *g-supplement* of  $a$  in  $L$ .  $b$  is a *g-supplement* of  $a$  in  $L$  if and only if  $1 = a \vee b$  and  $a \wedge b \ll_g b/0$ . If every element of  $L$  has a *g-supplement* in  $L$ , then  $L$  is called a *g-supplemented* lattice. Let  $L$  be a lattice and  $t$  be a maximal element of  $L - \{1\}$ . If  $t \trianglelefteq L$ , then  $t$  is called a *g-maximal* element of  $L$ . The meet of all *g-maximal* elements of  $L$  is called the *g-radical* of  $L$  and denoted by  $r_g(L)$ . If  $L$  have not any *g-maximal* elements, then we call  $r_g(L) = 1$ . Let  $L$  be a lattice. If every element of  $L$  distinct from 1 is *g-small* in  $L$ , then  $L$  is called a *g-hollow* lattice.

More details about (amply) supplemented lattices are in [1, 2, 5, 10]. More results about (amply) supplemented modules are in [6, 9, 15]. More details about weakly supplemented lattices are in [1]. More details about *g-small* elements and *g-supplemented* lattices are in [14]. More details about *g-small* submodules and *g-supplemented* modules are in [7, 8, 12].

**Lemma 1.** *Let  $L$  be a lattice and  $a, b, c, d \in L$ . Then the followings hold.*

- (i) *If  $a \leq b$  and  $b \ll_g L$ , then  $a \ll_g L$ .*
- (ii) *If  $a \ll_g b/0$ , then  $a \ll_g t/0$  for every  $t \in L$  with  $b \leq t$ .*
- (iii) *If  $a \ll_g L$ , then  $a \vee b \ll_g 1/b$ .*
- (iv) *If  $a \ll_g b/0$  and  $c \ll_g d/0$ , then  $a \vee c \ll_g (b \vee d)/0$ .*
- (v) *If  $a_i \ll_g b_i/0$  for  $a_i, b_i \in L$  ( $i = 1, 2, \dots, n$ ), then  $a_1 \vee a_2 \vee \dots \vee a_n \ll_g (b_1 \vee b_2 \vee \dots \vee b_n)/0$ .*
- (vi) *If  $a \leq b$  and  $b \ll_g L$ , then  $b \ll_g 1/a$ .*
- (vii) *If  $a \ll_g L$ , then  $a \leq r_g(L)$ .*
- (viii)  *$r_g(a/0) \leq r_g(L)$ .*

*Proof.* See [14, Lemma 1, Lemma 6 and Lemma 7]. □

**Lemma 2.** *Let  $L$  be a lattice and  $a, b, c \in L$ . If  $a \vee b = 1$  and  $(a \wedge b) \vee c = 1$ , then  $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$ .*

*Proof.* See [11, Lemma 2].  $\square$

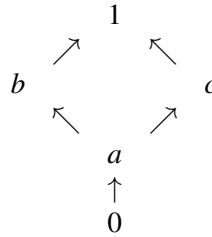
## 2. WEAKLY G-SUPPLEMENTED LATTICES

**Definition 1.** Let  $L$  be a lattice and  $a, b \in L$ . If  $1 = a \vee b$  and  $a \wedge b \ll_g L$ , then  $b$  is called a weak g-supplement of  $a$  in  $L$ . If every element of  $L$  has a weak g-supplement in  $L$ , then  $L$  is called a weakly g-supplemented lattice. (See also [13])

Every g-supplemented lattice is weakly g-supplemented. But the converse of this statement is not true in general (see Example 1). Hollow, local and g-hollow lattices are weakly g-supplemented. Every weakly supplemented lattice is weakly g-supplemented.

*Example 1.* Consider the  $\mathbb{Z}$ -module  ${}_{\mathbb{Z}}\mathbb{Q}$ . Let  $L$  be the set of all submodules of  ${}_{\mathbb{Z}}\mathbb{Q}$ . Then  $L$  is a complete modular lattice with the greatest element  $\mathbb{Q}$  and the smallest element  $0$  by the operation  $\subset$ . By [6, Example 20.12],  ${}_{\mathbb{Z}}\mathbb{Q}$  is weakly supplemented but not supplemented. Hence  $L$  is weakly supplemented but not supplemented. Since  $L$  is weakly supplemented, it is weakly g-supplemented. Since every nonzero element of  $L$  is essential in  $L$  and  $L$  is not supplemented,  $L$  is not g-supplemented.

*Example 2.* Consider the lattice  $L = \{0, a, b, c, 1\}$  given by the following diagram.



Then  $L$  is weakly g-supplemented but not g-hollow.

**Proposition 1.** Let  $L$  be a weakly g-supplemented lattice. If every nonzero element of  $L$  is essential in  $L$ , then  $L$  is weakly supplemented.

*Proof.* Clear from definitions.  $\square$

**Proposition 2.** Let  $L$  be a lattice and  $a, b, c \in L$ . If  $c \ll L$ , then  $b$  is a weak g-supplement of  $a$  in  $L$  if and only if  $b$  is a weak g-supplement of  $a \vee c$  in  $L$ .

*Proof.* ( $\implies$ ) Since  $b$  is a weak g-supplement of  $a$  in  $L$ ,  $1 = a \vee b$  and  $a \wedge b \ll_g L$ . Here  $1 = a \vee b = a \vee c \vee b$ . Let  $((a \vee c) \wedge b) \vee t = 1$  with  $t \leq L$ . Since  $1 = a \vee c \vee b$  and  $((a \vee c) \wedge b) \vee t = 1$ , by Lemma 2,  $a \vee c \vee (b \wedge t) = 1$ . Then by  $c \ll L$ ,  $a \vee (b \wedge t) = 1$ . Since  $b \vee t = 1$ , by Lemma 2,  $(a \wedge b) \vee t = 1$  and since  $a \wedge b \ll_g L$  and  $t \leq L$ ,  $t = 1$ . Hence  $(a \vee c) \wedge b \ll_g L$  and  $b$  is a weak g-supplement of  $a \vee c$  in  $L$ .

( $\impliedby$ ) Since  $b$  is a weak g-supplement of  $a \vee c$  in  $L$ ,  $1 = a \vee c \vee b$  and  $(a \vee c) \wedge b \ll_g L$ . Since  $1 = a \vee c \vee b$  and  $c \ll L$ ,  $1 = a \vee b$ . Since  $a \wedge b \leq (a \vee c) \wedge b$  and

$(a \vee c) \wedge b \ll_g L$ , by Lemma 1(i),  $a \wedge b \ll_g L$ . Hence  $b$  is a weak  $g$ -supplement of  $a$  in  $L$ , as desired.  $\square$

**Proposition 3.** *Let  $L$  be a lattice,  $a, b, c \in L$  and  $b \leq L$ . If  $c \ll_g L$ , then  $b$  is a weak  $g$ -supplement of  $a$  in  $L$  if and only if  $b$  is a weak  $g$ -supplement of  $a \vee c$  in  $L$ .*

*Proof.* ( $\implies$ ) Since  $b$  is a weak  $g$ -supplement of  $a$  in  $L$ ,  $1 = a \vee b$  and  $a \wedge b \ll_g L$ . Here  $1 = a \vee b = a \vee c \vee b$ . Let  $((a \vee c) \wedge b) \vee t = 1$  with  $t \leq L$ . Since  $1 = a \vee c \vee b$  and  $((a \vee c) \wedge b) \vee t = 1$ , by Lemma 2,  $a \vee c \vee (b \wedge t) = 1$ . Since  $b \leq L$  and  $t \leq L$ , we have  $b \wedge t \leq L$  and  $a \vee (b \wedge t) \leq L$ . Then by  $c \ll_g L$ ,  $a \vee (b \wedge t) = 1$ . Since  $b \vee t = 1$ , by Lemma 2,  $(a \wedge b) \vee t = 1$  and since  $a \wedge b \ll_g L$  and  $t \leq L$ ,  $t = 1$ . Hence  $(a \vee c) \wedge b \ll_g L$  and  $b$  is a weak  $g$ -supplement of  $a \vee c$  in  $L$ .

( $\impliedby$ ) Since  $b$  is a weak  $g$ -supplement of  $a \vee c$  in  $L$ ,  $1 = a \vee c \vee b$  and  $(a \vee c) \wedge b \ll_g L$ . Since  $b \leq L$ , we have  $a \vee b \leq L$ . Then by  $1 = a \vee c \vee b$  and  $c \ll_g L$ ,  $1 = a \vee b$ . Since  $a \wedge b \leq (a \vee c) \wedge b$  and  $(a \vee c) \wedge b \ll_g L$ , by Lemma 1(i),  $a \wedge b \ll_g L$ . Hence  $b$  is a weak  $g$ -supplement of  $a$  in  $L$ , as desired.  $\square$

**Lemma 3.** *Let  $L$  be a lattice and  $a, b \in L$ . If  $a \vee b$  has a weak  $g$ -supplement  $x$  in  $L$  and  $(a \vee x) \wedge b$  has a weak  $g$ -supplement  $y$  in  $b/0$ , then  $x \vee y$  is a weak  $g$ -supplement of  $a$  in  $L$ .*

*Proof.* Since  $x$  is a weak  $g$ -supplement of  $a \vee b$  in  $L$ ,  $1 = a \vee b \vee x$  and  $(a \vee b) \wedge x \ll_g L$ . Since  $y$  is a weak  $g$ -supplement of  $(a \vee x) \wedge b$  in  $b/0$ ,  $b = ((a \vee x) \wedge b) \vee y$  and  $(a \vee x) \wedge y = (a \vee x) \wedge b \wedge y \ll_g b/0$ . Then  $1 = a \vee b \vee x = a \vee x \vee ((a \vee x) \wedge b) \vee y = a \vee x \vee y$  and by Lemma 1(iv),  $a \wedge (x \vee y) \leq ((a \vee x) \wedge y) \vee ((a \vee y) \wedge x) \leq ((a \vee x) \wedge y) \vee ((a \vee b) \wedge x) \ll_g L$ . Hence  $x \vee y$  is a weak  $g$ -supplement of  $a$  in  $L$ .  $\square$

**Corollary 1.** *Let  $L$  be a lattice and  $a, b \in L$ . If  $a \vee b$  has a weak  $g$ -supplement in  $L$  and  $b/0$  is weakly  $g$ -supplemented, then  $a$  has a weak  $g$ -supplement in  $L$ .*

*Proof.* Clear from Lemma 3.  $\square$

**Lemma 4.** *Let  $1 = a \vee b$  with  $a, b \in L$ . If  $a/0$  and  $b/0$  are weakly  $g$ -supplemented, then  $L$  is also weakly  $g$ -supplemented.*

*Proof.* Let  $x$  be any element of  $L$ . Then  $0$  is a weak  $g$ -supplement of  $x \vee a \vee b$  in  $L$  and since  $b/0$  is weakly  $g$ -supplemented, by Corollary 1,  $x \vee a$  has a weak  $g$ -supplement in  $L$ . Since  $a/0$  is weakly  $g$ -supplemented, again by Corollary 1,  $x$  has a weak  $g$ -supplement in  $L$ . Hence  $L$  is weakly  $g$ -supplemented.  $\square$

**Corollary 2.** *Let  $1 = a_1 \vee a_2 \vee \dots \vee a_n$  with  $a_i \in L$  ( $1 \leq i \leq n$ ). If  $a_i/0$  is weakly  $g$ -supplemented for every  $i = 1, 2, \dots, n$ , then  $L$  is also weakly  $g$ -supplemented.*

*Proof.* Clear from Lemma 4.  $\square$

**Lemma 5.** *Let  $L$  be a lattice and  $a, b, c \in L$  with  $c \leq a$ . If  $b$  is a weak  $g$ -supplement of  $a$  in  $L$ , then  $b \vee c$  is a weak  $g$ -supplement of  $a$  in  $1/c$ .*

*Proof.* Since  $b$  is a weak  $g$ -supplement of  $a$  in  $L$ ,  $1 = a \vee b$  and  $a \wedge b \ll_g L$ . Since  $a \wedge b \ll_g L$ , by Lemma 1(iii),  $(a \wedge b) \vee c \ll_g 1/c$ . Hence  $1 = a \vee b = a \vee b \vee c$  and  $a \wedge (b \vee c) = (a \wedge b) \vee c \ll_g 1/c$  and  $b \vee c$  is a weak  $g$ -supplement of  $a$  in  $1/c$ .  $\square$

**Corollary 3.** *Let  $L$  be a weakly  $g$ -supplemented lattice. Then  $1/a$  is weakly  $g$ -supplemented for every  $a \in L$ .*

*Proof.* Clear from Lemma 5.  $\square$

**Lemma 6.** *Let  $L$  be a weakly  $g$ -supplemented lattice. Then  $1/r_g(L)$  is complemented.*

*Proof.* Let  $x$  be any element of  $1/r_g(L)$ . Since  $L$  is weakly  $g$ -supplemented,  $x$  has a weak  $g$ -supplement  $y$  in  $L$ . Here  $1 = x \vee y$  and  $x \wedge y \ll_g L$ . Since  $x \wedge y \ll_g L$ , by Lemma 1(vii),  $x \wedge y \leq r_g(L)$ . Hence  $1 = x \vee y \vee r_g(L)$  and  $x \wedge (y \vee r_g(L)) = (x \wedge y) \vee r_g(L) = r_g(L)$ . Therefore,  $y \vee r_g(L)$  is a complement of  $x$  in  $1/r_g(L)$  and  $1/r_g(L)$  is complemented.  $\square$

**Corollary 4.** *Let  $L$  be a weakly  $g$ -supplemented lattice. Then  $1/r_g(L)$  is  $\oplus$ -supplemented.*

*Proof.* Clear from [3, Definition 1] and Lemma 6.  $\square$

**Lemma 7.** *Let  $L$  be a lattice and  $a, b \in L$ . If  $1 = a \vee b$  and  $a \wedge b$  has a weak  $g$ -supplement  $x$  in  $b/0$ , then  $x$  is a weak  $g$ -supplement of  $a$  in  $L$ .*

*Proof.* Since  $x$  is a weak  $g$ -supplement of  $a \wedge b$  in  $b/0$ ,  $b = (a \wedge b) \vee x$  and  $a \wedge x = a \wedge b \wedge x \ll_g b/0$ . Then  $1 = a \vee b = a \vee (a \wedge b) \vee x = a \vee x$  and  $a \wedge x \ll_g L$ . Hence  $x$  is a weak  $g$ -supplement of  $a$  in  $L$ .  $\square$

**Corollary 5.** *Let  $L$  be a lattice,  $a, b \in L$  and  $1 = a \vee b$ . If  $b/0$  is weakly  $g$ -supplemented, then  $a$  has a weak  $g$ -supplement in  $L$ .*

*Proof.* Clear from Lemma 7.  $\square$

**Lemma 8.** *Let  $L$  be a lattice,  $a, b \in L$  and  $1 = a \vee b$ . If  $L$  is weakly  $g$ -supplemented, then  $a$  has a weak  $g$ -supplement  $c$  in  $L$  with  $c \leq b$ .*

*Proof.* Since  $L$  is weakly  $g$ -supplemented,  $a \wedge b$  has a weak  $g$ -supplement  $x$  in  $L$ . Here  $1 = (a \wedge b) \vee x$  and  $a \wedge b \wedge x \ll_g L$ . Since  $1 = (a \wedge b) \vee x$ , by modularity,  $b = b \wedge 1 = b \wedge ((a \wedge b) \vee x) = (a \wedge b) \vee (b \wedge x)$ . Let  $c = b \wedge x$ . Then  $1 = a \vee b = a \vee (a \wedge b) \vee (b \wedge x) = a \vee (b \wedge x) = a \vee c$  and  $a \wedge c = a \wedge b \wedge x \ll_g L$ . Hence  $c$  is a weak  $g$ -supplement of  $a$  in  $L$ . Moreover,  $c \leq b$ . Hence the desired result is obtained.  $\square$

**Corollary 6.** *Let  $L$  be a lattice. Then  $L$  is weakly  $g$ -supplemented if and only if for every  $a, b \in L$  with  $1 = a \vee b$ ,  $a$  has a weak  $g$ -supplement  $c$  in  $L$  with  $c \leq b$ .*

*Proof.* Clear from Lemma 8.  $\square$

Let  $x, y \in L$ . A relation  $\beta_*$  is defined on the elements of  $L$  by  $x\beta_*y$  if and only if for every  $t \in L$  with  $1 = x \vee t$  then  $1 = y \vee t$  and for every  $k \in L$  with  $1 = y \vee k$  then  $1 = x \vee k$ . (See [11, Definition 1]). More information about  $\beta_*$  relation are in [11]. More information about  $\beta^*$  relation on modules are in [4].

**Lemma 9.** *Let  $L$  be a lattice and  $a\beta_*b$  in  $L$ . If  $a$  and  $b$  have weak  $g$ -supplements in  $L$ , then they have the same weak  $g$ -supplements in  $L$ .*

*Proof.* Let  $x$  be a weak  $g$ -supplement of  $a$  in  $L$ . Then  $1 = a \vee x$  and  $a \wedge x \ll_g L$ . Since  $1 = a \vee x$  and  $a\beta_*b$ , we have  $1 = b \vee x$ . Let  $1 = (b \wedge x) \vee t$  with  $t \trianglelefteq L$ . Then by Lemma 2,  $1 = b \vee (x \wedge t)$  and since  $a\beta_*b$ , we have  $1 = a \vee (x \wedge t)$ . Since  $1 = x \vee t$  and  $1 = a \vee (x \wedge t)$ , by Lemma 2,  $1 = (a \wedge x) \vee t$ . Since  $a \wedge x \ll_g L$  and  $t \trianglelefteq L$ ,  $t = 1$ . Hence  $b \wedge x \ll_g L$  and since  $1 = b \vee x$ ,  $x$  is a weak  $g$ -supplement of  $b$  in  $L$ . Similarly, interchanging the roles of  $a$  and  $b$  we can prove that each weak  $g$ -supplement of  $b$  in  $L$  is also a weak  $g$ -supplement of  $a$  in  $L$ .  $\square$

**Corollary 7.** *Let  $L$  be a lattice and let  $a$  lie above  $b$  in  $L$ . If  $a$  and  $b$  have weak  $g$ -supplements in  $L$ , then they have the same weak  $g$ -supplements in  $L$ .*

*Proof.* By [11, Theorem 3],  $a\beta_*b$  and by Lemma 9, the desired result is obtained.  $\square$

**Lemma 10.** *Let  $L$  be a lattice. If every element of  $L$  is  $\beta_*$ -equivalent to a weak  $g$ -supplement element in  $L$ , then  $L$  is weakly  $g$ -supplemented.*

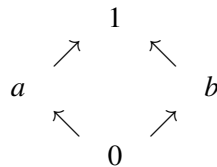
*Proof.* Let  $a \in L$ . By hypothesis, there exists a weak  $g$ -supplement element  $x$  in  $L$  such that  $a\beta_*x$ . Let  $x$  be a weak  $g$ -supplement of  $b$  in  $L$ . Then by definition,  $b$  is a weak  $g$ -supplement of  $x$  in  $L$ . Since  $a\beta_*x$ , by Lemma 9,  $b$  is a weak  $g$ -supplement of  $a$  in  $L$ . Hence  $L$  is weakly  $g$ -supplemented.  $\square$

**Corollary 8.** *Let  $L$  be a lattice. If every element of  $L$  lies above a weak  $g$ -supplement element in  $L$ , then  $L$  is weakly  $g$ -supplemented.*

*Proof.* Clear from [11, Theorem 3] and Lemma 10.  $\square$

*Example 3.* Let  $L$  be a nonzero complemented lattice. Here  $1 \ll_g L$ , but not  $1 \ll L$ .  $1$  is a weak  $g$ -supplement of  $1$  in  $L$ , but  $1$  is not a weak supplement of  $1$  in  $L$ .

*Example 4.* Consider the lattice  $L = \{0, a, b, 1\}$  given by the following diagram.



Then  $L$  is  $g$ -hollow but not hollow. Here  $1 \ll_g L$ , but not  $1 \ll L$ .  $1$  is a weak  $g$ -supplement of  $1$  in  $L$ , but  $1$  is not a weak supplement of  $1$  in  $L$ .

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*Authors' addresses***Berna Koşar**

(Corresponding author) Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Samsun, Turkey

E-mail address: [bernak@omu.edu.tr](mailto:bernak@omu.edu.tr)

**Celil Nebiyev**

Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Samsun, Turkey

E-mail address: [cnebiyev@omu.edu.tr](mailto:cnebiyev@omu.edu.tr)

**Hasan Hüseyin Ökten**

Amasya University, Technical Sciences Vocational School, Amasya, Turkey

E-mail address: [hokten@gmail.com](mailto:hokten@gmail.com)