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BOURGAIN-LEBESGUE SPACES

BAOHUA DONG, XUANXUAN HU, AND JINGSHI XU

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Abstract. Bourgain initially introduced a specific instance of Bourgain-Morrey spaces to investigate the restriction and multiplier problems in $\mathbb{R}^3$. Following this, the concept of Bourgain-type function spaces garnered considerable attention among researchers. In the paper, we aim to introduce Bourgain-Lebesgue spaces, and delve into the embedding properties, the Young inequality, dilation properties in the spaces. Additionally, we explore the boundedness properties within Bourgain-Lebesgue spaces concerning local Hardy-Littlewood maximal operators and their vector-valued counterparts.

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1. INTRODUCTION

Let $\nu \in \mathbb{Z}$ and $\vec{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. A dyadic cube $Q_{\nu\vec{m}}$ is defined by

$$Q_{\nu\vec{m}} := \prod_{i=1}^{n} \left[\frac{m_i}{2^\nu}, \frac{m_i+1}{2^\nu}\right),$$

denote $\mathcal{D}_\nu := \{Q_{\nu\vec{m}} : \vec{m} \in \mathbb{Z}^n\}$ and $\mathcal{D} := \bigcup_{\nu\in\mathbb{Z}} \mathcal{D}_\nu$. Let $0 < p \leqslant q < \infty$ and $0 < r \leqslant \infty$. The Bourgain-Morrey space $\mathcal{M}^q_{p,r}(\mathcal{D})$ is the set of all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}^q_{p,r}(\mathcal{D})} = \left\| \left\{ |Q_{\nu\vec{m}}|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{Q_{\nu\vec{m}}} |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \right\}_{\nu\in\mathbb{Z}, \vec{m}\in\mathbb{Z}^n} \right\|_{\ell^r} < \infty.$$

Here $\|\cdot\|_{\ell^r}$ is the norm of discrete Lebesgue space ℓ^r.

In [2], Bourgain introduced a function space to study the restriction and multiplier problems in $\mathbb{R}^3$. The function space can now be viewed as a special case of Bourgain-Morrey spaces. In [4], Hatano et al studied the Bourgain-Morrey space from the perspectives of harmonic analysis and functional analysis. They obtain some classical

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results related to the spaces, such as approximation properties in the spaces, interpolation properties between the spaces and the boundedness of operators in the spaces, as well as the dual of the spaces. Besides, there are some general spaces related to Bourgain-Morrey spaces, such as Triebel-Lizorkin-Bourgain-Morrey spaces [5] and Besov-Bourgain-Morrey spaces [11]. The Bourgain-Morrey spaces are important in partial differential equations, for more details we refer the reader to references [1, 6–9] and the references therein.

Our interests are successfully attracted by Example 2.9 in [4]. The example shows that the Bourgain-Morrey space $\mathcal{M}^q_{p,r}(\mathcal{D})$ is not trivial if and only if $0<p<q<r<\infty$ or $0<p\leqslant q<r=\infty$. In the case of $0<p\leqslant q<r=\infty$, the Bourgain-Morrey space $\mathcal{M}^q_{p,\infty}(\mathcal{D})$ coincides with the Morrey space $\mathcal{M}^q_p(\mathbb{R}^n)$ which is defined by

$$\mathcal{M}^q_p(\mathbb{R}^n) := \left\{ f\in L^p_{\text{loc}}(\mathbb{R}^n):\ \|f\|_{\mathcal{M}^q_p(\mathbb{R}^n)}<\infty \right\},$$

and

$$\|f\|_{\mathcal{M}^q_p(\mathbb{R}^n)} := \sup_{Q\in\mathcal{D}} |Q|^{\frac{1}{q}-\frac{1}{p}} \left(\int_Q |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}.$$

It is well know that $\mathcal{M}^q_p(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ when $p=q$, so the Bourgain-Morrey space $\mathcal{M}^p_{p,\infty}(\mathcal{D})$ is identical with $L^p(\mathbb{R}^n)$.

While in the case of $0<p<q<r<\infty$, the Bourgain-Morrey space $\mathcal{M}^q_{p,r}(\mathcal{D})$ can not be reduced to the Lebesgue space. In the setting, an interesting question is that how to introduce a corresponding space in the case $0<p=q<r<\infty$. We try to answer this question in the paper.

Inspired by the above works, in Section 2, we aim to introduce the Bourgain-Lebesgue space and delve into the embedding properties of the spaces. Besides, there are some examples show that Lebesgue spaces and the Bourgain-Lebesgue space are not contained within each other. In Section 3, the Young inequality and dilation properties are obtained in the Bourgain-Lebesgue space. Section 4 contains the boundedness and weak boundedness of local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces. In Section 5, we obtain the boundedness of vector-valued local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces.

At the end of the section, we need to explain some conventions on notations. The C is a positive constant and independent of the main parameters in the formula, but may vary from line to line. The symbol $f\lesssim g$ means $f\leqslant Cg$. The symbol $f\gtrsim g$ means $f\geq Cg$.

2. The Bourgain-Lebesgue space and its basic properties

In order to introduce the Bourgain-Lebesgue space, we have to throw the large cubes in $\mathcal{D}$ away (for more details see Example 1). Let $\mathbb{N}$ be the set of nonnegative integers, $\mathcal{B}:=\bigcup_{\nu\in\mathbb{N}}\mathcal{D}_\nu$. So $\mathcal{B}$ contains all the cubes in $\mathcal{D}$ with the length less than or equal to 1.

Definition 1. Let $0 < p < r \leqslant \infty$, the Bourgain-Lebesgue space is defined by

$$\mathcal{B}^r L^p(\mathbb{R}^n) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} := \begin{cases} \left(\sum_{Q \in \mathcal{B}} \|f\|^r_{L^p(Q)} \right)^{\frac{1}{r}}, & r \in (0,\infty), \\ \sup_{Q \in \mathcal{B}} \|f\|_{L^p(Q)}, & r = \infty. \end{cases}$$

It is easy to know that $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ are subspaces of $\mathcal{B}^\infty L^p(\mathbb{R}^n)$. $\mathcal{B}^\infty L^p(\mathbb{R}^n)$ is equivalent to the Wiener amalgam space $W(L^p, \ell^\infty)(\mathbb{R}^n)$ which is important to consider the problems related to multidimensional summation [10]. We also mention that the space $\mathcal{B}^\infty L^\infty(\mathbb{R}^n)$ is not contained in the Definition 1, because the norm of $\mathcal{B}^\infty L^\infty(\mathbb{R}^n)$ is equal to the norm of $L^\infty(\mathbb{R}^n)$.

Next example means that the condition "$p < r$" is needed in Definition 1.

Example 1. Let $0 < p < \infty$ and $0 < r < \infty$. Then $\chi_{[0,1)^n} \in \mathcal{B}^r L^p(\mathbb{R}^n)$ if and only if $0 < p < r < \infty$.

Next two embedding properties are obvious of Bourgain-Lebesgue spaces.

Proposition 1. *Let* $0 < p < r_1 \leqslant r_2 \leqslant \infty$. *Then* $\mathcal{B}^{r_1} L^p(\mathbb{R}^n) \subseteq \mathcal{B}^{r_2} L^p(\mathbb{R}^n)$ *with continuous embedding.*

Proof. The proposition is due to $l^{r_1} \subseteq l^{r_2}$ with $r_1 \leqslant r_2$. □

Proposition 2. *Let* $0 < p_1 \leqslant p_2 < r \leqslant \infty$. *Then* $\mathcal{B}^r L^{p_2}(\mathbb{R}^n) \subseteq \mathcal{B}^r L^{p_1}(\mathbb{R}^n)$ *with continuous embedding.*

Proof. The proposition is obtained by $\|f\|_{L^{p_1}(Q)} \leqslant \|f\|_{L^{p_2}(Q)}$ for $Q \in \mathcal{B}$. □

The following three examples show that Lebesgue spaces $L^p(\mathbb{R}^n)$ and Bourgain-Lebesgue spaces $\mathcal{B}^r L^p(\mathbb{R}^n)$ do not contain each other for $0 < p < r < \infty$.

Example 2. Let $0 < p < r < \infty$, $a \in (n,\infty)$ and $f(x) = |x|^{-\frac{a}{p}} \cdot \chi_{\mathbb{R}^n \setminus B(\vec{0},1)}(x)$, $x \in \mathbb{R}^n$. Then $f \in L^p(\mathbb{R}^n)$, but $f \notin \mathcal{B}^r L^p(\mathbb{R}^n)$.

Let $x \in \mathbb{R}^n$, we use $x_i \in \mathbb{R}$ to denote the i-th coordinate component of x, then $x = (x_1, x_2, \ldots, x_n)$.

Example 3. Let $0 < p < r < \infty$, $f(x) = x_1^{-\frac{1}{p}} \left(\sum_{k \in \mathbb{Z}^+} \chi_{[k,k+1)^n}(x) \right)$, $x \in \mathbb{R}^n$. Then $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$, but $f \notin L^p(\mathbb{R}^n)$.

Example 4. Let $0 < p < r < \infty$, $a \in (0,1)$ and

$$f(x) := \left[(1-a)^n \prod_{i=1}^n x_i^{-a} \right]^{\frac{1}{p}} \cdot \sum_{k \in \mathbb{Z}^+} \chi_{[k,k+1)^n}(x), \qquad x \in \mathbb{R}^n.$$

Then

(1) $f \notin L^p(\mathbb{R}^n)$ and $f \notin \mathcal{B}^r L^p(\mathbb{R}^n)$, $a \in \left(0, \frac{p}{nr}\right]$;
(2) $f \notin L^p(\mathbb{R}^n)$ but $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$, $a \in \left(\frac{p}{nr}, \frac{1}{n}\right]$;
(3) $f \in L^p(\mathbb{R}^n)$ and $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$, $a \in \left(\frac{1}{n}, 1\right)$.

At the end of the section, we show Bourgain-Lebesgue spaces are complete.

Theorem 1. *Let $1 \leqslant p < r \leqslant \infty$. Then $\mathcal{B}^r L^p(\mathbb{R}^n)$ is a Banach space.*

Proof. We only prove the theorem for $1 \leqslant p < r < \infty$ because the proof is similar to the theorem for $1 \leqslant p < r = \infty$. The fact is obvious that $\|\cdot\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}$ is a norm. Let $\{f_m\}_{m\in\mathbb{Z}^+} \subseteq \mathcal{B}^r L^p(\mathbb{R}^n)$ be a Cauchy sequence, it can deduce that $\{f_m\}_{m\in\mathbb{Z}^+}$ is a Cauchy sequence in $L^p(Q)$ independent of $Q \in \mathcal{B}$. So, there exist $\{g_Q\}_{Q\in\mathcal{B}} \subset L^p(\mathbb{R}^n)$ such that

$$\lim_{m\to\infty} \|f_m - g_Q\|_{L^p(Q)} = 0, \text{ uniformly for } Q \in \mathcal{B},$$

and

$$g_Q(x) = g_{Q_{(k)}}(x) \qquad \text{a.e.}, \ x \in Q_k,$$

where $Q_{(k)}$ is a k-th subgeneration of Q.

Let $Q \in \mathcal{D}_0$ and

$$f_Q(x) := \begin{cases} g_Q(x), & x \in Q, \\ 0, & x \notin Q, \end{cases}$$

we denote f by

$$f(x) := \sum_{Q\in\mathcal{D}_0} f_Q(x), \ x \in \mathbb{R}^n.$$

Now we prove that $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$. Due to the fact

$$\begin{aligned} \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &= \left(\sum_{Q\in\mathcal{B}} \|g_Q\|_{L^p(Q)}^r\right)^{\frac{1}{r}} = \left[\sum_{Q\in\mathcal{B}} \left(\lim_{m\to\infty} \|f_m\|_{L^p(Q)}\right)^r\right]^{\frac{1}{r}} \\ &= \lim_{m\to\infty} \|f_m\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}, \end{aligned}$$

there exist a constant C such that

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant C.$$

It means $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$. Besides,

$$\lim_{m\to\infty} \|f_m - f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} = \left[\sum_{Q\in\mathcal{B}} \left(\lim_{m\to\infty} \|f_m - f\|_{L^p(Q)}\right)^r\right]^{\frac{1}{r}} = 0.$$

To sum up, $\mathcal{B}^r L^p(\mathbb{R}^n)$ is a Banach space. □

3. THE CONVOLUTION OPERATION IN BOURGAIN-LEBESGUE SPACES

In the section, some convolution inequalities are obtained in Bourgain-Lebesgue spaces. We begin with the translation properties in the spaces.

Lemma 1. *Let* $1 \leqslant p < r < \infty$. *Then*

$$\frac{1}{2^n}\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}, \qquad y \in \mathbb{R}^n.$$

Proof. Let $\nu \in \mathbb{N}$ and $y \in \mathbb{R}^n$, there exist $\vec{m}_1(\nu,y), \vec{m}_2(\nu,y), \ldots, \vec{m}_{2^n}(\nu,y) \in \mathbb{Z}^n$ such that

$$Q_{\nu\vec{m}} - y \subset \bigcup_{k=1}^{2^n} Q_{\nu(\vec{m}+\vec{m}_k(\nu,y))},$$

for each $\vec{m} \in \mathbb{Z}^n$. Let $z = x - y$, then

$$\|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \left[\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n} \left(\sum_{k=1}^{2^n} \int_{Q_{\nu(\vec{m}+\vec{m}_k(\nu,y))}} |f(z)|^p \,\mathrm{d}z\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

On the other hand, by a similar way,

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} = \left[\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n} \left(\int_{Q_{\nu\vec{m}}+y} |f(x-y)|^p \,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \leqslant 2^n \|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}. \qquad \square$$

The following theorem states that convolution operation is well defined in Bourgain-Lebesgue spaces.

Theorem 2. *Let* $1 < p < r < \infty$, $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$. *Then*

$$\|g * f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

Proof. It is easy to know that

$$\begin{aligned}\|g * f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &\leqslant \left[\sum_{Q\in\mathcal{B}} \left(\int_{\mathbb{R}^n} \|f(\cdot - y)\|_{L^p(Q)} |g(y)| \,\mathrm{d}y\right)^r\right]^{\frac{1}{r}} \\ &\leqslant \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{B}} \|f(\cdot - y)\|_{L^p(Q)}^r\right)^{\frac{1}{r}} |g(y)| \,\mathrm{d}y,\end{aligned}$$

where the penultimate inequality is due to the Minkowski inequality for the L^p-norm and the last inequality is because of the Minkowski inequality for the ℓ^r-norm. By Lemma 1, we have

$$\|g * f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}^n} \|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} |g(y)| \,\mathrm{d}y \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}. \qquad \square$$

By the next lemma, we will show that the Young inequality is true in Bourgain-Lebesgue spaces. Let $a>0$, Q be a cube, denote aQ by the dilation of Q around its centre by a.

Lemma 2. *Let $0<p<r<\infty$ and $a>0$. There exists a positive constant $C_{a,n}$ related to a and n such that*

$$\left(\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\|f\|_{L^p(aQ_{\nu\vec{m}})}^r\right)^{\frac{1}{r}}\leqslant C_{a,n}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.$$

Proof. Let $a\in(0,1]$, the lemma is obviously correct because $aQ_{\nu\vec{m}}\subset Q_{\nu\vec{m}}$ for $\nu\in\mathbb{N}$ and $\vec{m}\in\mathbb{Z}^n$. Let $a\in(1,\infty)$, there exists $\beta_a\in\mathbb{Z}^+$ such that $2^{\beta_a-1}\leqslant a<2^{\beta_a}$. For any $Q_{\nu\vec{m}}\in\mathcal{B}$ with $\nu\in\{0,1,\dots,\beta_a-1\}$ and $\vec{m}\in\mathbb{Z}^n$, there are at most $\tau(a):=\left(2^{\beta_a}+1\right)^n$ cubes in $\mathcal{D}_0$, for example $Q_{0\vec{m}_1},Q_{0\vec{m}_2},\dots,Q_{0\vec{m}_{\tau(a)}}$, such that

$$aQ_{\nu\vec{m}}\subset\bigcup_{i=1}^{\tau(a)}Q_{0\vec{m}_i}.$$

So

$$\begin{aligned}
&\left(\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\|f\|_{L^p(aQ_{\nu\vec{m}})}^r\right)^{\frac{1}{r}}\\
&\leqslant\left[\sum_{\substack{\nu\in\{0,1,\dots,\beta_a-1\}\\ \vec{m}\in\mathbb{Z}^n}}\left(\int_{aQ_{\nu\vec{m}}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{\substack{\nu\in\mathbb{N}\setminus\{0,1,\dots,\beta_a-1\}\\ \vec{m}\in\mathbb{Z}^n}}\left(\int_{aQ_{\nu\vec{m}}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\
&\leqslant\sum_{i=1}^{\tau(a)}\left[\sum_{\substack{\nu\in\{0,1,\dots,\beta_a-1\}\\ \vec{m}\in\mathbb{Z}^n}}\left(\int_{Q_{0\vec{m}_i}}|f(x)|^p\,\mathrm{d}x\right]^{\frac{r}{p}}\right)^{\frac{1}{r}}+2^n\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}\\
&\leqslant\left(\tau(a)\beta_a+2^n\right)\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.
\end{aligned}$$

□

Now, we state Young's inequality for Bourgain-Lebesgue spaces.

Theorem 3 (The Young inequality)**.** *Let $1\leqslant p<r<\infty$, $1\leqslant p_0<r_0<\infty$, $1\leqslant p_1<r_1<\infty$ and*

$$\frac{1}{p}+1=\frac{1}{p_0}+\frac{1}{p_1},\qquad\frac{1}{r}+1=\frac{1}{r_0}+\frac{1}{r_1}.$$

Then, for $f \in \mathcal{B}^{r_0}L^{p_0}(\mathbb{R}^n)$ and $g \in \mathcal{B}^{r_1}L^{p_1}(\mathbb{R}^n)$, there exists

$$\|f * g\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{B}^{r_0}L^{p_0}(\mathbb{R}^n)} \|g\|_{\mathcal{B}^{r_1}L^{p_1}(\mathbb{R}^n)}.$$

Proof. Let $Q_{\nu\vec{m}} \in \mathcal{B}$, by the Minkowski inequality and the Young inequality for the L^p-norm,

$$\begin{aligned}\|f * g\|_{L^p(Q_{\nu\vec{m}})} &\leqslant \sum_{\vec{m}' \in \mathbb{Z}^n} \left\| \left(f\chi_{Q_{\nu\vec{m}'}}\right) * \left(g\chi_{Q_{\nu\vec{m}}-Q_{\nu\vec{m}'}}\right) \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{\vec{m}' \in \mathbb{Z}^n} \left\| f\chi_{Q_{\nu\vec{m}'}} \right\|_{L^{p_0}(\mathbb{R}^n)} \left\| g\chi_{Q_{\nu\vec{m}}-Q_{\nu\vec{m}'}} \right\|_{L^{p_1}(\mathbb{R}^n)},\end{aligned}$$

where $Q_{\nu\vec{m}} - Q_{\nu\vec{m}'} = \{x - x' : x \in Q_{\nu\vec{m}}, x' \in Q_{\nu\vec{m}'}\}$.

Due to the fact that $Q_{\nu\vec{m}} - Q_{\nu\vec{m}'} \subset 3Q_{\nu(\vec{m}-\vec{m}')}$, we have

$$\begin{aligned}\left(\sum_{\vec{m} \in \mathbb{Z}^n} \|f * g\|^r_{L^p(Q_{\nu\vec{m}})}\right)^{\frac{1}{r}} &\lesssim \left[\sum_{\vec{m} \in \mathbb{Z}^n} \left(\sum_{\vec{m}' \in \mathbb{Z}^n} \left\| f\chi_{Q_{\nu\vec{m}'}} \right\|_{L^{p_0}(\mathbb{R}^n)} \cdot \left\| g\chi_{3Q_{\nu(\vec{m}-\vec{m}')}} \right\|_{L^{p_1}(\mathbb{R}^n)}\right)^r\right]^{\frac{1}{r}} \\ &\leqslant \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|f\chi_{Q_{\nu\vec{m}}}\|^{r_0}_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{r_0}} \cdot \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|g\chi_{3Q_{\nu\vec{m}}}\|^{r_1}_{L^{p_1}(\mathbb{R}^n)}\right)^{\frac{1}{r_1}} \\ &= \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|f\|^{r_0}_{L^{p_0}(Q_{\nu\vec{m}})}\right)^{\frac{1}{r_0}} \cdot \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|g\|^{r_1}_{L^{p_1}(3Q_{\nu\vec{m}})}\right)^{\frac{1}{r_1}},\end{aligned}$$

where the second inequality is due to the Young inequality for the discrete Lebesgue space ℓ^r. Then

$$\begin{aligned}&\left\| \left\{ \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|f * g\|^r_{L^p(Q_{\nu\vec{m}})}\right)^{\frac{1}{r}} \right\}_{\nu \in \mathbb{N}} \right\|_{\ell^r} \\ &\leqslant \left\| \left\{ \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|f\chi_{Q_{\nu\vec{m}}}\|^{r_0}_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{r_0}} \right\}_{\nu \in \mathbb{N}} \right\|_{\ell^r} \cdot \left\| \left\{ \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|g\|^{r_1}_{L^{p_1}(3Q_{\nu\vec{m}})}\right)^{\frac{1}{r_1}} \right\}_{\nu \in \mathbb{N}} \right\|_{\ell^\infty} \\ &\leqslant \left\| \left\{ \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|f\chi_{Q_{\nu\vec{m}}}\|^{r_0}_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{r_0}} \right\}_{\nu \in \mathbb{N}} \right\|_{\ell^{r_0}} \cdot \left\| \left\{ \left(\sum_{\vec{m} \in \mathbb{Z}^n} \|g\|^{r_1}_{L^{p_1}(3Q_{\nu\vec{m}})}\right)^{\frac{1}{r_1}} \right\}_{\nu \in \mathbb{N}} \right\|_{\ell^{r_1}},\end{aligned}$$

where the last inequality is due to the embedding properties of the discrete Lebesgue space ℓ^r. By Lemma 2, we obtain

$$\|f * g\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{B}^{r_0}L^{p_0}(\mathbb{R}^n)} \|g\|_{\mathcal{B}^{r_1}L^{p_1}(\mathbb{R}^n)}. \qquad \square$$

Finally, we consider the dilation properties in Bourgain-Lebesgue spaces.

Theorem 4. *Let* $0<p<r<\infty$ *and* $t\in(0,1]$. *Then*

$$\|f(t\cdot)\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}\leqslant 2^n t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.$$

Proof. The theorem is correct when $t=1$, so we only need to prove the case when $t\in(0,1)$. For such t, there exists $\nu_t\in\mathbb{N}$ such that $2^{-\nu_t-1}\leqslant t<2^{-\nu_t}$. So there are $\vec{m}_1(\nu+\nu_t,t),\vec{m}_2(\nu+\nu_t,t),\dots,\vec{m}_{2^n}(\nu+\nu_t,t)\in\mathbb{Z}^n$ such that

$$\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu},t\frac{m_i+1}{2^\nu}\right)\subset\bigcup_{k=1}^{2^n}Q_{\nu+\nu_t,\vec{m}+\vec{m}_k(\nu+\nu_t,t)},$$

for each $\vec{m}\in\mathbb{Z}^n$. Let $x=ty$, $y\in Q_{\nu\vec{m}}$, then

$$\begin{aligned}\|f(t\cdot)\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}&\leqslant t^{-\frac{n}{p}}\left[\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\left(\int_{\bigcup_{k=1}^{2^n}Q_{\nu+\nu_t,\vec{m}+\vec{m}_k(\nu+\nu_t,t)}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\&\leqslant 2^n t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.\end{aligned}$$

□

The dilation properties become complicated in the spaces when $t\in(1,\infty)$.

Theorem 5. *Let* $0<p<r<\infty$ *and* $t\in(1,\infty)$. *Then there exists a positive constant* $C_{n,t}$ *related to* n *and* t *such that*

$$\|f(t\cdot)\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}\leqslant C_{n,t}t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.$$

Proof. For $t\in(1,\infty)$, there exists $\nu_t\in\mathbb{Z}^+$ such that $2^{\nu_t-1}\leqslant t<2^{\nu_t}$. Let $x=ty$, $y\in Q_{\nu\vec{m}}$, then

$$\begin{aligned}\|f(t\cdot)\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}\leqslant&\left[\sum_{\nu=0}^{\nu_t-1}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}}|f(ty)|^p\,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\&+\left[\sum_{\nu=\nu_t}^{\infty}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}}|f(ty)|^p\,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}.\end{aligned}$$

When $\nu\in\mathbb{N}\backslash\{0,\dots,\nu_t-1\}$, similar to the proof of Theorem 4, there are $\vec{m}_1(\nu-\nu_t,t),\vec{m}_2(\nu-\nu_t,t),\dots,\vec{m}_{2^n}(\nu-\nu_t,t)\in\mathbb{Z}^n$ such that

$$\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu},t\frac{m_i+1}{2^\nu}\right)\subset\bigcup_{k=1}^{2^n}Q_{\nu-\nu_t,\vec{m}+\vec{m}_k(\nu-\nu_t,t)},$$

for each $\vec{m}\in\mathbb{Z}^n$. Then

$$\begin{aligned}\left[\sum_{\nu=\nu_t}^{\infty}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}}|f(ty)|^p\,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}&\leqslant 2^n t^{-\frac{n}{p}}\left[\sum_{\nu=0}^{\infty}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\&\leqslant 2^n t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.\end{aligned}$$

When $\nu \in \{0, \dots, \nu_t - 1\}$, we know that

$$2^{\nu_t-\nu-1} \leqslant \ell\left(\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu}, t\frac{m_i+1}{2^\nu}\right)\right) < 2^{\nu_t-\nu}.$$

So, there are at most $\gamma(t) := (2^{\nu_t}+1)^n$ cubes in $\mathcal{D}_0$, for example $Q_{0\vec{m}_1}, Q_{0\vec{m}_2}, \dots, Q_{0\vec{m}_{\gamma(t)}}$, such that

$$\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu}, t\frac{m_i+1}{2^\nu}\right) \subset \bigcup_{i=1}^{\gamma(t)} Q_{0\vec{m}_i}.$$

Then

$$\left[\sum_{\nu=0}^{\nu_t-1}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}}|f(ty)|^p\,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \leqslant t^{-\frac{n}{p}}\sum_{i=1}^{\gamma(t)}\left[\sum_{\nu=0}^{\nu_t-1}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{0\vec{m}_i}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}$$
$$\leqslant \gamma(t)\nu_t t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.$$

Thus

$$\|f(t\cdot)\|_{\mathcal{B}^rL^p(\mathbb{R}^n)} \leqslant (\gamma(t)\nu_t + 2^n)t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}. \qquad \square$$

4. The boundedness of local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces

The Hardy-Littlewood maximal operator and its local versions are important in the theory of function spaces. In the section, we investigate the boundedness and weak boundedness of local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces. Let $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ and $Q \subset \mathbb{R}^n$ be a cube, the Hardy-Littlewood maximal operator $\mathcal{M}$ is as follows

$$(\mathcal{M}f)(x) := \sup_{Q\subset\mathbb{R}^n, x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

Let $\ell(Q)$ be the length of Q, we restrict $\ell(Q)$ to be less than or equal to one, then the local Hardy-Littlewood maximal operator $\mathcal{M}_{\mathrm{loc}}$ is defined by

$$(\mathcal{M}_{\mathrm{loc}}f)(x) := \sup_{\substack{Q\subset\mathbb{R}^n, x\in Q\\ \ell(Q)\leqslant 1}}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

In addition, the definition of the Hardy-Littlewood maximal operator associated with $\mathcal{B}$, denoted by $\mathcal{M}_{\mathcal{B}}$, is

$$(\mathcal{M}_{\mathcal{B}}f)(x) := \sup_{Q\in\mathcal{B}, x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

It is easy to know that

$$(\mathcal{M}_{\mathcal{B}} f)(x) \leqslant (\mathcal{M}_{\mathrm{loc}} f)(x) \leqslant (\mathcal{M} f)(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

so $\mathcal{M}_{\mathcal{B}}$ is a bounded sublinear operator in $L^p(\mathbb{R}^n), 1 < p \leqslant \infty$.

Next, we prove that $\mathcal{M}_{\mathcal{B}}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$ for $1 < p < r < \infty$.

Theorem 6. *Let $1 < p < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.*

Proof. Let $Q \in \mathcal{B}$,

$$f_1 := f\chi_Q, \ f_2 := f\chi_{\mathbb{R}^n \setminus Q}.$$

Then $f = f_1 + f_2$ and

$$\left\| \mathcal{M}_{\mathcal{B}} f_1 \right\|_{L^p(Q)} \lesssim \|f_1\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(Q)}.$$

It means

$$\left\| \mathcal{M}_{\mathcal{B}} f_1 \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left(\sum_{Q \in \mathcal{B}} \|f\|_{L^p(Q)}^r \right)^{\frac{1}{r}} = \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Let $x \in Q$, $k \in \mathbb{Z}^+$, Q_k be the k-th dyadic parent of Q, then $|Q_k| = 2^{nk}|Q|$ and

$$(\mathcal{M}_{\mathcal{B}} f_2)(x) \leqslant \sup_{Q_k \in \mathcal{B}} \frac{1}{|Q_k|} \int_{Q_k} |f(y)| \, \mathrm{d}y \leqslant \sum_{k=1}^{\infty} I_{Q_k} \left(\frac{1}{|Q_k|} \right)^{\frac{1}{p}} \|f\|_{L^p(Q_k)},$$

where $I_{Q_k} = 1$ if $Q_k \in \mathcal{B}$, $I_{Q_k} = 0$ if $Q_k \in \mathcal{D} \setminus \mathcal{B}$. So we have

$$\left\| \mathcal{M}_{\mathcal{B}} f_2 \right\|_{L^p(Q)} \leqslant \sum_{k=1}^{\infty} 2^{-\frac{nk}{p}} I_{Q_k} \|f\|_{L^p(Q_k)}.$$

Let $\mathcal{Q} := \{Q_k \in \mathcal{B} : \ Q \in \mathcal{B}\}$, then

$$\left\| \mathcal{M}_{\mathcal{B}} f_2 \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \sum_{k=1}^{\infty} 2^{\frac{nk}{r} - \frac{nk}{p}} \left(\sum_{Q_k \in \mathcal{Q}} \|f\|_{L^p(Q_k)}^r \right)^{\frac{1}{r}} \lesssim \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)},$$

where the penultimate inequality is due to the fact that there are at most 2^{nk} dyadic cubes in $\mathcal{B}$ such that their k-th dyadic parent is Q_k.

To sum up, we obtain

$$\left\| \mathcal{M}_{\mathcal{B}} f \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \mathcal{M}_{\mathcal{B}} f_1 \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} + \left\| \mathcal{M}_{\mathcal{B}} f_2 \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Thus we finish the proof. □

Denote $\alpha \in \{0,1,2\}^n$ by $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n), \ \alpha_j \in \{0,1,2\}, j = 1,2,\ldots,n$. Let

$$Q_{\nu \vec{m}}^{\alpha} := \prod_{j=1}^{n} \left[\frac{m_j + \frac{\alpha_j}{3}}{2^\nu}, \frac{m_j + 1 + \frac{\alpha_j}{3}}{2^\nu} \right),$$

$\mathcal{D}_\nu^\alpha := \{Q_{\nu\vec{m}}^\alpha : \vec{m} \in \mathbb{Z}^n\}$, and $\mathcal{B}_\alpha := \bigcup_{\nu\in\mathbb{N}} \mathcal{D}_\nu^\alpha$, as well as $\mathcal{D}_\alpha := \bigcup_{\nu\in\mathbb{Z}} \mathcal{D}_\nu^\alpha$. By the notations we know that $\mathcal{B}_{(0,0,\dots,0)} = \mathcal{B}$ and $\mathcal{D}_{(0,0,\dots,0)} = \mathcal{D}$.

The notation $\mathcal{M}_{\mathcal{B}_\alpha}$ means the Hardy-Littlewood maximal operator associated with $\mathcal{B}_\alpha$ and is defined by

$$(\mathcal{M}_{\mathcal{B}_\alpha} f)(x) := \sup_{Q\in\mathcal{B}_\alpha, x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad x \in \mathbb{R}^n.$$

By a similar methods of Theorem 6, we have

Corollary 1. *Let $1 < p < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}_\alpha}$ is bounded on $\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)$.*

To establish the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}_\alpha}$ in the space $\mathcal{B}^r L^p(\mathbb{R}^n)$, it is essential to demonstrate the norm equivalence between the spaces $\mathcal{B}^r L^p(\mathbb{R}^n)$ and $\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)$.

Lemma 3. *Let $1 < p < r < \infty$ and $\alpha \in \{0,1,2\}^n$. The norms of $\mathcal{B}^r L^p(\mathbb{R}^n)$ and $\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)$ are equivalent.*

Proof. Let $Q_{\nu\vec{m}}^\alpha \in \mathcal{B}_\alpha$, there exists $\vec{m}_1(\nu,\alpha), \vec{m}_2(\nu,\alpha), \dots, \vec{m}_{2^n}(\nu,\alpha) \in \mathbb{Z}^n$ such that

$$Q_{\nu\vec{m}}^\alpha \subset \bigcup_{k=1}^{2^n} Q_{\nu(\vec{m}+\vec{m}_k(\nu,\alpha))}.$$

Then we have

$$\|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)} \leqslant \sum_{k=1}^{2^n} \left(\sum_{\nu\in\mathbb{N}, \vec{m}\in\mathbb{Z}^n} \|f\|_{L^p\left(Q_{\nu(\vec{m}+\vec{m}_k(\nu,\alpha))}\right)}^r \right)^{\frac{1}{r}} \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

By a similar way,

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)}.$$

Thus

$$\frac{1}{2^n} \|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)} \leqslant \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)}.$$

Thus we finish the proof. □

Based on Lemma 3, we state the following theorem.

Theorem 7. *Let $1 < p < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}_\alpha}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.*

Local Hardy-Littlewood maximal operators can be dominated by a sum of a sequence of Hardy-Littlewood maximal operators $\{\mathcal{M}_{\mathcal{B}_\alpha}\}_{\alpha\in\{0,1,2\}^n}$ as follows

$$\mathcal{M}_{\mathrm{loc}} f \lesssim \sum_{\alpha\in\{0,1,2\}^n} \mathcal{M}_{\mathcal{B}_\alpha} f.$$

This fact leads to the conclusion that the local Hardy-Littlewood maximal operator is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.

Theorem 8. *Let $1 < p < r < \infty$. The local Hardy-Littlewood maximal operator $\mathcal{M}_{\text{loc}}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.*

We now aim to prove the local Hardy-Littlewood maximal operator $\mathcal{M}_{\text{loc}}$ is bounded from Bourgain-Lebesgue spaces to weak Bourgain-Lebesgue spaces.

Definition 2 (The weak Bourgain-Lebesgue space)**.** Let $0 < p < r < \infty$, the weak Bourgain-Lebesgue space $\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)$ is defined as

$$\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \ \|f\|_{\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)} := \left(\sum_{Q \in \mathcal{B}} \|f\|^r_{L^{p,\infty}(Q)} \right)^{\frac{1}{r}},$$
$$\|f\|_{L^{p,\infty}(Q)} = \inf\{C > 0 : \lambda^p |\{x \in Q : |f(x)| > \lambda\}| \leqslant C^p\}.$$

It is easy to see that $\|f\|_{\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)} \leqslant \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}$. By Theorem 8 we can state the following theorem.

Theorem 9. *Let $1 < p < r < \infty$. The local Hardy-Littlewood maximal operator $\mathcal{M}_{\text{loc}}$ is bounded from $\mathcal{B}^r L^p(\mathbb{R}^n)$ to $\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)$.*

We proceed to prove the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$ for $1 < r < \infty$.

Lemma 4 (see Lemma 2.1.5 in [3])**.** *Let $\{Q_1, Q_2, \dots, Q_k\}$ be a finite collection of open cubes in $\mathbb{R}^n$. Then there exists a finite subcollection $\{Q_{j_1}, \dots, Q_{x_{j_l}}\}$ of pairwise disjoint cubes such that*

$$\left| \bigcup_{i=1}^k Q_i \right| \leqslant 3^n \sum_{r=1}^l |Q_{j_r}|.$$

Inspired by Theorem 2.1.6 in [3] and with the help of Lemma 4, we obtain the following lemma.

Lemma 5. *Let $f \in L^1(\mathbb{R}^n)$ and $Q \subset \mathbb{R}^n$, then*

$$\left| \{x \in Q : \ (\mathcal{M}f)(x) > \alpha\} \right| \leqslant \frac{3^n}{\alpha} \int_{\{x \in Q: \ (\mathcal{M}f)(x) > \alpha\}} |f(y)| \, dy.$$

Proof. The proof is similar to Theorem 2.1.6 in [3], so we omit it here . □

By Lemma 5 we know that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$.

Theorem 10. *Let $1 < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$.*

Proof. Let $Q \in \mathcal{B}$, by Lemma 5, we have

$$\left|\left\{x \in Q : (\mathcal{M}f)(x) > \alpha\right\}\right| \leqslant \frac{3^n}{\alpha} \int_{\left\{x \in Q:\ (\mathcal{M}f)(x) > \alpha\right\}} |f(y)| \, \mathrm{d}y \leqslant \frac{3^n}{\alpha} \|f\|_{L^1(Q)}.$$

Thus

$$\left\|\mathcal{M}f\right\|_{L^{1,\infty}(Q)} \leqslant 3^n \|f\|_{L^1(Q)},$$

which means

$$\left\|\mathcal{M}f\right\|_{\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)} \leqslant 3^n \|f\|_{\mathcal{B}^r L^1(\mathbb{R}^n)}.$$

Thus we finish the proof. □

By the fact that $(\mathcal{M}_{\mathrm{loc}} f)(x) \leqslant (\mathcal{M}f)(x)$ for a.e. $x \in \mathbb{R}^n$ and Theorem 10, we have the following corollary.

Corollary 2. *Let $1 < r < \infty$. The local Hardy-Littlewood maximal operator $\mathcal{M}_{\mathrm{loc}}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$.*

By the way, the constant 3^n in Lemma 5 can be replaced by 1 if we only focus on the Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}}$.

Corollary 3. *Let $f \in L^1(\mathbb{R}^n)$ and $Q \in \mathcal{B}$, then*

$$\left|\left\{x \in Q : (\mathcal{M}_{\mathcal{B}} f)(x) > \alpha\right\}\right| \leqslant \frac{1}{\alpha} \int_{\left\{x \in Q:\ (\mathcal{M}_{\mathcal{B}} f)(x) > \alpha\right\}} |f(y)| \, \mathrm{d}y.$$

Remark 1. The method used in Theorem 6 fails to prove the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ in $\mathcal{B}^r L^p(\mathbb{R}^n)$. We do not know whether the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $\mathcal{B}^r L^p(\mathbb{R}^n)$.

5. The boundedness of vector-valued local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces

In the last section, we focus on the boundedness of vector-valued local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces.

Theorem 11. *Let $1 < p < r < \infty$, $0 < q < \infty$ and $\{f_i\}_{i \in \mathbb{Z}^+}$ be an sequence of functions contained in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\left\|\left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathcal{B}} f_i|^q\right)^{\frac{1}{q}}\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\|\left(\sum_{i=1}^{\infty} |f_i|^q\right)^{\frac{1}{q}}\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Proof. Let $Q \in \mathcal{B}$, $i \in \mathbb{Z}^+$,

$$f_{i,1} := f_i \chi_Q,\ f_{i,2} := f_i \chi_{\mathbb{R}^n \setminus Q}.$$

Then $f = f_{i,1} + f_{i,2}$. By the Fefferman-Stein vector valued inequality, for example see (5.6.25) in [3], we have

$$\left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,1} \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_{i,1}|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} = \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q)}.$$

So

$$\left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,1} \right|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Let $k \in \mathbb{Z}^+$, Q_k be the k-th dyadic parent of Q, then $|Q_k| = 2^{nk}|Q|$. For all $x \in Q$,

$$\left(\mathcal{M}_{\mathcal{B}} f_{i,2} \right)(x) = \sup_{Q_k \in \mathcal{B}} \frac{1}{|Q_k|} \int_{Q_k \setminus Q} |f_i(y)|\, \mathrm{d}y \leqslant \sum_{k=1}^{\infty} \frac{I_{Q_k}}{|Q_k|} \int_{Q_k} |f_i(y)|\, \mathrm{d}y,$$

where $I_{Q_k} = 1$ if $Q_k \in \mathcal{B}$, $I_{Q_k} = 0$ if $Q_k \in \mathcal{D} \setminus \mathcal{B}$. By the Minkowski inequality and the Hölder inequality, we have

$$\left(\sum_{i=1}^{\infty} \left| \left(\mathcal{M}_{\mathcal{B}} f_{i,2} \right)(x) \right|^q \right)^{\frac{1}{q}} \lesssim \sum_{k=1}^{\infty} \left(\frac{I_{Q_k}}{|Q_k|} \right)^{\frac{1}{p}} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q_k)}.$$

So

$$\left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,2} \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q)} \lesssim \sum_{k=1}^{\infty} \left(\frac{|Q| I_{Q_k}}{|Q_k|} \right)^{\frac{1}{p}} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q_k)}.$$

It means

$$\begin{aligned} \left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,2} \right|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &\lesssim \sum_{k=1}^{\infty} 2^{\frac{nk}{r} - \frac{nk}{p}} \left[\sum_{Q_k \in \mathcal{Q}} I_{Q_k} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q_k)}^r \right]^{\frac{1}{r}} \\ &\lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{B^r L^p(\mathbb{R}^n)}, \end{aligned}$$

where the inequalities are due to $\mathcal{Q} := \{Q_k \in \mathcal{B} : Q \in \mathcal{B}\}$ and the fact that there are at most 2^{nk} dyadic cubes in $\mathcal{B}$ such that their k-th dyadic parent is Q_k.

By the fact $|\mathcal{M}_{\mathcal{B}} f_i| \leqslant |\mathcal{M}_{\mathcal{B}} f_{i,1}| + |\mathcal{M}_{\mathcal{B}} f_{i,2}|$, we have

$$\left\| \left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathcal{B}} f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}. \qquad \square$$

By the similar argument in the proof of Theorem 11 and the equivalent norms of $\mathcal{B}^r L^p(\mathbb{R}^n)$ and $\mathcal{B}^r_\alpha L^p(\mathbb{R}^n)$, we have the following corollary.

Corollary 4. *Let $1 < p < r < \infty$, $0 < q < \infty$ and $\{f_i\}_{i\in\mathbb{Z}^+}$ be an sequence of functions contained in $L^p_{\text{loc}}(\mathbb{R}^n)$. Then*

$$\left\| \left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathcal{B}_\alpha} f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Then we have the boundedness of vector-valued local Hardy-Littlewood maximal operators on $\mathcal{B}^r L^p(\mathbb{R}^n)$.

Theorem 12. *Let $1 < p < r < \infty$, $0 < q \leqslant \infty$ and $\{f_i\}_{i\in\mathbb{Z}^+}$ be an sequence of functions contained in $L^p_{\text{loc}}(\mathbb{R}^n)$. Then*

$$\left\| \left(\sum_{i=1}^{\infty} |\mathcal{M}_{\text{loc}} f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

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Authors' addresses

Baohua Dong
Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, China
Center for Applied Mathematics of Jiangsu Province, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail address: baohuadong@nuist.edu.cn

Xuanxuan Hu
Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail address: huxvanxvan@163.com

Jingshi Xu
(**Corresponding author**) School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China
Center for Applied Mathematics of Guangxi (GUET), Guilin 541004, China
Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin 541004, China
E-mail address: jingshixu@126.com